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Research Article

Analysis of Nonlinear Duopoly Games with Product Differentiation: Stability, Global Dynamics, and Control

S. S. Askar[1,2] and A. Al-khedhairi[1]

[1] *Department of Statistics and Operations Researches, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to S. S. Askar; s.e.a.askar@hotmail.co.uk

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Many researchers have used quadratic utility function to study its influences on economic games with product differentiation. Such games include Cournot, Bertrand, and a mixed-type game called Cournot-Bertrand. Within this paper, a cubic utility function that is derived from a constant elasticity of substitution production function (CES) is introduced. This cubic function is more desirable than the quadratic one besides its amenability to efficiency analysis. Based on that utility a two-dimensional Cournot duopoly game with horizontal product differentiation is modeled using a discrete time scale. Two different types of games are studied in this paper. In the first game, firms are updating their output production using the traditional bounded rationality approach. In the second game, firms adopt Puu's mechanism to update their productions. Puu's mechanism does not require any information about the profit function; instead it needs both firms to know their production and their profits in the past time periods. In both scenarios, an explicit form for the Nash equilibrium point is obtained under certain conditions. The stability analysis of Nash point is considered. Furthermore, some numerical simulations are carried out to confirm the chaotic behavior of Nash equilibrium point. This analysis includes bifurcation, attractor, maximum Lyapunov exponent, and sensitivity to initial conditions.

1. Introduction

Complex dynamic behaviors are typically undesirable phenomena in economic games where searching for stable equilibrium states is a main interest. These complex behaviors are important when studying the characteristics of equilibrium states in such models. The complexity may arise from different sources. It may occur due to the form of utility function adopted to construct those models or from the demand function whether it is linear or nonlinear, or finally it may come from the amount of information that each player involved in those models knows about its rival. Many economically plausible models which have handled this complexity have been studied in literature. Cournot [1] introduced his first model about firms competition and since that time Cournot's model became a central concept of many studies that came later. In Cournot's model, it has been assumed that firms played against each other using quantities as their strategic variables. Quantities are not the only variables that firms can use. There are also prices that have been used by Bertrand [2] in studying another type of games named later as Bertrand games. The appearance of Cournot and Bertrand models has led to more investigations and analysis of such economic games and hence some important results have been obtained. Those important results have shown some complex dynamic characteristics such as bifurcation and chaos. For instance, recently, studies on such economical competition were suspected to lead to complex dynamic behavior such as bifurcation and chaos. In [3], complex dynamics characteristics have been detected in a simple monopoly model. Using a demand function with no inflection points, complex behaviors such as bifurcation and chaos have been investigated in [4]. Askar [5] has shown some important results about Cournot duopoly game that was formed by a concave demand function. Other investigations on those complex characteristics can be found in [6–12].

It has been discussed in [13–15] that quantities may have a degree of competition that is low if commodities are substitutes. Therefore, if firms are free to decide their key variables, they certainly favor to perform with quantities instead of prices. On the other hand, there are types of strategic variables which are a mix between quantities and prices. Games that adopted such type of variables are called Cournot-Bertrand. To the best of our knowledge, Shubik [16] introduced the first model of duopoly on which firms use quantities and prices as their strategic variables. Even though Shubik introduced an analytical scheme for analyzing the behavior of such games, he was not capable of deriving the equilibrium point of those games. Since then literature has contained few number of studies in which the Nash equilibrium (equilibrium solution) was explicitly obtained in an analytical form and its stability has been investigated. The Cournot-Bertrand competition requires a certain degree of differentiation among products offered by firms to avoid one firm dominating the market by its lower price. In [17–20], the authors have argued that, in certain cases, Cournot-Bertrand game may be optimal. Furthermore, Tremblay et al. [21] have demonstrated that empirical evidence has led to the fact that this kind of competition is abundant. Recently, C. H. Tremblay and V. J. Tremblay [22] have shown the static properties of the Nash equilibrium of a Cournot-Bertrand duopoly according to product differentiation. Naimzada and Tramontana [23] have studied the dynamic properties of a Cournot duopoly game with product differentiation using linearity of demand and cost objective functions. In [24], the authors have studied an economic market on which three heterogeneous firms are in conflict and the demand function they adopt is isoelastic. A dynamic duopoly game with heterogeneous players has been investigated in [25] by taking one of the two competed firms as an upper limiter on output, and the other one is a lower limiter. The influences of the limiter on the dynamic behavior of the firms have been shown. In [26], the dynamics of a duopoly Cournot game model has been analyzed. The game has been modeled based on different adjustment mechanisms and expectations on which one of the firms adopts the mechanism "one-period look-ahead" and selects his decision using estimations, while the other firm uses a bounded rational mechanism. A dynamic of a banking duopoly game using homogeneous and heterogeneous firms has been analyzed in [27] in order to identify the impact of capital requirements in the context of the Monti-Klein model. Fanti et al. have studied a nonlinear dynamic duopoly model on which price and product differentiation are augmented with managerial firms [28]. In [29], a heterogeneous duopoly Cournot game is studied where firms adopt two different approaches which are Local Monopolistic Approximation (LMA) and a gradient-based approach. In [30], an investment process has been considered and studied in a duopoly game where all the firms are heterogeneous firms.

