Research Article

Globally Asymptotic Stability of Stochastic Nonlinear Systems with Time-Varying Delays via Output Feedback Control

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We address the problem of globally asymptotic stability for a class of stochastic nonlinear systems with time-varying delays. By the backstepping method and Lyapunov theory, we design a linear output feedback controller recursively based on the observable linearization for a class of stochastic nonlinear systems with time-varying delays to guarantee that the closed-loop system is globally asymptotically stable in probability. In particular, we extend the deterministic nonlinear system to stochastic nonlinear systems with time-varying delays. Finally, an example and its simulations are given to illustrate the theoretical results.

1. Introduction

In recent years, the stochastic nonlinear system has received much attention and has enjoyed a good development, which has been widely applied in many fields such as engineering and finance [1]. Particularly, owing to the reason that a system by the output feedback is more flexible to respond to the information of control systems, the stochastic nonlinear system via the output feedback control has been more widely studied. Marino and Tomei [2] and Battilotti [3] designed linear observer to study the output feedback control for nonlinear systems. As a result, a large number of researchers have been focused on the stability analysis of stochastic nonlinear systems and the design of controller.

Since the backstepping method has been introduced in the nonlinear system, the theory of stochastic nonlinear systems has achieved a remarkable development. According to the different Lyapunov function, there are two ways to solve the problem. One is that Pan and Basar [4–6] considered a quadratic Lyapunov function. By using the backstepping method to design the controller, they discussed the risk-sensitive optimal control problem. The other is that Deng and

Krstić [7–10] used a quartic Lyapunov function to guarantee that the closed-loop system is globally asymptotically stable in probability. Based on their works, Liu et al. [11–13] employed the quartic Lyapunov function to design the output feedback control for stochastic nonlinear systems. M.-L. Liu and Y.-G. Liu [14] and Chen et al. [15] used such Lyapunov function to study the state feedback stability for stochastic nonlinear systems with time-varying delays. In [16], Du et al. discussed the global output feedback stability for a class of uncertain upper-triangular systems with the input delay. Finite-time stability for stochastic nonlinear systems in strict-feedback form was considered in [17]. More results can be found in [18–20].

In this paper, we are concerned with the globally asymptotic stability of stochastic nonlinear system with time-varying delays. We extend the results of the deterministic nonlinear systems in [21] to the stochastic nonlinear systems with time-varying delays by output feedback control and design a Lyapunov-Krasovskii functional to prove the globally asymptotic stability of the closed-system via the linear observer. It is obvious that the form of the linear controller is simpler than that used in [9]. And compared with the

C^1 output feedback controller obtained by the homogeneous observers in [20], the controller designed by the linear observers in this paper is smooth.

The rest of this paper is organized as follows. In Section 2, we present some definitions and establish a new inequality, which plays an important role in the proof of our main results. In Section 3, we construct a Lyapunov-Krasovskii functional and design a linear output feedback control to prove the globally asymptotic stability based on the linear observers. In Section 4, we use an example to illustrate our theoretical results. Finally, we conclude this paper with some general remarks.

2. Preliminaries

In this section, we will give the following notations, definitions, and some preliminary lemmas. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\text{Trace}\{X\}$ denotes the trace for square matrix X; $|X|$ denotes the Euclidean norm of a vector X. $\mathscr{C}([-d, 0]; \mathbb{R}^n)$ denotes the space of continuous $\mathbb{R}^n$-valued functions on $[-d, 0]$ endowed with the norm $\|\cdot\|$ defined by $\|f\| = \sup_{x \in [-d,0]} |f(x)|$ for $f \in \mathscr{C}([-d, 0]; \mathbb{R}^n)$; $\mathscr{C}^b_{\mathscr{F}_0}([-d, 0]; \mathbb{R}^n)$ denotes the family of all $\mathscr{F}_0$-measurable bounded $\mathscr{C}([-d, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -d \leq \theta \leq 0\}$. $\mathscr{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathscr{C}^{2,1}(\mathbb{R}^n \times [-d, \infty); \mathbb{R}^+)$ denotes the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times [-d, \infty)$ which are $\mathscr{C}^2$ in x and $\mathscr{C}^1$ in t; $\mathscr{C}^{2,1}$ denotes the family of all functions which are $\mathscr{C}^2$ in the first argument and $\mathscr{C}^1$ in the second argument. $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous and strictly increasing and vanish at zero; $\mathscr{K}_\infty$ denotes the set of all functions which are of class $\mathscr{K}$ and unbounded. $\mathscr{K}\mathscr{L}$ is the set of all functions $\beta(s, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which are of $\mathscr{K}$ for each fixed t and decrease to zero as $t \to \infty$ for each fixed s.

Consider the following stochastic time-varying delay system:

$$
\begin{aligned}
dx(t) = {} & f(x(t), x(t - d(t)), t)\, dt \\
& + g(x(t), x(t - d(t)), t)\, d\omega(t), \quad \forall t \geq 0,
\end{aligned}
\tag{1}
$$

with initial date $\{x(\theta) : -d \leq \theta \leq 0\} = \xi \in \mathscr{C}^b_{\mathscr{F}_0}([-d, 0]; \mathbb{R}^n)$, where $d(t) : \mathbb{R}_+ \to [0, d]$ is a Borel measurable function and $\omega(t)$ is an r-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, P)$ with Ω being a sample space, $\mathscr{F}$ being a filtration, and P being a probability measure. $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times r}$ are assumed to be locally Lipschitz in $(x(t), x(t - d(t)))$ uniformly in t and satisfy $f(0, 0, t) \equiv 0$, $g(0, 0, t) \equiv 0$.

Definition 1 (see [22]). For any given $V(x(t), t) \in \mathscr{C}^{2,1}$ associated with system (1), the differential operator $\mathscr{L}$ is defined as

$$
\mathscr{L}V(x(t), t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x^T} f + \frac{1}{2} \text{Trace}\left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\}, \tag{2}
$$

where $(1/2)\text{Trace}\{g^T(\partial^2 V/\partial x^2)g\}$ is called as the Hessian term of $\mathscr{L}$.

Definition 2 (see [11, 15]). For system (1), the equilibrium $x = 0$ is said to be globally asymptotically stable in probability, if, for any $\varepsilon > 0$, there exists a class of $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$
P\{|x(t)| < \gamma(\|\xi\|)\} \geq 1 - \varepsilon,
$$
$$
\forall t \geq 0, \ \forall \xi \in \mathscr{C}^b_{\mathscr{F}_0}([-d, 0]; \mathbb{R}^n) \setminus \{0\},
\tag{3}
$$

and for any $\xi \in \mathscr{C}^b_{\mathscr{F}_0}([-d, 0]; \mathbb{R}^n)$, one has

$$
P\left\{ \lim_{t \to \infty} |x(t)| = 0 \right\} = 1. \tag{4}
$$

Definition 3 (see [22]). For fixed coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and real numbers $r_i > 0$, $i = 1, \ldots, n$.

(1) The dilation $\Delta_\epsilon(x)$ is defined by $\Delta_\epsilon(x) = (\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n)$ for any $\epsilon > 0$; $r_1, \ldots, r_n$ are called the weights of the coordinates. For simplicity, we define dilation weight $\Delta = (r_1, \ldots, r_n)$.

