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Research Article

$(\in, \in \vee q)$-Fuzzy *BCK*-Submodules

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We introduce the concept of generalized fuzzy module of *BCK*-algebras and present some fundamental properties. Also, we introduce the concept of generalized interval-valued fuzzy *BCK*-submodule and present some fundamental properties.

1. Introduction

In 1966, Iseki and Imai [1, 2] introduced *BCK*-algebra. This notion was originated from two different ways: (1) set theory and (2) classical and no classical propositional calculi. Certain algebraic structures, for example, Boolean-algebra and *MV*-algebras, are introduced as *BCK*-algebras [3]. Every module is an action of ring on certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. *BCK*-module is an action of *BCK*-algebra on commutative group. In 1994, the notion of *BCK*-module was introduced by Abujabal et al. [4]. They established isomorphism theorems and studied some properties of *BCK*-modules. The theory of *BCK*-modules was further developed by Perveen et al. [5].

After the introduction of fuzzy sets by Zadeh [6], there have been a number of generalizations of this fundamental concept. In 1991, Xi [7] applied fuzzy set theory to *BCK*-algebras and introduced the notion of fuzzy subalgebras (ideals) of the *BCK*-algebras, and since then some authors studied fuzzy subalgebras and fuzzy ideals. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [8]. In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. With this objective in view, Jun [9] introduced the concept of (α, β)-fuzzy ideals of a *BCK*/*BCI*-algebra and investigated related results. In [10] Zadeh made an extension of the concept of fuzzy set by an interval-valued fuzzy set. In 2000, Jun [11] applied interval-valued fuzzy set theory to *BCK*-algebras and introduced the notion of interval-valued fuzzy subalgebras (ideals) of the *BCK*-algebras. In this paper, we introduce the concept of generalized fuzzy *BCK*-submodules and some basic properties are obtained and we define the concept of generalized interval-valued fuzzy *BCK*-submodules and some basic properties are obtained.

2. Preliminaries

Definition 1 (see [8]). By a *BCK*-algebra one means an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(*BCK*-1) $((x * y) * (x * z)) * (z * y) = 0$,

(*BCK*-2) $(x * (x * y)) * y = 0$,

(*BCK*-3) $x * x = 0$,

(*BCK*-4) $0 * x = 0$,

(*BCK*-5) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

A partial ordering "$\leq$" is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A *BCK*-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$, and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2 (see [12]). Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), one means that an abelian group M with an operation $X \times M \rightarrow M$ with $(x, m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

(i) $(x \wedge y)m = x(ym)$,

(ii) $x(m + n) = xm + xn$,

(iii) $0m = 0$.

Moreover, if X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

A mapping $\mu : X \rightarrow [0, 1]$ is called a fuzzy set in a BCK-algebra X. For any fuzzy set μ in X and any $t \in [0, 1]$, we define set $U(\mu; t) = \mu^t = \{x \in X \mid \mu(x) \geq t\}$, which is called upper t-level cut of μ.

Definition 3 (see [12]). A fuzzy subset μ of M is said to be a fuzzy BCK-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold:

(FBCKM1) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}$,

(FBCKM2) $\mu(-m) = \mu(m)$,

(FBCKM3) $\mu(xm) \geq \mu(m)$.

Definition 4 (see [13]). A fuzzy set μ in a set X of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if } y = x \\ 0, & \text{if } y \neq x, \end{cases} \quad (1)$$

is said to be a fuzzy point with support x and value t and is denoted by x_t; we say that a fuzzy point x_t belongs to a fuzzy set μ and write $x_t \in \mu$, if $\mu(x) \geq t$. A fuzzy point x_t is quasi-coincident with a fuzzy set μ, if $\mu(x) + t > 1$. In this case we write $x_t q\mu$. $x_t \in \vee q\mu$ means that $x_t \in \mu$ or $x_t q\mu$, $x_t \in \wedge q\mu$ means that $x_t \in \mu$ and $x_t q\mu$.

Theorem 5 (see [12]). *Let $\mu \in \mathcal{F}(M)$, where $\mathcal{F}(M)$ is the set of all fuzzy subsets of BCK-module M. Then μ is a fuzzy BCK-submodule of M if and only if*

(i) $\mu(0) \geq \mu(m)$,

(ii) $\mu(xm - yn) \geq \min\{\mu(m), \mu(n)\}$.