In monopoly and duopoly markets, there are two important methods that have been used to tackle those types of markets. They are called bounded rationality and Puu's incomplete information. In bounded rationality approach, firms estimate their marginal profit so that they can update their output productions. Depending on this estimation of the profit and using discrete time steps they build a discrete dynamical system that describes the dynamic behavior of the game. In addition, the firms adopting such local adjustment mechanism are not requested to know the demand and the cost functions used in the market [31]. Instead they need to know if there is some small changes in the production by the marginal profit estimation, will that lead to a response from the market? This mechanism is sometimes called myopic [31] and has been extensively used by many authors, mainly with continuous time [32]. However, it is discussed elsewhere [31] that a decision process based on discrete time scales is more realistic because in real economic systems it may not be possible to revise the production decisions at every time instant. On the other hand, an alternative and different approach has been introduced recently by Puu [33]. It depends on imperfect information about the competed firms in the market and is called Puu's incomplete information. It is characterized by its applicability in economic markets as it is realistically accepted. It requires the firms to know the quantity and the profit in the past two times steps. Furthermore, it does not need any information about the form of the profit function in order to estimate the quantity produced in the next time step. One of the disadvantages of this mechanism is its singularity when approaching the equilibrium point and this has been reported in [34]. It has been investigated that systems based on Puu's techniques are numerically unstable when approaching the equilibrium position [34]. Moreover such systems have serious instabilities in the case of duopoly. The authors in [34] have modified those systems based on Puu's techniques with a change in the quantities produced by 10% per time step to avoid singularities in such systems.

Investors' utility functions may assume some complex forms; however, most recent theoretical and applied discussions have dealt with relatively simple forms such as quadratic utility functions [13, 14]. The reason for choosing such forms is to manage the investment decision rules in a simple form. Utility functions should possess some specific and amenable characteristics such as continuity and differentiability. Even though there are different functional forms of utility, still there is little guidance for researchers to select among them. Regardless of such functional forms, utility functions must possess some important aspects such as strictly positive marginal utility of income, estimating the parameters of the function in an easy way to manipulate it analytically [35, 36].

The main aim of this paper is to analyze the influences of a cubic utility function on a Cournot duopoly game with differentiated products. We claim that cubic utility function may be amenable and desirable to some extent compared to quadratic form. Our main results concern stability and instability of the fixed points of the proposed model including the routes that lead to different types of bifurcations. Our studies in this paper include analyzing two types of games depending on two important and different methods, the bounded rationality and Puu's methods. Furthermore, we develop a new control model based on the parameters adjustment approach in order to protect the system from instability and chaos.

The paper is organized as follows. In Section 2, two different games are introduced. The first one consists of two bounded rational Cournot duopoly games and the second one describes a competition between a bounded rational and naive Cournot duopoly players. For both games, the Nash equilibrium point is obtained and its stability is investigated. Under some conditions, it is shown that the Nash point is locally asymptotically stable and then it loses its stability via bifurcation. In Section 3, a game between two duopolistic firms is introduced in which both firms adopt Puu's approach to update their productions in the next time step. For Puu's game, Nash point is computed and its stability is investigated. In Section 4, a control method to suppress chaos in Puu's system is developed. Finally, we end the paper with conclusions to show the significance of our results.