(2) A function $V \in \mathscr{C}(\mathbb{R}^n, R)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $V(\Delta_\epsilon(x)) = \epsilon^\tau V(x_1, \ldots, x_n)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, $\epsilon > 0$.

(3) A vector field $h \in \mathscr{C}(\mathbb{R}^n, \mathbb{R}^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in R$ such that $h_i(\Delta_\epsilon(x)) = \epsilon^{\tau + r_i} h_i(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, $\epsilon > 0$.

(4) A homogeneous q-norm is defined as $\|x\|_{\Delta, q} = (\Sigma_{i=1}^n |x_i|^{q/r_i})^{1/q}$ for any $x \in \mathbb{R}^n$, where $q > 1$ is a constant. For simplicity, in this paper, we choose $q = 2$ and write $\|x\|_\Delta$ for $\|x\|_{\Delta, 2}$.

Lemma 4 (see [11, 15, 22]). *For system (1), if there exist a function $V(x(t), t) \in \mathscr{C}^{2,1}(R^n \times [-d, \infty); R_+)$, two class $\mathscr{K}_\infty$ functions α_1, α_2, and a class $\mathscr{K}$ function α_3 such that*

$$
\alpha_1(|x(t)|) \leq V(x(t), t) \leq \alpha_2\left(\sup_{-d \leq \sigma \leq 0} |x(\sigma + t)| \right), \tag{5}
$$

$$
\mathscr{L}V(x(t), t) \leq -\alpha_3(|x(t)|), \tag{6}
$$

then there exists a unique solution on $[-d, \infty)$ for each $x_0 \in \mathbb{R}^n$ and the equilibrium $x(t) = 0$ is globally asymptotically stable in probability.

Lemma 5 (see [22]). *Suppose that $V : R \to R$ is a homogeneous function of degree τ with respect to the dilation weight Δ; then*

(i) *$\partial V / \partial x_i$ is homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weight of x_i;*

(ii) *there is a constant $\overline{\gamma}$ such that $V(x(t)) \leq \overline{\gamma} \|x(t)\|_\Delta^\tau$. Moreover, if $V(x(t))$ is positive definite, then $V(x(t)) \geq \underline{\gamma} \|x(t)\|_\Delta^\tau$, where $\underline{\gamma}$ is a positive constant.*

Lemma 6 (see [1]). *Let $V : D \to R$ be a continuous positive definite on a domain $D = R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathscr{K}$ functions α_1 and α_2, defined on $[0, r]$, such that*

$$
\alpha_1(|x(t)|) \leq V(x(t)) \leq \alpha_2(|x(t)|) \tag{7}
$$

for all $x \in B_r$. If $D = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x(t))$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

Lemma 7. *For any constants $a > 0$ and $b \in \mathbb{R}$, one has that, for any $x, y \in \mathbb{R}$,*

$$-ax^4 + bxy^3 \leq k_1 a^{-1/3} b^{4/3} y^4, \tag{8}$$

$$-ax^4 + bx^3 y \leq k_2 b^4 a^{-3} y^4, \tag{9}$$

where $k_1 = 4^{-1/3} - 4^{-4/3} > 0$ and $k_2 = (3/4)^3 - (3/4)^4 > 0$.

Proof. We first prove (8). Let $Z_1(x) = -ax^4 + bxy^3$, where y is a parameter. Then, we have

$$Z_1'(x) = -4ax^3 + by^3 = 0, \quad x = \left(\frac{b}{4a}\right)^{1/3} y. \tag{10}$$

It is clear that, for any $x \in (-\infty, (b/4a)^{1/3} y)$, $Z_1'(x) > 0$ and $x \in ((b/4a)^{1/3} y, +\infty)$, $Z_1'(x) < 0$. With the sufficient condition of extreme value, $x = (b/4a)^{1/3} y$ is the maximum point of function $Z_1(x)$. Therefore, we get

$$Z_1(x) \leq -a\left[\left(\frac{b}{4a}\right)^{1/3} y\right]^4 + b\left[\left(\frac{b}{4a}\right)^{1/3} y\right] y^3$$
$$= \left(4^{-1/3} - 4^{-4/3}\right) a^{-1/3} b^{4/3} y^4. \tag{11}$$

We now prove (9). Similarly, letting $Z_2(x) = -ax^4 + bx^3 y$, where y is a parameter, we have

$$Z_2'(x) = -4ax^3 + 3bx^2 y = 0, \quad x = \frac{3by}{4a} \text{ or } x = 0. \tag{12}$$

Obviously, for any $x \in (-\infty, 3by/4a)$, $Z_2'(x) > 0$ and $x \in (3by/4a, +\infty)$, $Z_2'(x) < 0$. With the sufficient condition of extreme value, $x = 3by/4a$ is the maximum point of function $Z_2(x)$. Therefore, we obtain

$$Z_2(x) \leq -a\left(\frac{3by}{4a}\right)^4 + b\left(\frac{3by}{4a}\right)^3 y$$
$$= \left[\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4\right] b^4 a^{-3} y^4. \tag{13}$$

$\square$

Lemma 8 (Cauchy-Schwartz's inequality). *For any vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, one has*

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2$$
$$\leq \left(x_1^2 + \cdots + x_n^2\right)\left(y_1^2 + \cdots + y_n^2\right). \tag{14}$$

Lemma 9. *For a sequence of numbers $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$, one has*

$$|a_1 + a_2 + \cdots + a_n| \geq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2},$$

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$\geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

$$\geq \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}. \tag{15}$$

Lemma 10 (see [17]). *For any given real numbers c, d and any real-valued functions $f(x, y) > 0$, $g(x, y, z) \geq 0$, the following inequality holds:*

$$g(x, y, z) |x|^c |y|^d$$
$$\leq \frac{c}{c+d} f(x, y) |x|^{c+d}$$
$$+ \frac{d}{c+d} (g(x, y, z))^{(c+d)/d} (f(x, y))^{-c/d} |y|^{c+d}, \tag{16}$$

where $x, y, z \in \mathbb{R}$. Particularly when one takes $f(x, y) = g(x, y, z) = 1$, $c = 3$, and $d = 4$, then the inequality will become

$$x^3 y \leq \frac{3}{4} x^4 + \frac{1}{4} y^4. \tag{17}$$

3. The Output Feedback Model and Control Design

In this section, we will design a linear observer system for a class of stochastic nonlinear systems with time-varying delays. Using the backstepping method, a simple linear controller will be constructed to guarantee that the closed-loop stochastic system is globally asymptotically stable in probability.