3. $(\in, \in \vee q)$-Fuzzy *BCK*-Submodules

Definition 6. A fuzzy subset μ of M is said to be an $(\in, \in \vee q)$-fuzzy BCK-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$ the following axioms hold:

(i) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2), 0.5\}$,

(ii) $\mu(-m) \geq \min\{\mu(m), 0.5\}$,

(iii) $\mu(xm) \geq \min\{\mu(m), 0.5\}$.

Example 7. Let $X = \{0, a, b, c\}$ and consider the following equation:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

(2)

Then $(X, *)$ is a bounded implicative BCK-algebra and so is a BCK-module over itself. Now let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 > t_1 > t_2$. Define $\mu : X \rightarrow [0, 1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$, and $\mu(b) = \mu(c) = t_2$. Then μ is an $(\in, \in \vee q)$-fuzzy BCK-submodule of X.

Theorem 8. *A fuzzy subset μ of M is an $(\in, \in \vee q)$-fuzzy BCK-submodule if and only if*

(i) $\mu(xm) \geq \min\{\mu(m), 0.5\}$,

(ii) $\mu(m_1 - m_2) \geq \min\{\mu(m_1), \mu(m_2), 0.5\}$.

Proof. From the definition of $(\in, \in \vee q)$-fuzzy BCK-submodule,

(i) $\mu(xm) \geq \min\{\mu(m), 0.5\}$,

(ii) $\mu(m_1 - m_2) = \mu(m_1 + (-m_2)) \geq \min\{\mu(m_1), \mu(-m_2), 0.5\}$.

But we know that $\mu(-m) = \mu(m)$.

Then $\mu(m_1 - m_2) \geq \min\{\mu(m_1), \mu(m_2), 0.5\}$.

Conversely, we have $\mu(xm) = \mu(xm - y0) \geq \min\{\mu(m), \mu(0), 0.5\}$.

Since $\mu(0) \geq \min\{\mu(m), 0.5\}$,

then $\mu(xm) \geq \min\{\mu(m), 0.5\}$.

And $\mu(m_1 - m_2) = \mu(1 \cdot m_1 - 1 \cdot m_2) \geq \min\{\mu(m_1), \mu(m_2), 0.5\}$.

This proves that μ is an $(\in, \in \vee q)$-fuzzy BCK-submodule.

□

Theorem 9. *Let $\mu \in \mathcal{F}(M)$, where $\mathcal{F}(M)$ is the set of all fuzzy subsets of BCK-module M. Then μ is an $(\in, \in \vee q)$-fuzzy BCK-submodule of M if and only if*

(i) $\mu(0) \geq \min\{\mu(m), 0.5\}$,

(ii) $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\}$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy BCK-submodule:

(i) Consider $\mu(xm) \geq \min\{\mu(m), 0.5\}$, put $x = 0$, and then $\mu(0) \geq \min\{\mu(m), 0.5\}$.

Consider $\mu(xm - yn) \geq \min\{\mu(xm), \mu(yn), 0.5\} \geq \min\{\mu(m), \mu(n), 0.5\}$.

Hence, $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\}$.

Conversely, we have $\mu(xm - y0) \geq \min\{\mu(m), \mu(0), 0.5\} \geq \min\{\mu(m), 0.5\}$

and $\mu(m-n) = \mu(1\cdot m - 1\cdot n) \geq \min\{\mu(m), \mu(n), 0.5\}$.

This proves that μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule.

□

Theorem 10. *A fuzzy set μ in X is an $(\in, \in \vee q)$-fuzzy BCK-submodule if and only if $\mu_t \neq \phi$, $t \in (0, 0.5]$ is BCK-submodule.*

Proof. Let $\mu_t \neq \phi$, $t \in (0, 0.5]$, $t < 0.5$, and $\mu_t = \{x \in X : \mu(x) \geq t\}$; let $m, n \in \mu_t$; then $\mu(m), \mu(n) \geq t$. Since μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule, $\mu(m-n) \geq \min\{\mu(m), \mu(n), 0.5\} = t$, and $\mu(m-n) \geq t$, then $m - n \in \mu_t \rightarrow$ (i); let $m \in \mu_t, x \in X$; then $\mu(m) \geq t$; then $\mu(xm) \geq \min\{\mu(m), 0.5\} \geq \min\{t, 0.5\} = t$; then $\mu(xm) \geq t$. Hence, $xm \in \mu_t \rightarrow$ (ii) and from (i) and (ii) we get that μ_t is *BCK*-submodule.

