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Research Article

On the Periods of Biperiodic Fibonacci and Biperiodic Lucas Numbers

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This paper is concerned with periods of Biperiodic Fibonacci and Biperiodic Lucas sequences taken as modulo prime and prime power. By using Fermat's little theorem, quadratic reciprocity, many results are obtained.

1. Introduction

Fibonacci sequence and Lucas sequence are well-known sequences among integer sequences. The Fibonacci numbers satisfy the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Binet's Formula for the Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ are roots of the characteristic equation $x^2 - x - 1 = 0$. The positive root α is known as "golden ratio." Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art [1–3]. In particular, Edson and Yayenie [4] introduced the Biperiodic Fibonacci sequence as follows:

$$\begin{aligned} q_0 &= 0, \\ q_1 &= 1, \\ q_n &= \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \ge 2, \end{aligned} \tag{2}$$

where a and b are two nonzero real numbers. We take a and b as integers. If we take $a = b = 1$, we get the Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$.

If we take $a = b = 2$, we get the Pell sequence $\{0, 1, 2, 5, 12, 29, 70, \ldots\}$. Similarly, if we take $a = b = k$, we get the k–Fibonacci sequence [5–8]. Binet's Formula for the Biperiodic Fibonacci sequence is given as

$$\begin{aligned} q_m &= \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor m/2 \rfloor}}\right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \\ &= \frac{1}{a^{\lfloor (m-1)/2 \rfloor} b^{\lfloor (m+1)/2 \rfloor}} \frac{\alpha^m - \beta^m}{\alpha - \beta}, \end{aligned} \tag{3}$$

where $\alpha, \beta = (ab \pm \sqrt{a^2b^2 + 4ab})/2$ are the roots of the characteristic equation $x^2 - abx - ab = 0$ and $\xi(m) = m - 2\lfloor m/2 \rfloor$. Moreover,

$$\begin{aligned} (\alpha\beta) &= -ab, \\ (\alpha + \beta) &= ab. \end{aligned} \tag{4}$$

This sequence and its properties can be found in [1, 4].

Another well-known sequence is the Lucas sequence which satisfies the same recurrence relation as the Fibonacci sequence $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. Binet's Formula for the Lucas sequence is

$$L_n = \alpha^n + \beta^n, \tag{5}$$

where α and β are defined in (1). Bilgici defined generalization of Lucas sequence similar to the Biperiodic Fibonacci sequence as follows:

$$l_0 = 2, \quad l_1 = a, \quad l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2, \tag{6}$$

where a and b are two nonzero real numbers. We take a and b as integers. This sequence and other generalizations of Lucas sequence with their properties can be found in [9, 10]. If we take $a = b = 1$, we get the Lucas sequence $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$. Also, if we take $a = b = k$, we get the k–Lucas sequence [11].

$$l_n = \left(\frac{a^{\xi(n)}}{(ab)^{\lfloor (n+1)/2 \rfloor}} \right) \alpha^n + \beta^n = \frac{1}{a^{\lfloor n/2 \rfloor} b^{\lfloor (n+1)/2 \rfloor}} \left(\alpha^n + \beta^n \right), \tag{7}$$

where α, β, and ξ are defined in (3).

On the other hand several researchers have made significant studies about the period of the recurrence sequences [2]. Wall [12] defined the period-length of the recurring series obtained by reducing a Fibonacci series by a modulus m. As an example, the Fibonacci sequence mod3 is

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, \ldots \tag{8}$$

and has period 8. Vinson [3] and Robinson [13] both extended Wall's study. Moreover, they studied the Fibonacci sequence for prime moduli and showed that for primes $p \equiv 1, 4 \pmod 5$ the period-length of the Fibonacci sequence mod$p$ divides $p - 1$, while for primes $p \equiv 2, 3 \pmod 5$ the period-length of the Fibonacci sequence mod p divides $2(p+1)$.

Gupta et al. [14] give alternative proofs of this results that also use the Fibonacci matrix. They place the roots of its characteristic polynomial in an appropriate splitting fields. Renault [15] examined the behaviour of the (a, b)-Fibonacci sequence under a modulus.

Lucas studied the (a, b)-Fibonacci sequence extensively. He assigned $\Delta = a^2 + 4b$ and deduced that if Δ is quadratic residue (that is, a nonzero perfect square) mod p, then $\alpha(p) \mid p - 1$. If Δ is quadratic nonresidue then $\alpha(p) \mid p + 1$. Also, Rogers and Campbell studied the period of the Fibonacci sequence mod$j$ [16]. They investigated the Fibonacci sequence modulo p prime and then generalized to prime powers.