Consider the following stochastic nonlinear system with time-varying delay:

$$dx_1 = \left(x_2 + f_1\left(t, \overline{x}_1, \overline{x}_1(t - d(t))\right)\right) dt$$
$$+ g_1^T\left(t, \overline{x}_1, \overline{x}_1(t - d(t))\right) dw,$$
$$dx_2 = \left(x_3 + f_2\left(t, \overline{x}_2, \overline{x}_2(t - d(t))\right)\right) dt$$
$$+ g_2^T\left(t, \overline{x}_2, \overline{x}_2(t - d(t))\right) dw,$$
$$\vdots$$
$$dx_n = \left(u + f_n\left(t, \overline{x}_n, \overline{x}_n(t - d(t))\right)\right) dt$$
$$+ g_n^T\left(t, \overline{x}_n, \overline{x}_n(t - d(t))\right) dw,$$
$$y = x_1, \tag{18}$$

where the initial data $\{x(\theta) : -d \leq \theta \leq 0\} = \xi \in \mathcal{C}_{\mathcal{F}_0}^b([-d, 0]; \mathbb{R}^n)$, $d(t) : \mathbb{R}_+ \to [0, d]$ is a Borel measurable

function and $d'(t) \leq d' < 1$; ω is an r-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. $\overline{x}_i = (x_1, \ldots, x_i)$ is the system state, $\overline{x}_i(t - d(t)) = (x_1(t - d(t)), \ldots, x_i(t - d(t)))$ is the time-delayed state vectors, $u \in \mathbb{R}$ is the system input, and $y \in \mathbb{R}$ is the system output. $f_i : \mathbb{R}_+ \times \mathbb{R}^i \times \mathbb{R}^i \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}_+ \times \mathbb{R}^i \times \mathbb{R}^i \rightarrow \mathbb{R}^r$ are assumed to be locally Lipschitz in $(x(t), x(t - d(t)))$ uniformly in t with $f_i(t, 0, 0) = 0$, $g_i(t, 0, 0) = 0$, $i = 1, \ldots, n$.

Remark 11. It should be pointed out that Zhai and Zha [23] considered the global adaptive output feedback control for system (18) without the diffusion terms. And Duan and Xie [24] discussed the globally asymptotic stability for system (18) without time-varying delay. Compared with these works, we focus on the output feedback stabilization of stochastic nonlinear systems with time-varying delay (see system (18)).

Assumption 12. There exist nonnegative constants $l_1, l_2 \in \mathbb{R}_+$ such that

$$
\begin{aligned}
\left| f_i \left(t, \overline{x}_i, \overline{x}_i \left(t - \theta(t) \right) \right) \right| &\leq l_1 \left(|x_1| + \cdots + |x_i| \right. \\
&\quad + \left| x_1 \left(t - \theta(t) \right) \right| + \cdots + \left. \left| x_i \left(t - \theta(t) \right) \right| \right), \\
\left| g_i \left(t, \overline{x}_i, \overline{x}_i \left(t - \theta(t) \right) \right) \right| &\leq l_2 \left(|x_1| + \cdots + |x_i| \right. \\
&\quad + \left| x_1 \left(t - \theta(t) \right) \right| + \cdots + \left. \left| x_i \left(t - \theta(t) \right) \right| \right).
\end{aligned} \tag{19}
$$

The linear observer system is designed as

$$
\begin{aligned}
\dot{\widehat{x}}_1 &= \widehat{x}_2(t) + La_1 \left(x_1 - \widehat{x}_1 \right), \\
\dot{\widehat{x}}_2 &= \widehat{x}_3(t) + L^2 a_2 \left(x_1 - \widehat{x}_1 \right), \\
&\vdots \\
\dot{\widehat{x}}_{n-1} &= \widehat{x}_n(t) + L^{n-1} a_{n-1} \left(x_1 - \widehat{x}_1 \right), \\
\dot{\widehat{x}}_n &= u + L^n a_n \left(x_1 - \widehat{x}_1 \right),
\end{aligned} \tag{20}
$$

where $L \geq 1$ is a constant to be determined and $a_i > 0$, $i = 1, \ldots, n$, are coefficients of the Hurwitz polynomial

$$
p(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \tag{21}
$$

The observation error $\varepsilon_i = (x_i - \widehat{x}_i)/L^{i-1}$ satisfies

$$
\begin{aligned}
d\varepsilon = L &\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 0 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \varepsilon dt + \begin{bmatrix} f_1 \\ \dfrac{f_2}{L} \\ \vdots \\ \dfrac{f_{n-1}}{L^{n-2}} \\ \dfrac{f_n}{L^{n-1}} \end{bmatrix} dt \\
&+ \begin{bmatrix} g_1^T \\ \dfrac{g_2^T}{L} \\ \vdots \\ \dfrac{g_{n-1}^T}{L^{n-2}} \\ \dfrac{g_n^T}{L^{n-1}} \end{bmatrix} dw = LA\varepsilon dt + F dt + G dw,
\end{aligned} \tag{22}
$$

where A is a Hurwitz matrix. Therefore, there is a positive-definite matrix P such that

$$
A^T P + PA = -I. \tag{23}
$$

Theorem 13. *Suppose Assumption 12 holds. Then, the equilibrium at origin of closed-loop stochastic nonlinear system (18) and (20) with the linear controller (50) below is globally asymptotically stable in probability and $P\{\lim_{t \rightarrow \infty} |x(t)| = 0\} = 1$. Furthermore, the closed-loop system has a unique solution on $[-d, \infty)$ for each $\xi \in \mathbb{R}^n$.*

Proof. Note $z = (x, \widehat{x})$, $x_i(t) = x_i$, and $\widehat{x}_i(t) = \widehat{x}_i$ for simplicity. Consider the following Lyapunov function: $V_0 = ((n+1)/2)(\varepsilon^T P \varepsilon)^2$. Then by Lemma 10, a direct computation yields

$$
\begin{aligned}
\mathscr{L}V_0 &= (n+1)\left(\varepsilon^T P \varepsilon \right)\left(L\varepsilon^T \left(A^T P + PA \right) \varepsilon + 2\varepsilon^T PF \right) \\
&\quad + \frac{1}{2} \operatorname{tr}\left(G^T (n+1)\left(4P\varepsilon\varepsilon^T P + 2\varepsilon^T P\varepsilon P \right) G \right) \\
&\leq -(n+1)\lambda_{\min} L |\varepsilon|^4 + 2(n+1)\lambda_{\max}^2 |\varepsilon|^3 |F| \\
&\quad + 3(n+1) r \sqrt{r} \lambda_{\max}^2 |\varepsilon|^2 |G|^2 \\
&\leq -(n+1)\lambda_{\min} L |\varepsilon|^4 \\
&\quad + \frac{3(n+1)}{2}\left(1 + r\sqrt{r} \right) \lambda_{\max}^2 |\varepsilon|^4 \\
&\quad + \frac{n+1}{2}\lambda_{\max}^2 |F|^4 + \frac{3(n+1)}{2} r\sqrt{r} \lambda_{\max}^2 |G|^4,
\end{aligned} \tag{24}
$$

where $\lambda_{\min}$ denotes the minimum eigenvalue of the matrix P and $\lambda_{\max}$ denotes the maximum eigenvalue of the matrix P.