2. The Dynamic Model

In this section, we assume an economic market where two types of agents are presented: firms and consumers. The competition in this market may be divided into two important sectors. The first sector produces the numeraire commodity $y \geqslant 0$, while the other sector consists of two duopolistic firms, usually called firm 1 and firm 2. Both firms' decision variables are either prices p_i or production quantities q_i, $i = 1, 2$. Existence of a continuum identical consumers that have preferences towards q_1, q_2, and y is assumed. These preferences may be represented by a separable utility function $V(q_1, q_2, y) : \mathbb{R}_+^3 \to \mathbb{R}_+$ given by $V(q_1, q_2, y) = U(q_1, q_2) + y$, where $U(q_1, q_2) : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a twice differentiable function. The representative consumer has an exogenously given income, $M \geqslant 0$, and thus maximizes his/her preferences with respect to the quantities produced, q_1, q_2, according to a budget constraint, $p_1 q_1 + p_2 q_2 + y = M$. According to that the consumer's optimization problem is given by the following form:

$$\max_{q_1, q_2} U(q_1, q_2) - p_1 q_1 - p_2 q_2 + M. \tag{1}$$

Solving problem (1) gives the inverse demand function, $p_i = \partial U / \partial q_i = p_i(q_1, q_2)$, $i = 1, 2$, which represents the prices of commodities produced by the firms as functions of quantities. In order to have explicit demand functions, a specific utility function is required. In this paper, the following cubic utility function is considered:

$$U(q_1, q_2) = (a - d q_1 q_2)(q_1 + q_2) - \frac{1}{3}\left(q_1^3 + q_2^3\right). \tag{2}$$

It is easy to check that U is strictly concave at $q_1 = q_2 = \sqrt{a/(1+3d)}$, $d \in (-1/3, 1)$. Therefore, it is strictly quasiconcave and so is strictly convex to the origin level curves. This means that by setting the total differential $dU = 0$ we get the marginal rate of technical substitution, MRTS $= -dq_2/dq_1 = (a - (q_1^2 + 2q_1q_2 + dq_2^2))/(a - (dq_1^2 + 2q_1q_2 + q_2^2))$, which is strictly convex to the origin. It means that the consumer can substitute one input for the other and continue to produce the same level of output. Here $a > 0$ captures the size of the market demand, while $-1 < d < 1$ represents the degree of horizontal product differentiation. Using (1) and (2), one gets

$$\begin{aligned} p_1 &= a - q_1^2 - d q_2^2 - 2 d q_1 q_2, \\ p_2 &= a - d q_1^2 - q_2^2 - 2 d q_1 q_2. \end{aligned} \tag{3}$$

If $d = 1$, the two inverse demand functions become identical, which is the case of homogenous goods. This leads to MRTS $= -dq_2/dq_1 = 1$ and hence $dq_2 = -dq_1$ which means that the consumer can substitute one good for the other and continue to remain on the same indifference curve. If $d = 0$, it implies that the market has two monopolistic firms. Assuming negative values of the parameter d implies complementarity between the two firms. Now, we construct our proposed games.

First, it is assumed that both firms play based on their marginal profits. The profit function for firm i is $\pi_i = (p_i - c) q_i$, $i = 1, 2$, where c is a fixed marginal cost. By substituting (3) in π_i, $i = 1, 2$, profits of firm 1 and firm 2 are, respectively, given by

$$\begin{aligned} \pi_1(q_1, q_2) &= \left(a - c - q_1^2 - d q_2^2 - 2 d q_1 q_2\right) q_1, \\ \pi_2(q_1, q_2) &= \left(a - c - d q_1^2 - q_2^2 - 2 d q_1 q_2\right) q_2. \end{aligned} \tag{4}$$