2. Period of Biperiodic Fibonacci Sequence

In this section, we investigate the Biperiodic Fibonacci sequence modulo p prime and then generalize to prime powers.

Definition 1. The period of the Biperiodic Fibonacci sequence modulo a positive integer j is the smallest positive integer m such that

$$q_m \equiv 0 \pmod j, \quad q_{m+1} \equiv 1 \pmod j. \tag{9}$$

By the definition above, the only members that can possibly come back to the starting point are multiples of m. This can be summed up in the statement that if m is the period of $q_n \pmod j$, then,

$$q_k \equiv 0 \pmod j, \quad q_{k+1} \equiv 1 \pmod j \quad \Updownarrow \quad m \mid k, \quad \text{for any } k \in Z. \tag{10}$$

Theorem 2. *Let p be a prime and let n be a positive integer. If*

$$a \equiv 1 \pmod p \tag{11}$$

then,

$$a^{p^n} \equiv 1 \pmod{p^{n+1}}. \tag{12}$$

We remark that the proof of the Theorem 2 can be seen in [16].

Theorem 3. *Let p be a prime, let k be a positive integer, and let α and β be the fundamental roots of the Biperiodic Fibonacci sequence. If m is the period of $q_n \pmod p$,*

$$\alpha^{mp^{k-1}} \equiv \beta^{mp^{k-1}} \equiv 1 \pmod{p^k}. \tag{13}$$

Proof. For $a \neq 0$ and p is a prime integer, we have

$$q_m = \left(\frac{\left(a^{1-\xi(m)}\right)}{(ab)^{\lfloor m/2 \rfloor}} \right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \equiv 0 \pmod p, \tag{14}$$

and then

$$\alpha^m \equiv \beta^m \pmod p. \tag{15}$$

Also we obtain

$$\begin{aligned} q_m &\equiv q_{m+1} - q_1 \pmod p \\ &\equiv \left(\frac{a^{1-\xi(m+1)}}{(ab)^{\lfloor (m+1)/2 \rfloor}} \right) \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} - \frac{\alpha - \beta}{\alpha - \beta} \pmod p \\ &\equiv 0 \pmod p. \end{aligned} \tag{16}$$

If m is even, then

$$q_m \equiv \left(\frac{1}{(ab)^{m/2}} \right) \frac{\alpha\left(\alpha^m - 1\right) - \beta\left(\beta^m - 1\right)}{\alpha - \beta} \pmod p. \tag{17}$$

From $\alpha^m \equiv \beta^m \pmod{p}$, we get

$$q_m \equiv \left(\frac{1}{(ab)^{m/2}}\right)(\alpha^m - 1) \equiv 0 \pmod{p}. \quad (18)$$

Thus, $\alpha^m \equiv \beta^m \equiv 1 \pmod{p}$. From Theorem 2,

$$\alpha^{mp^{k-1}} \equiv \beta^{mp^{k-1}} \equiv 1 \pmod{p^k}. \quad (19)$$

If m is odd,

$$q_m \equiv \left(\frac{a}{(ab)^{(m+1)/2}}\right)\left(\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} - \frac{\alpha - \beta}{\alpha - \beta}\right) \pmod{p}. \quad (20)$$

For $\alpha^m \equiv \beta^m \pmod{p}$ and $a/(ab)^{(m+1)/2} \neq 0$,

$$q_m \equiv \left(\frac{a}{(ab)^{(m+1)/2}}\right)(\alpha^m - 1) \equiv 0 \pmod{p}. \quad (21)$$

Therefore, $\alpha^m \equiv \beta^m \equiv 1 \pmod{p}$. From Theorem 2,

$$\alpha^{mp^{k-1}} \equiv \beta^{mp^{k-1}} \equiv 1 \pmod{p}. \quad (22)$$

□

Theorem 4. *Let p be an odd prime, let m denote the period of $q_n \pmod{p}$, and let $\triangle = a^2b^2 + 4ab$ be a nonzero quadratic residue* $\bmod p$*; then $m \mid p - 1$.*

Proof. As is known from Fermat's little theorem,

$$\begin{aligned} \alpha^{p-1} &\equiv 1 \pmod{p}, \\ \beta^{p-1} &\equiv 1 \pmod{p}. \end{aligned} \quad (23)$$