Conversely, let $t = \min\{\mu(x), \mu(y), 0.5\}, x, y \in \mu_t$, $\mu(x), \mu(y) \geq t$; since μ_t is *BCK*-submodule, then $x - y \in \mu_t$; this implies that $\mu(x-y) \geq t = \min\{\mu(x), \mu(y), 0.5\}$. Hence, $\mu(x-y) \geq \min\{\mu(x), \mu(y), 0.5\}$; let $s = \min\{\mu(m), 0.5\}$; let $x \in \mu_t, m \in \mu_s$; then $xm \in \mu_s$; this implies that $\mu(xm) \geq s = \min\{\mu(m), 0.5\}$. Hence, $\mu(xm) \geq \min\{\mu(m), 0.5\}$ proving μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule. □

Theorem 11. *Let μ be a fuzzy set in X. Then $\mu_t \neq \phi$ is BCK-submodule for all $t \in (0.5, 1]$ if and only if μ satisfies*

(i) $\forall x \in X, m \in M, \max\{\mu(xm), 0.5\} \geq \mu(m)$,

(ii) $\forall x \in X, m, n \in M, \max\{\mu(m-n), 0.5\} \geq \min\{\mu(m), \mu(n)\}$.

Proof. Assume that μ_t is *BCK*-submodule of X for all $t \in (0.5, 1]$ if there is $a \in X$ such that condition (i) is not valid implying that $(\exists a \in X)\ (\max\{\mu(xm), 0.5\} < \mu(a))$; then $\mu(a) \in (0.5, 1]$ and $a \in U(\mu, \mu(a)) = \mu_{\mu_a}$. But $\mu(xm) \leq \mu(a)$ implies $xm \notin \mu_{\mu_a}$, a contradiction. Hence, (i) is valid. Suppose that $\max\{\mu(a-b), 0.5\} < \min\{\mu(a), \mu(b)\} = s$ for some $a, b \in X$. Then $s \in (0.5, 1]$ and $a, b \in \mu_s$ but $a - b \notin \mu_s$ since $\mu(a-b) < s$. This is a contradiction, and therefore (ii) is valid.

Conversely, assume that μ satisfies conditions (i) and (ii). Let $t \in (0.5, 1]$ for any $x \in \mu_t$; we have $\max\{\mu(xm), 0.5\} \geq \mu(m) \geq t > 0.5$, so $\mu(xm) \geq t$; thus, $xm \in \mu_t$; let $m, n \in X$ be such that $m \in \mu_t$ and $n \in \mu_t$; then $\max\{\mu(m-n), 0.5\} \geq \min\{\mu(m), \mu(n)\} \geq t > 0.5$, and thus $\mu(m-n) \geq t$ implies that $m - n \in \mu_t$. Hence, μ_t is *BCK*-submodule. □

Theorem 12. *Every fuzzy BCK-submodule is an $(\in, \in \vee q)$-fuzzy BCK-submodule.*

Proof. Let μ be fuzzy *BCK*-submodule $(m \in M, x, y \in X, t \in (0, 1])$; then we have $\mu(0) \geq \mu(m) \geq \min\{\mu(m), 0.5\}$; hence, $\mu(0) \geq \min\{\mu(m), 0.5\}$ (i) and $\mu(xm - yn) \geq \min\{\mu(m), \mu(n)\} \geq \min\{\mu(m), \mu(n), 0.5\}$; then $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\}$ (ii); from (i) and (ii) we get that μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule. □

The following example shows that the converse of Theorem 12 is not true in general.