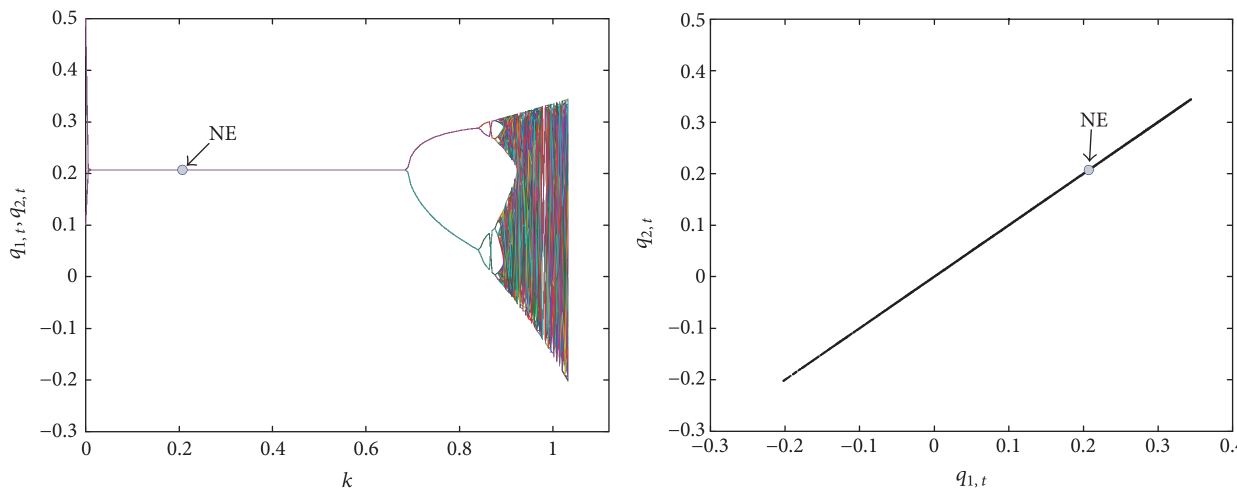
Suppose a dynamic competition between the two firms takes place. For a discrete time $t \in \mathbb{Z}_+$, (4) can be rewritten in the form

$$\begin{aligned} \pi_{1,t}(q_{1,t}, q_{2,t}) &= \left(a - c - q_{1,t}^2 - d q_{2,t}^2 - 2 d q_{1,t} q_{2,t}\right) q_{1,t}, \\ \pi_{2,t}(q_{1,t}, q_{2,t}) &= \left(a - c - d q_{1,t}^2 - q_{2,t}^2 - 2 d q_{1,t} q_{2,t}\right) q_{2,t}. \end{aligned} \tag{5}$$

Therefore, the marginal profits are obtained as follows:

$$\begin{aligned} \frac{\partial \pi_{1,t}(q_{1,t}, q_{2,t})}{\partial q_{1,t}} &= a - c - 3 q_{1,t}^2 - d q_{2,t}^2 - 4 d q_{1,t} q_{2,t}, \\ \frac{\partial \pi_{2,t}(q_{1,t}, q_{2,t})}{\partial q_{2,t}} &= a - c - d q_{1,t}^2 - 3 q_{2,t}^2 - 4 d q_{1,t} q_{2,t}. \end{aligned} \tag{6}$$

Now, we may set up the information by which each player should know about its competitor. If each player has a complete knowledge of the profit function (i.e., demand and cost functions), then he/she will use some kind of expectations against his/her competitor's decision. Those expectations may be naive, rational, or adaptive expectations, or, alternatively, some weighted sum of previous decision to set the quantities at time $t + 1$. On the other hand, if a player does not have a complete knowledge of the profit function, he/she can use some local estimation of the marginal profit in order to follow the steepest slope of the profit function [6]. Such limited information makes players unable to completely solve the optimization problem $\max_{q_1, q_2} \pi_{i,t+1}(q_{1,t+1}, q_{2,t+1})$ by considering expectations about the quantity that the competitor will choose for the next period, but they are able to get a correct estimate of their own local slope, that is, the partial derivatives of the profit computed at the current state of production. This will help each player to increase



(a) (b)

FIGURE 1: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = 0.8$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = 0.8$.

or decrease the quantity produced at time $t + 1$ depending on whether its own marginal profit at time t is positive or negative. Assuming limited information, the adjustment mechanism of quantities over time is described as follows:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k\frac{\partial \pi_{1,t}}{\partial q_{1,t}}, \\ q_{2,t+1} &= q_{2,t} + k\frac{\partial \pi_{2,t}}{\partial q_{2,t}}, \end{aligned} \tag{7}$$

where $k > 0$ is a constant that captures the speed at which firm i adjusts its quantity with respect to the consequent marginal change in its profit. Now the following games are considered.