Thus, we have

$$\begin{aligned} q_{p-1} &\equiv \left(\frac{1}{a^{\lfloor (p-2)/2 \rfloor} b^{\lfloor (p-1)/2 \rfloor}}\right) \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta} \equiv 0 \pmod{p}, \\ q_p &= \left(\frac{1}{a^{\lfloor (p-1)/2 \rfloor} b^{\lfloor p/2 \rfloor}}\right) \frac{\alpha^p - \beta^p}{\alpha - \beta} \\ &\equiv \left(\frac{1}{(ab)^{(p-1)/2}}\right) \frac{\alpha - \beta}{\alpha - \beta} \equiv \frac{1}{(ab)^{(p-1)/2}} \pmod{p}. \end{aligned} \quad (24)$$

From $ab = -\alpha\beta$, we have

$$(ab)^{(p-1)/2} = \left((-\alpha\beta)^{p-1}\right)^{1/2} \equiv 1 \pmod{p}, \quad (25)$$

and then

$$q_p \equiv 1 \pmod{p}. \quad (26)$$

So that

$$\begin{aligned} q_{p-1} &\equiv 0 \pmod{p}, \\ q_p &\equiv 1 \pmod{p}. \end{aligned} \quad (27)$$

Therefore, (10) implies that $m \mid p - 1$. □

Lemma 5. *If $\triangle$ is a quadratic nonresidue* $\bmod p$*, then $(\sqrt{\triangle})^p = -\sqrt{\triangle}$.*

Lemma 6. *Let α and β be the two roots of $x^2 - abx - ab = 0$ in $F = F_{p^2}$. $\triangle$ is a quadratic nonresidue* mod p*; then $\beta^{p+1} = \alpha^{p+1}$.*

Theorem 7. *Let p be an odd prime, let m denote the period of $q_n \pmod{p}$, and let $\triangle$ be a quadratic nonresidue* $\bmod p$*; then $m \mid 2p + 2$.*

Proof. From the Binomial theorem, we get $(\alpha - \beta)^p \equiv \alpha^p - \beta^p \pmod{p}$. It follows that

$$\begin{aligned} q_p &= \left(\frac{1}{a^{\lfloor (p-1)/2 \rfloor} b^{\lfloor p/2 \rfloor}}\right) \frac{\alpha^p - \beta^p}{\alpha - \beta} \\ &= \left(\frac{1}{(ab)^{(p-1)/2}}\right) \frac{\alpha^p - \beta^p}{\alpha - \beta} \\ &\equiv \left(\frac{1}{(ab)^{(p-1)/2}}\right) \frac{(\alpha - \beta)^p}{\alpha - \beta} \pmod{p} \\ &\equiv \left(\frac{1}{(ab)^{(p-1)/2}}\right) \frac{\left(\sqrt{\triangle}\right)^p}{\sqrt{\triangle}} \pmod{p} \\ &\equiv \left(\frac{1}{(ab)^{(p-1)/2}}\right) \frac{-\sqrt{\triangle}}{\sqrt{\triangle}} \pmod{p} \\ &\equiv -1 \pmod{p}, \\ q_{p+1} &= \left(\frac{1}{a^{\lfloor p/2 \rfloor} b^{\lfloor (p+1)/2 \rfloor}}\right) \frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta} \\ &= \left(\frac{1}{a^{(p-1)/2} b^{(p+1)/2}}\right) \frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta}. \end{aligned} \quad (28)$$

From Lemma 6, we obtain

$$q_{p+1} \equiv 0 \pmod{p}. \quad (29)$$

Thus, $m \nmid p + 1$. Also, we have

$$\begin{aligned} q_{2p+2} &= \left(\frac{1}{a^{\lfloor (2p+1)/2 \rfloor} b^{\lfloor (2p+2)/2 \rfloor}}\right) \frac{\alpha^{2p+2} - \beta^{2p+2}}{\alpha - \beta} \\ &= \left(\frac{1}{(ab)^p}\right) \frac{\left(\alpha^{p+1}\right)^2 - \left(\beta^{p+1}\right)^2}{\alpha - \beta} \equiv 0 \pmod{p}, \\ q_{2p+3} &= \left(\frac{1}{a^{\lfloor (2p+2)/2 \rfloor} b^{\lfloor (2p+3)/2 \rfloor}}\right) \frac{\alpha^{2p+3} - \beta^{2p+3}}{\alpha - \beta} \\ &= \left(\frac{1}{(ab)^{p+1}}\right) \frac{\left(\alpha^{p+1}\right)^2 \alpha - \left(\beta^{p+1}\right)^2 \beta}{\alpha - \beta} \end{aligned}$$

$$= \left(\frac{1}{(ab)^{p+1}} \left(\alpha^{p+1}\right)^2 \right) \frac{\alpha-\beta}{\alpha-\beta} \equiv \frac{\left(\alpha^{p+1}\right)^2}{(ab)^{p+1}} \equiv \frac{\left(\alpha^{p+1}\right)^2}{(-\alpha\beta)^{p+1}} \equiv \frac{\alpha^{p+1}}{\beta^{p+1}} \equiv 1 \pmod{p}. \tag{30}$$