Example 13. Let $X = \{0, a, b, c, d, 1\}$ be the set along with binary operation $*$ defined on it by (3); then $(X, *, 0)$ forms a bounded commutative, nonimplicative *BCK*-algebra. Now for the subset $M = \{0, a, c, d\}$ of X, define an operation $+$ as $x + y = (x * y) \vee (y * x)$. By (4), it follows that $(M, +)$ forms a commutative group. Define scaler multiplication $(X, M) \rightarrow M$ by $xm = x \wedge m$ for all $x \in X$ and $m \in M$(see (5)):

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
1	1	d	c	b	a	0

(3)

$+$	0	a	c	d
0	0	a	c	d
a	a	0	d	c
c	c	d	0	a
d	d	c	a	0

(4)

$\wedge$	0	a	c	d
0	0	0	0	0
a	0	a	0	a
b	0	a	0	a
c	0	0	c	c
d	0	a	c	d
1	0	a	c	d

(5)

Then $(M, +)$ forms an X-module. Now let $t_0, t_1, t_2 \in [0, 1]$ be such that $t_0 > t_1 > t_2$. Define $\mu : M \rightarrow [0, 1]$ by $\mu(0) = t_0$, $\mu(a) = t_1$, and $\mu(c) = \mu(d) = t_2$. Then μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule of X, but it is not fuzzy *BCK*-submodule of X if $\mu < 0.5$.

Theorem 14. *Let μ be an $(\in, \in \vee q)$-fuzzy BCK-submodule of X such that $\mu(m), \mu(n) < 0.5$ for all $m, n \in X$. Then μ is a fuzzy BCK-submodule of X.*

Proof. Since μ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule of X, then it satisfies these two conditions:

(i) $\mu(0) \geq \min\{\mu(m), 0.5\}$,

(ii) $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\}$.

Now we want to show that μ is an $(\in, \in)$-fuzzy *BCK*-submodule of X.

Since $\mu(m) < 0.5$,

then $\mu(0) \geq \min\{\mu(m), 0.5\} \geq \mu(m)$.

Hence, $\mu(0) \geq \mu(m)$,

and since $\mu(m), \mu(n) < 0.5$,

then $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\} \geq \min\{\mu(m), \mu(n)\}$.

Hence, $\mu(xm - yn) \geq \min\{\mu(m), \mu(n)\}$.

Then μ is a fuzzy BCK-submodule of X.

□

Lemma 15. *Let μ be a $(\in, \in \vee q)$-fuzzy BCK-submodule of module M. Let $m, n \in M$, $x, y \in X$ such that $\mu(m) < \mu(n)$; then*

(i) $\mu(xm - yn) \geq \mu(m)$ *if* $\mu(m) < 0.5$,

(ii) $\mu(xm), \mu(xn) \geq 0.5$ *if* $\mu(m) \geq 0.5$.

Proof. Since μ is a fuzzy submodule of M, we have $\mu(xm) \geq \min\{\mu(m), 0.5\}$.

(i) Let $\mu(m) < 0.5$; then $\mu(xm - yn) \geq \min\{\mu(m), \mu(n), 0.5\} \geq \min\{\mu(m), 0.5\}$ since $\mu(m) < \mu(n) = \mu(m)$, since $\mu(m) < 0.5$.

Then $\mu(xm - yn) \geq \mu(m)$.

(ii) If $\mu(m) \geq 0.5$, then $\mu(xm) \geq \min\{\mu(m), 0.5\} = \mu(m)$. But $\mu(m) \geq 0.5$.

This implies that $\mu(xm) \geq 0.5$.

If $\mu(m) \geq 0.5$, then $\mu(xn) \geq \min\{\mu(n), 0.5\}$, but $\mu(m) < \mu(n) = \mu(m)$, and $\mu(m) \geq 0.5$.

Then $\mu(xn) \geq 0.5$.

□

Definition 16. Let μ and ν be two $(\in, \in \vee q)$-fuzzy BCK-submodules and the intersection of μ and ν is defined as follows:

$$\mu(x) \cap \nu(x) = (\mu \cap \nu)(x) = \min\{\mu(x), \nu(x), 0.5\}. \quad (6)$$

Proposition 17. *Let $\{\mu_i : i \in \wedge\}$ be a nonempty family of an $(\in, \in \vee q)$-fuzzy BCK-submodule of M. Then $\cap_{i \in \wedge} \mu_i$ is an $(\in, \in \vee q)$-fuzzy BCK-submodule of M.*

Proof. Since μ_i is an $(\in, \in \vee q)$-fuzzy BCK-submodule of M, then μ_i satisfies conditions of $(\in, \in \vee q)$-fuzzy BCK-submodule. Let $i = 1, 2, \ldots, n$; let μ_1 and μ_2 be an $(\in, \in \vee q)$-fuzzy BCK-submodule; then μ_1 and μ_2 satisfy conditions of $(\in, \in \vee q)$-fuzzy BCK-submodule.