Bounded Rational Cournot Duopoly Game. By taking into account (6), system (7) becomes as follows:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k\left(a - c - 3q_{1,t}^2 - dq_{2,t}^2 - 4dq_{1,t}q_{2,t}\right), \\ q_{2,t+1} &= q_{2,t} + k\left(a - c - dq_{1,t}^2 - 3q_{2,t}^2 - 4dq_{1,t}q_{2,t}\right). \end{aligned} \tag{8}$$

The above system describes our proposed game. Its equilibrium point can be obtained by setting in system (8) $(q_{1,t+1}, q_{2,t+1}) = (q_{1,t}, q_{2,t})$; then the following proposition holds.

Proposition 1. *The system given by (8) admits a unique positive fixed point at* $-3/5 < d < 1$. *It is given by* NE $= (\sqrt{(a-c)/(3+5d)}, \sqrt{(a-c)/(3+5d)})$ *and* $a > c$.

It is interesting to study the stability and instability of this fixed point. The Jacobian matrix of system (8) at this point is given by the following:

$$J : \begin{bmatrix} 1 - 2k(3+2d)\sqrt{\dfrac{a-c}{3+5d}} & -6kd\sqrt{\dfrac{a-c}{3+5d}} \\ -6kd\sqrt{\dfrac{a-c}{3+5d}} & 1 - 2k(3+2d)\sqrt{\dfrac{a-c}{3+5d}} \end{bmatrix} \tag{9}$$

whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 - 2k(3-d)\sqrt{\frac{a-c}{3+5d}}, \\ \lambda_2 &= 1 - 2k(3+5d)\sqrt{\frac{a-c}{3+5d}}. \end{aligned} \tag{10}$$

Therefore, this fixed point is asymptotically stable under the condition $0 < k < 1/\sqrt{(a-c)(3+5d)}$.

Some numerical evidences are provided to illustrate the above results given in Proposition 1. They are carried out by assuming the system's parameters as follows: $a = 1$, $c = 0.7$, $d = 0.8$. The results graphically show that the behavior of map (8) changes from stability to chaotic state for different values of the reaction coefficient k. Figure 1 shows that the fixed point is asymptotically stable for certain values of k and at the same time positive values of the quantities produced are guaranteed. After that it becomes unstable due to bifurcation appearing and the negative values of quantities obtained. In Figure 1(b) the attractor behavior of the two firms at those parameters is depicted. The bifurcation diagram shows that the system moves from stability through a sequence of a period doubling bifurcation to chaos. In Figure 2(a), one can easily see that a bifurcation is reported too and the region of stability increases as the differentiation parameter d decreases in the case substitutability ($0 < d < 1$). Figure 3 shows that when d approaches zero (in the case of substitutability) the stability region increases and then becomes unstable due to bifurcation. Moreover, the simulation experiments have confirmed that when choosing values of $d \in [0.8, 1]$ no results can be obtained and this is because both firms' outputs become negative and that has no sense in economy.

On the other hand, when choosing negative values of $d \in (-0.54, 0)$, the behavior of the fixed point of system (8) becomes asymptotically stable for some values of the