By Lemmas 8 and 9 and Assumption 12, one gets

$$
|F|^4 \le l_1^4 n^5 \left(x_1^4 + \frac{1}{L^4} x_2^4 + \cdots + \frac{1}{L^{4n-4}} x_n^4 \right)
$$
$$
+ l_1^4 n^5 \left(x_1^4 (t - \theta(t)) + \frac{1}{L^4} x_2^4 (t - \theta(t)) + \cdots \right.
$$
$$
\left. + \frac{1}{L^{4n-4}} x_n^4 (t - \theta(t)) \right),
$$
$$
|G|^4 \le l_2^4 n^5 \left(x_1^4 + \frac{1}{L^4} x_2^4 + \cdots + \frac{1}{L^{4n-4}} x_n^4 \right)
$$
$$
+ l_2^4 n^5 \left(x_1^4 (t - \theta(t)) + \frac{1}{L^4} x_2^4 (t - \theta(t)) + \cdots \right.
$$
$$
\left. + \frac{1}{L^{4n-4}} x_n^4 (t - \theta(t)) \right).
$$

(25)

Substituting (25) into (24), we have

$$
\mathscr{L} V_0 \le -(n+1)\lambda_{\min} L |\varepsilon|^4
$$
$$
+ \frac{3(n+1)}{2} (1 + r\sqrt{r}) \lambda_{\max}^2 |\varepsilon|^4
$$
$$
+ \frac{(n+1)n^5}{2} \lambda_{\max}^2 \left(l_1^4 + 3r\sqrt{r} l_2^4 \right) \sum_{i=1}^n \frac{x_i^4 (t - \theta_t)}{L^{4i-4}}
$$
$$
+ \frac{(n+1)n^5}{2} \lambda_{\max}^2 \left(l_1^4 + 3r\sqrt{r} l_2^4 \right) \sum_{i=1}^n \frac{x_i^4}{L^{4i-4}}
$$

(26)

$$
= -(n+1)\lambda_{\min} L |\varepsilon|^4
$$
$$
+ \frac{3(n+1)}{2} (1 + r\sqrt{r}) \lambda_{\max}^2 |\varepsilon|^4 + \alpha_1 \sum_{i=1}^n \frac{x_i^4}{L^{4i-4}}
$$
$$
+ \alpha_1 \sum_{i=1}^n \frac{x_i^4 (t - d(t))}{L^{4i-4}},
$$

where

$$
\alpha_1 = \frac{(n+1)n^5}{2} \lambda_{\max}^2 \left(l_1^4 + 3r\sqrt{r} l_2^4 \right).
$$

(27)

Take

$$
W = \frac{\alpha_1}{1 - d'} \sum_{i=1}^n \int_{t-d(t)}^t \frac{x_i^4(s)}{L^{4i-4}} ds;
$$

(28)

then

$$
\mathscr{L}(V_0 + W)
$$
$$
\le -\left((n+1)\lambda_{\min} L - \frac{3(n+1)}{2} (1 + r\sqrt{r}) \lambda_{\max}^2 \right) |\varepsilon|^4
$$

(29)

$$
+ \alpha_1 \left(1 + \frac{1}{1 - d'} \right) \sum_{i=1}^n \frac{x_i^4(t)}{L^{4i-4}}.
$$

Substituting $x_i = \hat{x}_i + L^{i-1}\varepsilon_i$, $i = 1, \ldots, n$, into (29), one gets

$$
\mathscr{L}(V_0 + W) \le -((n+1)\lambda_{\min} L - c_0) |\varepsilon|^4
$$
$$
+ c_1 \sum_{i=1}^n \frac{\hat{x}_i^4}{L^{4i-4}},
$$

(30)

where

$$
c_0 = \frac{3(n+1)}{2} (1 + r\sqrt{r}) \lambda_{\max}^2 + 8n\alpha_1 \left(1 + \frac{1}{1 - d'} \right),
$$
$$
c_1 = 8\alpha_1 \left(1 + \frac{1}{1 - d'} \right).
$$

(31)

Next, we will combine the backstepping design method with the mathematical induction to design the linear output feedback control.

Step 1. Construct the Lyapunov function $V_1 = V_0 + W + (1/4)\hat{x}_1^4$. A direct calculation gives

$$
\mathscr{L} V_1 = \mathscr{L} V_0 + \mathscr{L} W + \hat{x}_1^3 \left(\hat{x}_2 + L a_1 \varepsilon_1 \right)
$$
$$
\le -((n+1)\lambda_{\min} L - c_0) |\varepsilon|^4 + c_1 \sum_{i=1}^n \frac{\hat{x}_i^4}{L^{4i-4}} + \hat{x}_1^3 \hat{x}_2
$$
$$
+ L a_1 \hat{x}_1^3 \varepsilon_1
$$

(32)

$$
\le -(n\lambda_{\min} L - c_0) |\varepsilon|^4 + c_1 \sum_{i=1}^n \frac{\hat{x}_i^4}{L^{4i-4}}
$$
$$
+ k_1 \lambda_{\min}^{-1/3} a_1^{4/3} L \hat{x}_1^4 + \hat{x}_1^3 \hat{x}_2,
$$

where k_1 is defined in Lemma 7.

Define $\xi_2 = \hat{x}_2 - \hat{x}_2^*$ with $\hat{x}_2^*$ being a virtual control. Then, we have

$$
\mathscr{L} V_1 \le -(n\lambda_{\min} L - c_0) |\varepsilon|^4 + c_1 \left(\frac{\hat{x}_3^4}{L^8} + \cdots + \frac{\hat{x}_n^4}{L^{4n-4}} \right)
$$
$$
+ \left(c_1 + k_1 \lambda_{\min}^{-1/3} a_1^{4/3} L \right) \hat{x}_1^4 + \frac{c_1}{L^4} \left(\xi_2 + \hat{x}_2^* \right)^4
$$
$$
+ \hat{x}_1^3 \left(\xi_2 + \hat{x}_2^* \right)
$$

(33)

$$
\le -(n\lambda_{\min} L - c_0) |\varepsilon|^4 + c_1 \left(\frac{\hat{x}_3^4}{L^8} + \cdots + \frac{\hat{x}_n^4}{L^{4n-4}} \right)
$$
$$
+ \left(c_1 + k_1 \lambda_{\min}^{-1/3} a_1^{4/3} L \right) \hat{x}_1^4 + \frac{8c_1}{L^4} \xi_2^4 + \frac{8c_1}{L^4} \left(\hat{x}_2^* \right)^4
$$
$$
+ \hat{x}_1^3 \xi_2 + \hat{x}_1^3 \hat{x}_2^*.
$$

Choose the virtual controller

$$
\hat{x}_2^* = -L \left(n + c_1 + k_1 \lambda_1^{-1/3} a_1^{4/3} \right) \hat{x}_1 = -L b_1 \hat{x}_1,
$$

(34)

where $b_1 > 0$ is independent of L. Hence,

$$
\mathscr{L} V_1 \le -(n\lambda_{\min} L - c_0) |\varepsilon|^4 + c_1 \left(\frac{\hat{x}_3^4}{L^8} + \cdots + \frac{\hat{x}_n^4}{L^{4n-4}} \right)
$$
$$
+ \left(8c_1 b_1^4 - nL \right) \hat{x}_1^4 + \frac{8c_1}{L^4} \xi_2^4 + \hat{x}_1^3 \xi_2.
$$

(35)