Now we want to prove that $(\mu_1 \cap \mu_2)$ is an $(\in, \in \vee q)$-fuzzy BCK-submodule.

(i) Consider $(\mu_1 \cap \mu_2)(0) = \min\{\mu_1(0), \mu_2(0), 0.5\} \geq \min\{\mu_1(m), \mu_2(m), 0.5\} = (\mu_1 \cap \mu_2)(m)$.

Hence,

$$(\mu_1 \cap \mu_2)(0) \geq (\mu_1 \cap \mu_2)(m). \quad (7)$$

(ii) Consider $(\mu_1 \cap \mu_2)(xm - yn) = \min\{\mu_1(xm - yn), \mu_2(xm - yn), 0.5\} \geq \min\{\min\{\mu_1(m), \mu_1(n), 0.5\}, \min\{\mu_2(m), \mu_2(n), 0.5\}, 0.5\} = \min\{\min\{\mu_1(m), \mu_2(m), 0.5\}, \min\{\mu_1(n), \mu_2(n), 0.5\}, 0.5\} = \min\{(\mu_1 \cap \mu_2)(m), (\mu_1 \cap \mu_2)(n), 0.5\}$.

Hence,

$$(\mu_1 \cap \mu_2)(xm - yn) \geq \min\{(\mu_1 \cap \mu_2)(m), (\mu_1 \cap \mu_2)(n), 0.5\}. \quad (8)$$

From (i) and (ii) we get that $(\mu_1 \cap \mu_2)$ is an $(\in, \in \vee q)$-fuzzy BCK-submodule.

Furthermore, $\cap_{i \in \wedge} \mu_i$ is an $(\in, \in \vee q)$-fuzzy BCK-submodule.

□

Similarly, we can prove the generalization of previous proposition.

Theorem 18. *Let $\{\mu_i : i \in \wedge\}$ be a family of an $(\in, \in \vee q_k)$-fuzzy BCK-submodule of M. Then $\cap_{i \in \wedge} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy BCK-submodule of M.*

The following example shows that the union of two $(\in, \in \vee q)$-fuzzy BCK-submodules of X may not be an $(\in, \in \vee q)$-fuzzy BCK-submodule of X.

Example 19. Let $X = \{0, a, b, c\}$ be BCK-algebra which is given in Example 7 and let μ be an $(\in, \in \vee q)$-fuzzy BCK-submodule of X which is $\mu(0) = 0.6$, $\mu(a) = 0.5$, $\mu(b) = \mu(c) = 0.3$. Let ν be a fuzzy set in X defined by $\nu(0) = 0.5, \nu(a) = 0.4, \nu(b) = \nu(c) = 0.2$. Then ν is an $(\in, \in \vee q)$-fuzzy BCK-submodule of X. The union of μ and ν is given by $\mu(x) \cup \nu(x) = (\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$. Hence, $(\mu \cup \nu)(0) = 0.6, (\mu \cup \nu)(a) = 0.5, (\mu \cup \nu)(b) = (\mu \cup \nu)(c) = 0.3$. Then

$$(\mu \cup \nu)(m_1 + m_2) \geq \min\{(\mu \cup \nu)(m_1), (\mu \cup \nu)(m_2), 0.5\}. \quad (9)$$

Suppose that $m_1 = 0$ and $m_2 = b$; then $0 + b = b$:

$$\begin{aligned} &(\mu \cup \nu)(b) \geq \min\{(\mu \cup \nu)(0), (\mu \cup \nu)(b), 0.5\} \\ &0.3 \geq \min\{\max\{\mu(0), \mu(b)\}, \max\{\nu(0), \nu(b)\}, 0.5\} \\ &0.3 \geq \min\{\max\{0.6, 0.3\}, \max\{0.5, 0.2\}, 0.5\} \\ &0.3 \geq \min\{0.6, 0.5, 0.5\} \\ &0.3 \geq 0.5, \end{aligned} \quad (10)$$

a contradiction; hence, $(\mu \cup \nu)$ is not $(\in, \in \vee q)$-fuzzy BCK-submodule of X.