Thus, from (10), $m \mid 2p+1$. □

Theorem 8. *Let p be a prime, let m denote the period of $q_n \pmod{p}$, and let m' denote the period of $q_n \pmod{p^k}$. If mp^{k-1} is even then $m' \mid mp^{k-1}$ and if mp^{k-1} is odd then $m' \mid 4mp^{k-1}$.*

Proof. We have shown that $\alpha^{mp^{k-1}} \equiv \beta^{mp^{k-1}} \equiv 1 \pmod{p^k}$ in Theorem 3. So that

$$\begin{aligned} q_{mp^{k-1}} &= \left(\frac{a^{1-\xi(mp^{k-1})}}{(ab)^{\lfloor mp^{k-1}/2 \rfloor}} \right) \frac{\alpha^{mp^{k-1}} - \beta^{mp^{k-1}}}{\alpha-\beta} \equiv 0 \pmod{p^k}, \\ q_{mp^{k-1}+1} &= \left(\frac{a^{1-\xi(mp^{k-1}+1)}}{(ab)^{\lfloor (mp^{k-1}+1)/2 \rfloor}} \right) \frac{\alpha^{mp^{k-1}+1} - \beta^{mp^{k-1}+1}}{\alpha-\beta} \\ &\equiv \frac{a^{1-\xi(mp^{k-1}+1)}}{(ab)^{\lfloor (mp^{k-1}+1)/2 \rfloor}} \pmod{p^k}. \end{aligned} \tag{31}$$

If mp^{k-1} is even, then

$$q_{mp^{k-1}+1} \equiv \frac{1}{(ab)^{mp^{k-1}/2}} \pmod{p^k}. \tag{32}$$

From $\alpha\beta = -ab$ and by using Theorem 3, it follows that

$$(ab)^{mp^{k-1}/2} = (-\alpha\beta)^{mp^{k-1}/2} = \left((-\alpha\beta)^{mp^{k-1}} \right)^{1/2} \equiv 1 \pmod{p^k}. \tag{33}$$

Then,

$$q_{mp^{k-1}+1} \equiv 1 \pmod{p^k}. \tag{34}$$

Thus, from (10), $m' \mid mp^{k-1}$.

If mp^{k-1} is odd then

$$\begin{aligned} q_{4mp^{k-1}} &= \left(\frac{a^{1-\xi(4mp^{k-1})}}{(ab)^{\lfloor 4mp^{k-1}/2 \rfloor}} \right) \frac{\alpha^{4mp^{k-1}} - \beta^{4mp^{k-1}}}{\alpha-\beta} \equiv 0 \pmod{p^k}, \\ q_{4mp^{k-1}+1} & \\ &= \left(\frac{a^{1-\xi(4mp^{k-1}+1)}}{(ab)^{\lfloor (4mp^{k-1}+1)/2 \rfloor}} \right) \frac{\alpha^{4mp^{k-1}+1} - \beta^{4mp^{k-1}+1}}{\alpha-\beta} \\ &\equiv \frac{1}{(ab)^{2mp^{k-1}}} \pmod{p^k}. \end{aligned} \tag{35}$$

Since $\alpha\beta = -ab$ and $\alpha^{mp^{k-1}} \equiv \beta^{mp^{k-1}} \equiv 1 \pmod{p^k}$, then,

$$\begin{aligned} (ab)^{2mp^{k-1}} &= (-\alpha\beta)^{2mp^{k-1}} = \left((-\alpha\beta)^{mp^{k-1}} \right)^2 \equiv 1 \pmod{p^k}, \\ q_{4mp^{k-1}+1} &\equiv 1 \pmod{p^k}. \end{aligned} \tag{36}$$

Thus, $m' \mid 4mp^{k-1}$. □

3. Period of Biperiodic Lucas Sequence

In this section, we investigate the Biperiodic Lucas sequence modulo p a prime similar to Biperiodic Fibonacci sequence.