Step k. Suppose that, at Step k, there exists a smooth Lyapunov function V_k which is positive definite, radially unbounded, and twice continuously differentiable, satisfying

$$
\begin{aligned}
\mathscr{L}V_k \leq & - \left((n+1-k)\lambda_{\min}L - c_0 \right) |\varepsilon|^4 \\
& - \sum_{j=1}^{k} \frac{1}{L^{4j-4}} \left((n+1-k)L - 8c_1 b_j^4 \right) \xi_j^4 \\
& + c_1 \left(\frac{1}{L^{4k+4}} \widehat{x}_{k+2}^4 + \cdots + \frac{1}{L^{4n-4}} \widehat{x}_n^4 \right) + \frac{8c_1}{L^{4k}} \xi_{k+1}^4 \\
& + \frac{1}{L^{4k-4}} \xi_k^3 \xi_{k+1},
\end{aligned}
\tag{36}
$$

with a set of virtual controllers $\widehat{x}_1^*, \widehat{x}_2^*, \ldots, \widehat{x}_k^*$, defined by $\widehat{x}_1^* = 0$, $\widehat{x}_i^* = -Lb_{i-1}\xi_{i-1}$ with $b_{i-1} > 0$ being independent of the gain constant L, where $\xi_i = \widehat{x}_i - \widehat{x}_i^*$, $i = 1, \ldots, n$.

From the above definition, we have

$$
\begin{aligned}
\xi_{k+1} & = \widehat{x}_{k+1} + Lb_k \xi_k \\
& = \widehat{x}_{k+1} + Lb_k \widehat{x}_k + L^2 b_k b_{k-1} \widehat{x}_{k-1} + \cdots \\
& \quad + L^k b_k b_{k-1} \cdots b_1 \widehat{x}_1.
\end{aligned}
\tag{37}
$$

A direct computation yields

$$
\begin{aligned}
\mathscr{L}\left(\frac{1}{4L^{4k}} \xi_{k+1}^4 \right) & = \frac{1}{L^{4k}} \xi_{k+1}^3 \left(\widehat{x}_{k+2} + L^{k+1} a_{k+1} \varepsilon_1 \right. \\
& \left. + Lb_k \sum_{i=1}^{k} \frac{\partial \xi_k}{\partial x_i} \left(\widehat{x}_{i+1} + L^i a_i \varepsilon_1 \right) \right) = \frac{1}{L^{4k}} \xi_{k+1}^3 \left(\widehat{x}_{k+2} \right. \\
& + L^{k+1} a_{k+1} \varepsilon_1 \\
& \left. + \sum_{i=1}^{k} L^{k-i+1} b_k \cdots b_i \left(\xi_{i+1} - Lb_i \xi_i + L^i a_i \varepsilon_1 \right) \right) = \frac{1}{L^{4k}} \\
& \cdot \xi_{k+1}^3 \left(\widehat{x}_{k+2} + L^{k+1} d_0 \varepsilon_1 + L^{k+1} d_1 \xi_1 + L^k d_2 \xi_2 + \cdots \right. \\
& \left. + L^2 d_k \xi_k + L d_{k+1} \xi_{k+1} \right),
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
d_0 & = a_{k+1} + \left(b_k \cdots b_1 a_1 \right) + \left(b_k \cdots b_2 a_2 \right) + \cdots + b_k a_k, \\
d_1 & = -b_k \cdots b_1^2, \\
d_2 & = b_k \cdots b_1 - b_k \cdots b_2^2, \\
& \quad \vdots \\
d_k & = b_k b_{k-1} - b_k^2, \\
d_{k+1} & = b_k > 0.
\end{aligned}
\tag{39}
$$

Obviously, $d_0, d_1, \ldots, d_{k+1}$ are suitable real numbers and they are independent of the gain constant L.

Let us consider the following Lyapunov function:

$$
V_{k+1} = V_k + \frac{1}{4L^{4k}} \xi_{k+1}^4.
\tag{40}
$$

Then, combining (36) and (38), we obtain

$$
\begin{aligned}
\mathscr{L}V_{k+1} \leq & - \left((n+1-k)\lambda_{\min}L - c_0 \right) |\varepsilon|^4 \\
& - \sum_{j=1}^{k} \frac{1}{L^{4j-4}} \left((n+1-k)L - 8c_1 b_j^4 \right) \xi_j^4 \\
& + c_1 \left(\frac{1}{L^{4k+4}} \widehat{x}_{k+2}^4 + \cdots + \frac{1}{L^{4n-4}} \widehat{x}_n^4 \right) + \frac{8c_1}{L^{4k}} \xi_{k+1}^4 \\
& + \frac{1}{L^{4k-4}} \xi_k^3 \xi_{k+1} + \frac{1}{L^{4k}} \xi_{k+1}^3 \left(\widehat{x}_{k+2} + L^{k+1} d_0 \varepsilon_1 \right. \\
& \left. + L^{k+1} d_1 \xi_1 + \cdots + L^2 d_k \xi_k + L d_{k+1} \xi_{k+1} \right) \\
\leq & - \left((n-k)\lambda_{\min}L - c_0 \right) |\varepsilon|^4 \\
& - \sum_{j=1}^{k} \frac{1}{L^{4j-4}} \left((n-k)L - 8c_1 b_j^4 \right) \xi_j^4 + c_1 \left(\frac{1}{L^{4k+8}} \widehat{x}_{k+3}^4 \right. \\
& \left. + \cdots + \frac{1}{L^{4n-4}} \widehat{x}_n^4 \right) + \frac{8c_1}{L^{4k+4}} \xi_{k+2}^4 + \frac{8c_1}{L^{4k+4}} \left(\widehat{x}_{k+2}^* \right)^4 \\
& + \frac{1}{L^{4k}} \xi_{k+1}^3 \widehat{x}_{k+2}^* + \frac{1}{L^{4k}} \xi_{k+1}^3 \xi_{k+2} + \frac{d_{k+1} + 8 \left(c_1/L \right)}{L^{4k-1}} \\
& \cdot \xi_{k+1}^4 + \left(-\lambda_{\min} L \varepsilon_1^4 + \frac{1}{L^{3k-1}} d_0 \xi_{k+1}^3 \varepsilon_1 \right) + \left(-L\xi_1^4 \right. \\
& \left. + \frac{d_1}{L^{3k-1}} \xi_{k+1}^3 \xi_1 \right) + \left(-\frac{1}{L^3} \xi_2^4 + \frac{d_2}{L^{3k}} \xi_{k+1}^3 \xi_2 \right) + \cdots \\
& + \left(-\frac{1}{L^{4k-9}} \xi_{k-1}^4 + \frac{d_{k-1}}{L^{4k-3}} \xi_{k+1}^3 \xi_{k-1} \right) + \left(-\frac{1}{L^{4k-5}} \xi_k^4 \right. \\
& \left. + \frac{d_k}{L^{4k-2}} \xi_{k+1}^3 \xi_k + \frac{1}{L^{4k-4}} \xi_k^3 \xi_{k+1} \right).
\end{aligned}
\tag{41}
$$