Definition 20. Let $k \in \mathbb{N}$; μ_i is an $(\in, \in \vee q)$-fuzzy BCK-submodule for $i \in \mathbb{N}_k = \{1, 2, \ldots, k\}$ and $x \in X$. Then define $\sum_{i \in \mathbb{N}_k} \mu_i$ as follows:

$$\left(\sum_{i \in \mathbb{N}_k} \mu_i\right)(m) = \sup_{m = \sum_{i \in \mathbb{N}_k} a_i} \min\{\mu_1(a_1), \mu_2(a_2), \ldots, \mu_k(a_k), 0.5\}. \quad (11)$$

Proposition 21. *Let μ_i be an $(\in, \in \vee q)$-fuzzy BCK-submodule. Then $\sum_{i\in\mathbb{N}_k} \mu_i + \sum_{i\in\mathbb{N}_k} \mu_i \leq \sum_{i\in\mathbb{N}_k} \mu_i$.*

Proof. One has

$$\begin{aligned}
&\sum_{i\in\mathbb{N}_k} \mu_i + \sum_{i\in\mathbb{N}_k} \mu_i = \sum_{i\in\mathbb{N}_k} (\mu_i + \mu_i) \\
&\quad = \sup_{m=\sum_{i\in\mathbb{N}_k} a_i} \min\{(\mu_1 + \mu_1)(a_1), (\mu_2 + \mu_2) \\
&\quad \cdot (a_2), \ldots, (\mu_k + \mu_k)(a_k), 0.5\} \\
&\quad \leq \sup_{m=\sum_{i\in\mathbb{N}_k} a_i} \min\{\mu_1(a_1), \mu_2(a_2), \ldots, \mu_k(a_k), 0.5\} \\
&\quad \leq \sum_{i\in\mathbb{N}_k} \mu_i.
\end{aligned} \tag{12}$$

□

Theorem 22. *Let μ_i be an $(\in, \in \vee q)$-fuzzy BCK-submodule, for $i \in \mathbb{N}_k$; then $\sum_{i\in\mathbb{N}_k} \mu_i$ is an $(\in, \in \vee q)$-fuzzy BCK-submodule.*

Proof. Since μ_i is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule, then the following conditions hold:

(i) $\mu_i(0) \geq \min\{\mu_i(m), 0.5\}$, for $i \in \mathbb{N}_k$,

(ii) $\mu_i(xm - yn) \geq \min\{\mu_i(m), \mu_i(n), 0.5\}$, for $i \in \mathbb{N}_k$.

Now we want to prove that $\sum_{i\in\mathbb{N}_k} \mu_i$ satisfies conditions (i) and (ii).

We know that

$$\begin{aligned}
&\left(\sum_{i\in\mathbb{N}_k} \mu_i\right)(m) \\
&\quad = \sup_{m=\sum_{i\in\mathbb{N}_k} a_i} \min\{\mu_1(a_1), \mu_2(a_2), \ldots, \mu_k(a_k), 0.5\}.
\end{aligned} \tag{13}$$

Now we want to prove that $\sum_{i\in\mathbb{N}_k} \mu_i$ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule.

Prove (i):

$$\begin{aligned}
&\sum_{i\in\mathbb{N}_k} \mu_i(0) \\
&\quad = \sup_{m=\sum_{i\in\mathbb{N}_k} 0_i} \min\{\mu_1(0_1), \mu_2(0_2), \ldots, \mu_k(0_k), 0.5\} \\
&\quad \geq \min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_k(x_k), 0.5\}.
\end{aligned} \tag{14}$$

Since μ_i is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule then (i) holds.

Prove (ii):

$$\begin{aligned}
&\sum_{i\in\mathbb{N}_k} \mu_i(xm - yn) = \sup_{xm-yn=\sum_{i\in\mathbb{N}_k} a_i} \min\{\mu_1(a_1), \mu_2(a_2), \\
&\quad \ldots, \mu_k(a_k), 0.5\} \geq \min\{\mu_1(m_1), \mu_1(n_1), \mu_2(m_2), \\
&\quad \mu_2(n_2), \ldots, \mu_k(m_k), \mu_k(n_k), 0.5\}.
\end{aligned} \tag{15}$$

Since μ_i is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule, then (ii) holds.