Definition 9. The period of the Biperiodic Lucas sequence modulo a positive integer j is the smallest positive integer t such that

$$\begin{aligned} l_t &\equiv 2 \pmod{j}, \\ l_{t+1} &\equiv a \pmod{j}. \end{aligned} \tag{37}$$

For the same reasons as the Biperiodic Fibonacci sequence we have that if t is the period of $l_n \pmod{j}$, then

$$\begin{gathered} l_k \equiv 0 \pmod{j}, \\ l_{k+1} \equiv 1 \pmod{j} \\ \Updownarrow \\ t \mid k. \end{gathered} \tag{38}$$

for any $k \in Z$,

Theorem 10. *Let p be an odd prime, let t denote the period of $l_n \pmod{p}$, and let $\triangle$ be a nonzero quadratic residue* mod p; *then $t \mid p-1$.*

Proof. We use Fermat's little theorem to get

$$\begin{aligned} \alpha^{p-1} &\equiv 1 \pmod{p}, \\ \beta^{p-1} &\equiv 1 \pmod{p}, \end{aligned} \tag{39}$$

so

$$l_{p-1} = \left(\frac{1}{a^{\lfloor (p-1)/2 \rfloor} b^{\lfloor p/2 \rfloor}} \right) \left(\alpha^{p-1} + \beta^{p-1} \right) \equiv \frac{2}{(ab)^{(p-1)/2}} \pmod{p}. \tag{40}$$

We know $(ab)^{(p-1)/2} \equiv 1 \pmod{p}$; thus

$$l_{p-1} \equiv 2 \pmod{p}. \tag{41}$$

Also, we have

$$\begin{aligned} l_p &= \left(\frac{1}{a^{\lfloor p/2\rfloor}b^{\lfloor (p+1)/2\rfloor}}\right)(\alpha^p + \beta^p) \\ &\equiv \left(\frac{1}{a^{(p-1)/2}b^{(p+1)/2}}\right)(\alpha + \beta) \pmod{p} \\ &\equiv \frac{ab}{(ab)^{(p-1)/2}} \equiv \frac{ab}{b} \pmod{p} \\ &\equiv a \pmod{p}. \end{aligned} \tag{42}$$

By using (10), we get $t \mid p - 1$. □

Theorem 11. *Let p be an odd prime, let t denote the period of l_n (mod$p$), and let $\triangle$ be a quadratic nonresidue* mod$p$*; then $t \mid 2p + 2$.*

Proof. From Lemma 6 and (3), we get

$$\begin{aligned} l_{2p+2} &= \left(\frac{1}{a^{\lfloor (2p+2)/2\rfloor}b^{\lfloor (2p+3)/2\rfloor}}\right)\left(\alpha^{2p+2} + \beta^{2p+2}\right) \\ &\equiv \left(\frac{1}{(ab)^{p+1}}\right)\left(\left(\alpha^{p+1}\right)^2 + \left(\beta^{p+1}\right)^2\right) \pmod{p} \\ &\equiv \frac{2\left(\beta^{p+1}\right)^2}{(ab)^{p+1}} \pmod{p} \\ &\equiv \frac{2\left(\beta^{p+1}\right)^2}{\alpha^{p+1}\beta^{p+1}} \pmod{p} \\ &\equiv \frac{2\beta^{p+1}}{\alpha^{p+1}} \pmod{p} \\ &\equiv 2 \pmod{p}, \\ l_{2p+3} &= \left(\frac{1}{a^{\lfloor (2p+3)/2\rfloor}b^{\lfloor (2p+4)/2\rfloor}}\right)\left(\alpha^{2p+3} + \beta^{2p+3}\right) \\ &= \left(\frac{1}{a^{p+1}b^{p+2}}\right)\left(\alpha^{2p+3} + \beta^{2p+3}\right) \\ &= \left(\frac{1}{a^{p+1}b^{p+2}}\right)\left(\left(\alpha^{p+1}\right)^2\alpha + \left(\beta^{p+1}\right)^2\beta\right) \\ &= \frac{1}{(\alpha+\beta)\left(\alpha^{p+1}\right)^2}(ab)^{p+1}b \\ &= \frac{1}{(ab)\left(\alpha^{p+1}\right)^2}(ab)^{p+1}b = \frac{1}{a\left(\alpha^{p+1}\right)^2}\alpha^{p+1}\beta^{p+1} \\ &= \frac{1}{a\alpha^{p+1}}\beta^{p+1} = \frac{1}{a} \pmod{p}. \end{aligned} \tag{43}$$

Thus, from (10), $m \mid 2p + 2$. □

Competing Interests

The authors declare that they have no competing interests.

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