It follows from Lemma 7 that

$$
\begin{aligned}
-\lambda_{\min} L \varepsilon_1^4 + \frac{1}{L^{3k-1}} d_0 \xi_{k+1}^3 \varepsilon_1 & \leq k_1 d_0^{4/3} \lambda_{\min}^{-1/3} \frac{\xi_{k+1}^4}{L^{4k-1}}, \\
-L\xi_1^4 + \frac{d_1}{L^{3k-1}} \xi_{k+1}^3 \xi_1 & \leq k_1 d_1^{4/3} \frac{\xi_{k+1}^4}{L^{4k-1}}, \\
-\frac{1}{L^3} \xi_2^4 + \frac{d_2}{L^{3k}} \xi_{k+1}^3 \xi_2 & \leq k_1 d_2^{4/3} \frac{\xi_{k+1}^4}{L^{4k-1}}, \\
& \quad \vdots \\
-\frac{1}{L^{4k-9}} \xi_{k-1}^4 + \frac{d_{k-1}}{L^{4k-3}} \xi_{k+1}^3 \xi_{k-1} & \leq k_1 d_{k-1}^{4/3} \frac{\xi_{k+1}^4}{L^{4k-1}}.
\end{aligned}
\tag{42}
$$

In particular, the last term of (41) yields

$$
\begin{aligned}
&-\frac{1}{L^{4k-5}}\xi_k^4 + \frac{d_k}{L^{4k-2}}\xi_{k+1}^3\xi_k + \frac{1}{L^{4k-4}}\xi_k^3\xi_{k+1} \\
&\leq \left[-\frac{1}{2L^{4k-5}}\xi_k^4 + \frac{d_k}{L^{4k-2}}\xi_{k+1}^3\xi_k\right] \\
&\quad + \left[-\frac{1}{2L^{4k-5}}\xi_k^4 + \frac{1}{L^{4k-4}}\xi_k^3\xi_{k+1}\right] \\
&\leq \left(k_1\left(\frac{1}{2}\right)^{-1/3}d_k^{4/3} + k_2\left(\frac{1}{2}\right)^{-3}\right)\frac{\xi_{k+1}^4}{L^{4k-1}}.
\end{aligned}
\tag{43}
$$

Hence,

$$
\begin{aligned}
\mathscr{L}V_{k+1} \leq &-\left((n-k)\lambda_{\min}L - c_0\right)|\varepsilon|^4 \\
&- \sum_{j=1}^{k}\frac{1}{L^{4j-4}}\left((n-k)L - 8c_1b_j^4\right)\xi_j^4 \\
&+ c_1\left(\frac{1}{L^{4k+8}}\widehat{x}_{k+3}^4 + \cdots + \frac{1}{L^{4n-4}}\widehat{x}_n^4\right) \\
&+ \frac{8c_1}{L^{4k+4}}\xi_{k+2}^4 + \frac{8c_1}{L^{4k+4}}\left(\widehat{x}_{k+2}^*\right)^4 \\
&+ \frac{1}{L^{4k}}\xi_{k+1}^3\widehat{x}_{k+2}^* + \frac{1}{L^{4k}}\xi_{k+1}^3\xi_{k+2} \\
&+ \frac{1}{L^{4k-1}}\left(d_0' + d_1' + \cdots + d_{k+1}'\right)\xi_{k+1}^4,
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
d_0' &= k_1 d_0^{4/3}\lambda_{\min}^{-1/3}, \\
d_1' &= k_1 d_1^{4/3}, \\
&\vdots \\
d_{k-1}' &= k_1 d_{k-1}^{4/3}, \\
d_k' &= k_1\left(\frac{1}{2}\right)^{-1/3}d_k^{4/3} + k_2\left(\frac{1}{2}\right)^{-3}, \\
d_{k+1}' &= d_{k+1} + 8c_1.
\end{aligned}
\tag{45}
$$

Now, we choose the following linear controller:

$$
\begin{aligned}
\widehat{x}_{k+2}^* &= -L\left(n - k + d_0' + d_1' + \cdots + d_{k+1}'\right)\xi_{k+1} \\
&= -Lb_{k+1}\xi_{k+1},
\end{aligned}
\tag{46}
$$

with $b_{k+1} > 0$ being independent of L.

So it follows from (44) that

$$
\begin{aligned}
\mathscr{L}V_{k+1} \leq &-\left((n-k)\lambda_{\min}L - c_0\right)|\varepsilon|^4 \\
&- \sum_{j=1}^{k+1}\frac{1}{L^{4j-4}}\left((n-k)L - 8c_1b_j^4\right)\xi_j^4 \\
&+ c_1\left(\frac{1}{L^{4k+8}}\widehat{x}_{k+3}^4 + \cdots + \frac{1}{L^{4n-4}}\widehat{x}_n^4\right) \\
&+ \frac{8c_1}{L^{4k+4}}\xi_{k+2}^4 + \frac{1}{L^{4k}}\xi_{k+1}^3\xi_{k+2}.
\end{aligned}
\tag{47}
$$

Step n. Using the inductive argument step by step, at the $(n-1)$th step, we get

$$
\begin{aligned}
\mathscr{L}V_{n-1} \leq &-\left(2\lambda_{\min}L - c_0\right)|\varepsilon|^4 \\
&- \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(2L - 8c_1b_j^4\right)\xi_j^4 + \frac{8c_1}{L^{4n-4}}\xi_n^4 \\
&+ \frac{1}{L^{4n-8}}\xi_{n-1}^3\xi_n.
\end{aligned}
\tag{48}
$$

Noting the function $V_n = V_{n-1} + (1/4L^{4n-4})\xi_n^4$, we have

$$
\begin{aligned}
\mathscr{L}V_n \leq &-\left(2\lambda_{\min}L - c_0\right)|\varepsilon|^4 \\
&- \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(2L - 8c_1b_j^4\right)\xi_j^4 + \frac{8c_1}{L^{4n-4}}\xi_n^4 + \frac{1}{L^{4n-8}} \\
&\cdot \xi_{n-1}^3\xi_n + \frac{1}{L^{4n-4}}\xi_n^3\widehat{x}_{n+1} + \frac{1}{L^{4n-4}}\left(L^n d_0\varepsilon_1 + L^n d_1\xi_1\right. \\
&+ \cdots + L^2 d_{n-1}\xi_{n-1} + Ld_n\xi_n\left.\right)\xi_n^3 \leq -\left(\lambda_{\min}L - c_0\right) \\
&\cdot |\varepsilon|^4 - \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(L - 8c_1b_j^4\right)\xi_j^4 + \frac{1}{L^{4n-4}}\xi_n^3\widehat{x}_{n+1} \\
&+ \left(\frac{8c_1}{L^{4n-4}} + \frac{1}{L^{4n-5}}d_n\right)\xi_n^4 + \frac{1}{L^{4n-5}}\left(d_0' + d_1' + \cdots\right. \\
&+ d_{n-1}'\left.\right)\xi_n^4 \leq -\left(\lambda_{\min}L - c_0\right)|\varepsilon|^4 \\
&- \sum_{j=1}^{n-1}\frac{1}{L^{4j-4}}\left(L - 8c_1b_j^4\right)\xi_j^4 + \frac{1}{L^{4n-4}}\xi_n^3\widehat{x}_{n+1} \\
&+ \frac{1}{L^{4n-5}}\left(d_0' + d_1' + \cdots + d_{n-1}' + 8c_1 + d_n\right)\xi_n^4.
\end{aligned}
\tag{49}
$$