This implies that $\sum_{i\in\mathbb{N}_k} \mu_i$ is an $(\in, \in \vee q)$-fuzzy *BCK*-submodule.

□

4. Some Kinds of $(\in, \in \vee q)$-Interval-Valued Fuzzy *BCK*-Submodule

For any $\overline{F}(x) = [F^-(x), F^+(x)]$ and $\overline{t} = [t^-, t^+]$ we define $\overline{F}(x) + \overline{t} = [F^-(x) + t^-, F^+(x) + t^+]$, for all $x \in X$. In particular, if $F^-(x) + t^- > 1$, we write $\overline{F}(x) + \overline{t} > [1, 1]$. Let $x \in X$ and $\overline{t} \in D[0, 1]$. An interval-valued fuzzy set $\overline{G}$ of a *BCK*-algebra X is said to be an interval-valued fuzzy point $x_{\overline{t}}$, with support x and interval-valued $\overline{t}$, if

$$\overline{G}(y) = \begin{cases} \overline{t}\,(\neq [0,0]) & \text{if } y = x \\ [0,0] & \text{if } y \neq x \end{cases} \tag{16}$$

for all $y \in X$; we say $x_{\overline{t}}$ belongs to (resp., is quasi-coincident with) an interval-valued fuzzy set $\overline{F}$, written by $x_{\overline{t}} \in \overline{F}$ (resp., $x_{\overline{t}} q \overline{F}$), if $\overline{F}(x) \geq \overline{t}$ (resp., $\overline{F}(x) + \overline{t} > [1, 1]$); if $x_{\overline{t}} \in \overline{F}$ or $x_{\overline{t}} q \overline{F}$, then we write $x_{\overline{t}} \in \vee q \overline{F}$.

Definition 23 (see [7]). An interval-valued fuzzy set of X is $\overline{F} : X \to D[0, 1]$, where one denotes, for each $x \in X$, $\overline{F}(x) = [F^-(x), F^+(x)] \in D[0, 1]$.

Definition 24 (see [7]). Let $\overline{F}$ be an interval-valued fuzzy set of X. Then for every $[0, 0] < \overline{t} \leq [1, 1]$, the crisp set $\overline{F}_{\overline{t}} = \{x \in X \mid \overline{F}(x) \geq \overline{t}\}$ is called the level subset of $\overline{F}$.

Definition 25. An interval-valued fuzzy subset $\overline{F}$ of M is said to be an $(\in, \in \vee q)$-interval-valued fuzzy *BCK*-submodule of X for all $[0, 0] < \overline{t} \leq [1, 1]$ and for all $m, m_1, m_2 \in M$ and $x \in X$ the following axioms hold:

(i) $\overline{F}(m_1 + m_2) \geq r\min\{\overline{F}(m_1), \overline{F}(m_2), [0.5, 0.5]\}$,

(ii) $\overline{F}(-m) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

(iii) $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$.

Example 26. Consider the *BCK*-algebra $X = \{0, a, b, c\}$ as in Example 7. Define an interval-valued fuzzy set $\overline{F}$ of X by $\overline{F}(0) = [0.4, 0.5]$, $\overline{F}(a) = [0.3, 0.2]$, and $\overline{F}(b) = \overline{F}(c) = [0.1, 0.2]$. Hence, $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy *BCK*-submodule of X.

Theorem 27. *Let $\overline{F} \in \mathscr{F}(M)$, where $\mathscr{F}(M)$ is the set of all interval-valued fuzzy subsets of BCK-module M. Then $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule of M if and only if*

(i) $\overline{F}(0) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

(ii) $\overline{F}(xm - yn) \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$.

Proof. Let $\overline{F}$ be an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule:

(i) $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$, and $x = 0$ implies that

$\overline{F}(0) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

$\overline{F}(xm - yn) \geq r\min\{\overline{F}(xm), \overline{F}(yn), [0.5, 0.5]\} \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$.

Then $\overline{F}(xm - yn) \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$.

Conversely, we have $\overline{F}(xm) = \overline{F}(xm - y0) \geq r\min\{\overline{F}(m), \overline{F}(0), [0.5, 0.5]\} \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$.