We now design the linear controller

$$
\begin{aligned}
u = \widehat{x}_{n+1} = -Lb_n\xi_n = &-Lb_n\left(\widehat{x}_n + Lb_{n-1}\widehat{x}_{n-1}\right. \\
&+ L^2 b_{n-1}b_{n-2}\widehat{x}_{n-2} + \cdots + L^{n-1}b_{n-1}b_{n-2}\cdots b_1\widehat{x}_1\left.\right),
\end{aligned}
\tag{50}
$$

where $b_n = 1 + d_0' + d_1' + \cdots + d_{n-1}' + 8c_1 + d_n > 0$ is a real number independent of the gain parameter L. Thus,

$$
\mathscr{L}V_n \leq -\alpha_3\left(\varepsilon, \xi_1, \ldots, \xi_n\right),
\tag{51}
$$

where $\alpha_3(\varepsilon, \xi_1, \ldots, \xi_n) := (\lambda_{\min} L - c_0)|\varepsilon|^4 + \sum_{j=1}^{n-1}(1/L^{4j-4})(L - 8c_1 b_j^4)\xi_j^4 + (1/L^{4n-5})\xi_n^4$.

Choosing the gain constant $L = \max\{1, c_0/\lambda_{\min}, 8c_1 b_1^4, \ldots, 8c_1 b_{n-1}^2\}$, it is obvious that $\alpha_3(\varepsilon, \xi_1, \ldots, \xi_n)$ is a positive-definite and proper function. Thus, we see that (6) is true.

Now, we prove (5). From the above process, we have constructed the Lyapunov function

$$V_n = \overline{V} + \frac{\alpha_1}{1 - d'}\sum_{i=1}^{n}\int_{t-d(t)}^{t}\frac{x_i^4(s)}{L^{4i-4}}ds, \tag{52}$$

where α_1 and d' are positive parameters and

$$\overline{V} = \frac{n+1}{2}\left(\varepsilon^T P\varepsilon\right)^2 + \sum_{i=1}^{n}\frac{\xi_i^4}{4L^{4i-4}}. \tag{53}$$

It is easy to verify that $\overline{V}$ is $\mathscr{C}^2$ on z. Suppose that $U(z)$ is continuous, positive definite, and radially unbounded. Then by Lemmas 5 and 6, there exist positive constants $\underline{\gamma}$ and $\overline{\gamma}$ and two class $\mathscr{K}_\infty$ functions β_1 and α_{11} such that

$$\beta_1(|z|) \leq U(z) \leq \alpha_{11}(|z|),$$
$$\underline{\gamma}\|z\|_\Delta^4 \leq U(z) \leq \overline{\gamma}\|z\|_\Delta^4. \tag{54}$$

It follows from $d(t): \mathbb{R}_+ \to [0, d]$ that

$$\begin{aligned}
\frac{\alpha_1}{1-d'}\sum_{i=1}^{n}\int_{t-d(t)}^{t}\frac{x_i^4(s)}{L^{4i-4}}ds &\leq \gamma\int_{t-d(t)}^{t}\sum_{i=1}^{n}\frac{x_i^4(s)}{L^{4i-4}}ds \\
&\leq \overline{\gamma}\int_{t-d(t)}^{t}\|z(s)\|_\Delta^4\,ds \leq \widetilde{\gamma}\int_{t-d(t)}^{t}\alpha_{11}(|z(s)|)\,ds \\
&= \widetilde{\gamma}\int_{-d(t)}^{0}\alpha_{11}(|z(\sigma+t)|)\,d(\sigma+t) \\
&\leq \widetilde{\gamma}\int_{-d}^{0}\alpha_{11}(|z(\sigma+t)|)\,d(\sigma+t) \\
&\leq \widehat{\gamma}\sup_{-d\leq\sigma\leq 0}\alpha_{11}(|z(\sigma+t)|) \\
&\leq \alpha_{22}\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right),
\end{aligned} \tag{55}$$

where $\gamma = \alpha_1/(1-d')$, $\widetilde{\gamma}$ and $\widehat{\gamma}$ are positive constants, and α_{22} is a class $\mathscr{K}_\infty$ function. Noting that

$$|z(t)| \leq \sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|, \tag{56}$$

so

$$\alpha_{11}(|z(t)|) \leq \alpha_{11}\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right). \tag{57}$$

Defining

$$\begin{aligned}
\beta_2\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right) \\
:= \alpha_{11}\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right) \\
+ \alpha_{22}\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right),
\end{aligned} \tag{58}$$

one gets

$$\beta_1(|z(t)|) \leq V_n(z(t)) \leq \beta_2\left(\sup_{-d\leq\sigma\leq 0}|z(\sigma+t)|\right). \tag{59}$$

Therefore, by Lemma 4, the equilibrium of closed-loop nonlinear stochastic system (18), (20), and (50) is globally asymptotically stable and the closed-loop system has a unique solution on $[-d, \infty)$ for each $\xi \in \mathbb{R}^n$. $\square$

Remark 14. Letting $f_i = 0$ $(i = 1\ldots, n)$ and $d(t) \equiv 0$ in system (18), then the model will reduce to that introduced in [9]. Therefore, our result extends and improves those given in [9].

Remark 15. It should be pointed out that, compared with the C^1 output feedback controller in [20], we see that the linear controller (50) based on the observable linearization is a smooth one.

Remark 16. In this paper, combining the backstepping design with mathematical induction, we extend the method in [21] to the stochastic nonlinear system with time-varying delay. Although the globally asymptotic stability of stochastic nonlinear system has been discussed by Liu and Zhang in [12], they did not consider the term of time-varying delay. Therefore, our conclusion is proposed for the first time.

Remark 17. In [22], Liu and Xie considered the state feedback stability of stochastic feedforward nonlinear systems with time-varying delays. They introduced the homogeneous domination approach to construct a state feedback controller. It is generally known that in real lives the output feedback control is easier to achieve than the state feedback control and has great theory significance and utility value. Therefore, in this paper we combine the backstepping method with the mathematical induction to design a Lyapunov-Krasovskii functional and construct a linear output feedback controller for stochastic nonlinear systems with time-varying delay. Moreover, we impose different restricted conditions on functions f_i and g_i $(i = 1, \ldots, n)$ to ensure the global stability of stochastic nonlinear systems with time-varying delays.

4. An Example

In this section, we will use a simulation example to illustrate our main results.

Consider the following stochastic nonlinear system:

$$dx_1 = \left(x_2 + \frac{1}{16}x_1\sin x_2^2\right)dt$$

$$+ \frac{1}{16}\cos\left(x_2\left(t - d\left(t\right)\right)\right)dw, \qquad (60)$$

$$dx_2 = u\,dt$$

$$y = x_1,$$

where the time-varying delay $d(t) = \cos(t)$. Obviously, we have $|f_1(x)| \leq (1/16)(|x_1| + |x_2(t - \cos(t))|)$, $|g_1(x)| \leq (1/16)(|x_1| + |x_2(t - \cos(t))|)$, which implies that Assumption 12 holds. Moreover, both of them are locally Lipschitz. The linear observer system is designed as

$$\dot{\widehat{x}}_1 = \widehat{x}_2 + L\left(x_1 - \widehat{x}_1\right),$$

$$\dot{\widehat{x}}_2 = u + L^2\left(x_1 - \widehat{x}_1\right) \qquad (61)$$

with a suitable choice of the parameter L. The observation errors

$$\varepsilon_1 = x_1 - \widehat{x}_1,$$

$$\varepsilon_2 = x_2 - \widehat{x}_2 \qquad (62)$$

satisfy

$$d\varepsilon = L\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\varepsilon\,dt + \begin{bmatrix} f_1 \\ \dfrac{f_2}{L} \end{bmatrix}dt + \begin{bmatrix} g_1^T \\ \dfrac{g_2^T}{L} \end{bmatrix}dw \qquad (63)$$

$$= LA\varepsilon\,dt + F\,dt + G\,dw.$$

For the above matrix A, there exists a positive-definite matrix P satisfying $A^T P + PA = -I$, where

$$P = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{3}{2} \end{bmatrix}. \qquad (64)$$

It is easy to get the corresponding eigenvalues $\lambda_{\min} = (5 - \sqrt{5})/4$ and $\lambda_{\max} = (5 + \sqrt{5})/4$.

Consider the Lyapunov-Krasovskii functional

$$V = \frac{3}{2}\left(\varepsilon^T P\varepsilon\right)^2 + \frac{\alpha_1}{1 + \sin x}\sum_{i=1}^{2}\int_{t-\cos t}^{t}\frac{x_i^4(s)}{L^{4i-4}}ds + \frac{1}{4}\widehat{x}_1^4$$

$$+ \frac{1}{4L^4}\left(\widehat{x}_2 + Lb_1\widehat{x}_1\right)^4. \qquad (65)$$

Then, we take $\alpha_1 = 0.02$, $b_1 = 2$, $b_2 = 20$, $L = 160$, and the linear control $u = -3200(\widehat{x}_2 + Lb_1\widehat{x}_1)$. By Theorem 13, we see that closed-loop system (60), (61) and the linear controller u are globally asymptotically stable in probability and there is a unique solution on $[-d, \infty)$. (The state response and control input with initial conditions $x_1(0) = 0.5$, $x_2(0) = 50$, $\widehat{x}_1(0) = 0.6$, and $\widehat{x}_2(0) = -100$ are presented in Figures 1(a), 1(b), and 2, where hatx_1 and hatx_2 denote $\widehat{x}_1$ and $\widehat{x}_2$.)
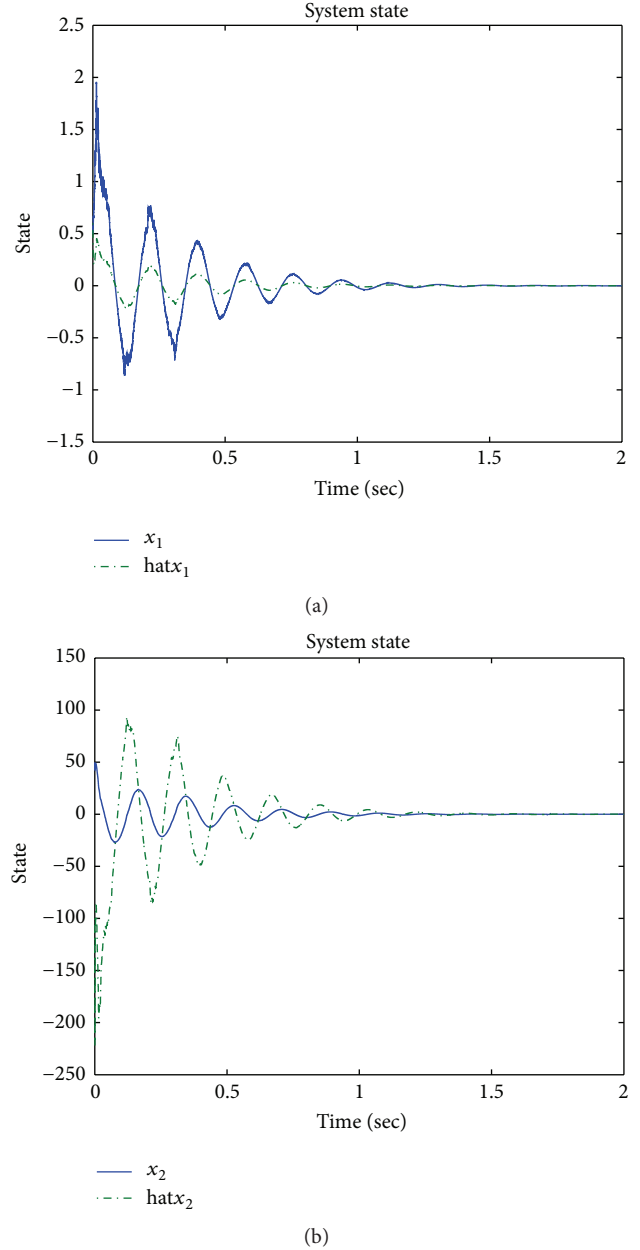
(a)

(b)

FIGURE 1: The state response.

5. Conclusion

In this paper, we combine the backstepping method with the mathematical induction to prove that a class of stochastic nonlinear systems with time-varying delays by the output feedback control is globally asymptotically stable in probability. It should be pointed out that we not only extend the results in deterministic nonlinear systems to more complex stochastic case but also add time-varying delays to stochastic nonlinear systems. For the term of time-varying delays, we design a Lyapunov-Krasovskii functional and a linear output feedback controller to render the closed-loop system globally asymptotically stable.
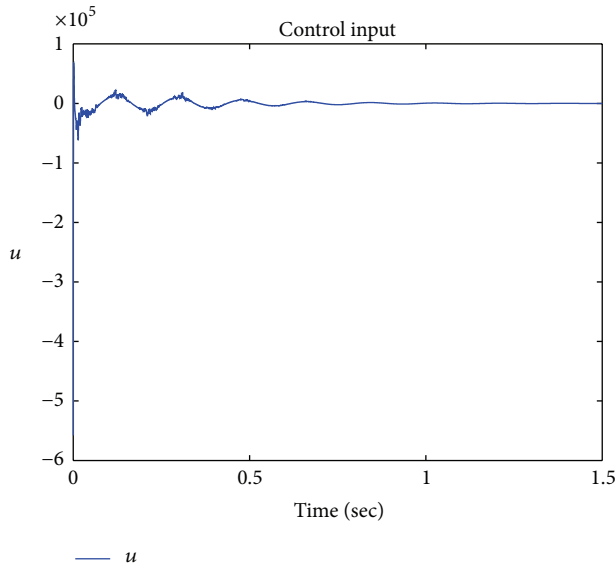
FIGURE 2: The control input.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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