Consider $\overline{F}(m - n) = \overline{F}(1 \cdot m - 1 \cdot n) \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$.

Hence, $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule.

□

Theorem 28. *An interval-valued fuzzy subset $\overline{F}$ of M is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule if and only if*

(i) $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

(ii) $\overline{F}(m_1 - m_2) \geq r\min\{\overline{F}(m_1), \overline{F}(m_2), [0.5, 0.5]\}$.

Proof. From the definition of $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule,

(i) $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

(ii) $\overline{F}(m_1 - m_2) = \overline{F}(m_1 + (-m_2)) \geq r\min\{\overline{F}(m_1), \overline{F}(-m_2), [0.5, 0.5]\}$.

But we know that $\overline{F}(-m) = \overline{F}(m)$.

Then $\overline{F}(m_1 - m_2) \geq r\min\{\overline{F}(m_1), \overline{F}(m_2), [0.5, 0.5]\}$.

Conversely, we have $\overline{F}(xm) = \overline{F}(xm - y0) \geq r\min\{\overline{F}(m), \overline{F}(0), [0.5, 0.5]\}, \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$.

Use Theorem 27, since $\overline{F}(0) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$,

and $\overline{F}(m_1 - m_2) = \overline{F}(1 \cdot m_1 - 1 \cdot m_2) \geq r\min\{\overline{F}(m_1), \overline{F}(m_2), [0.5, 0.5]\}$.

Hence, $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule.

□

Theorem 29. *An interval-valued fuzzy subset $\overline{F}$ of M is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule if and only if $\overline{F}_{\bar{t}}(\neq \phi)$ is BCK-submodule of X for all $[0, 0] < \bar{t} \leq [0.5, 0.5]$.*

Proof. Let $\overline{F}$ be an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule of X and $[0, 0] < \bar{t} \leq [0.5, 0.5]$ for any $m, n \in \overline{F}_{\bar{t}}$; then $\overline{F}(m), \overline{F}(n) \geq \bar{t}$. Since $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule, then $\overline{F}(m - n) \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\} = \bar{t}$, which implies that $\overline{F}(m - n) \geq \bar{t}$. Hence, $m - n \in \overline{F}_{\bar{t}}$; let $m \in \overline{F}_{\bar{t}}$, $x \in X$; then $\overline{F}(m) \geq \bar{t}$. Furthermore, $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\} \geq r\min\{\bar{t}, [0.5, 0.5]\} = \bar{t}$, and then $\overline{F}(xm) \geq \bar{t}$. Hence, $xm \in \overline{F}_{\bar{t}}$; hence $\overline{F}_{\bar{t}}$ is BCK-submodule.

Conversely, let $\bar{t} = r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}, m, n \in \overline{F}_{\bar{t}}$; then $\overline{F}(m), \overline{F}(n) \geq \bar{t}$; since $\overline{F}_{\bar{t}}$ is BCK-submodule, then $m - n \in \overline{F}_{\bar{t}}$ implies that $\overline{F}(m - n) \geq \bar{t} = r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$; then $\overline{F}(m-n) \geq r\min\{\overline{F}(m), \overline{F}(n), [0.5, 0.5]\}$; let $s = r\min\{\overline{F}(m), [0.5, 0.5]\}$, and let $x \in \overline{F}_{\bar{t}}, m \in \overline{F}_{\bar{s}}$; then $xm \in \overline{F}_{\bar{s}}$ implies that $\overline{F}(xm) \geq s = r\min\{\overline{F}(m), [0.5, 0.5]\}$; then $\overline{F}(xm) \geq r\min\{\overline{F}(m), [0.5, 0.5]\}$.

Hence, $\overline{F}$ is an $(\in, \in \vee q)$-interval-valued fuzzy BCK-submodule. □

5. Conclusions

In this paper, we introduced the concept of generalized fuzzy BCK-submodules and some basic properties were obtained and we defined the concept of generalized interval-valued fuzzy BCK-submodules and some basic properties were obtained. Other topics in this area which could be a further research are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy BCK-submodules, translations of $(\in, \in \vee q)$-fuzzy BCK-submodule, and other generalizations of fuzzy set theory to BCK-submodules.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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