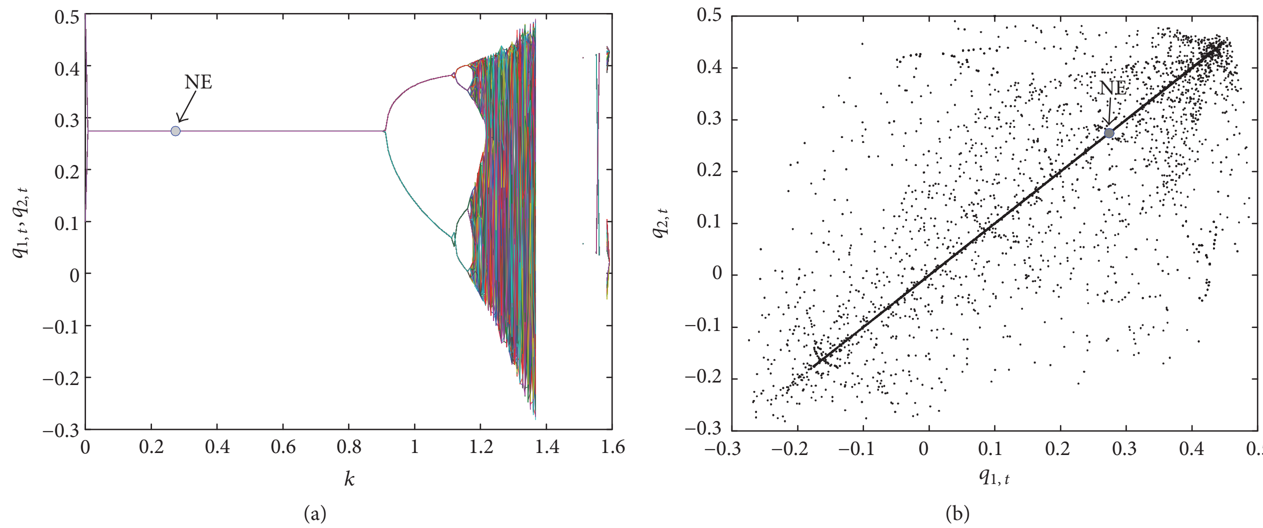


Figure 2: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = 0.2$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = 0.2$.

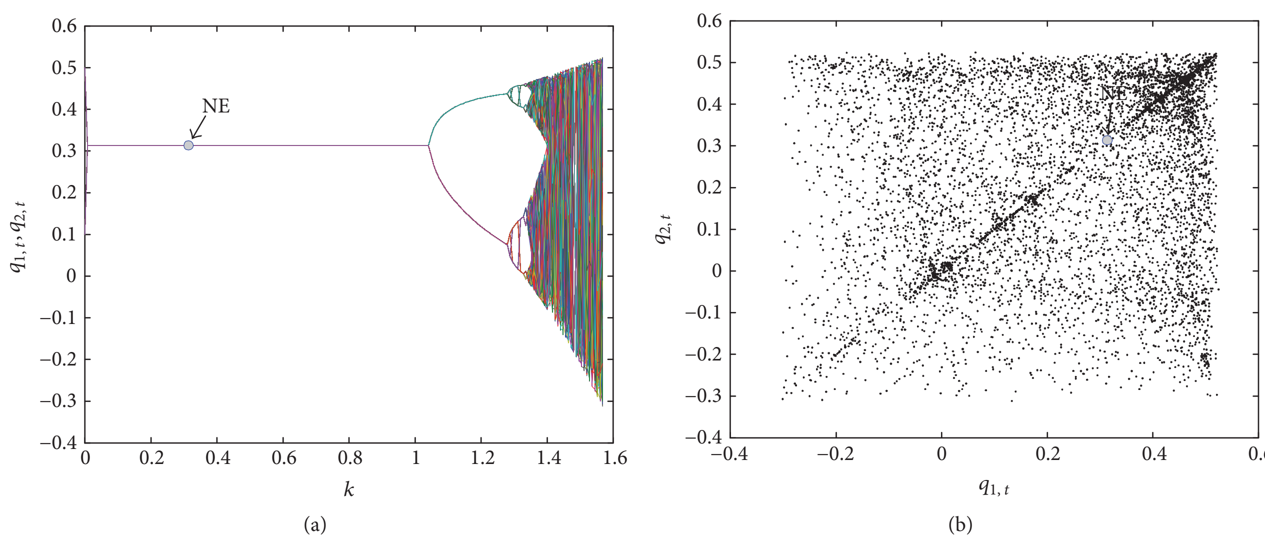


Figure 3: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = 0.01$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = 0.01$.

parameter k. Figures 4, 5, and 6 show the behavior of the system when negative values of d are chosen. Comparing the above attractors, there are some changes in their structures in both cases, the case of substitutability ($d \in (0, 0.8)$) and the case of complementarity ($d \in (-3/5, 0)$). Furthermore, Figure 4 shows that in the latter case when d increases ($d \in (-3/5, 0)$), and no matter what the two firms choose initially, the stability region of the fixed point increases. In Figure 7(a), the maximum Lyapunov exponents are plotted. The sensitivity of system (8) to the initial conditions is also depicted in Figure 7. The two orbits are initially started and slightly deviated from $(q_{1,0}, q_{2,0}) = (0.1, 0.5)$ and $(q_{1,0} + 0.0001, q_{2,0}) = (0.1001, 0.5)$.

Bounded Rational versus Naive Cournot Duopoly Game. In this game, some heterogeneous expectations are assumed. We assume that the first firm adopts bounded rational expectations against the quantity produced by its opponent in the future. The second firm has a naive expectation in the sense that it expects that the competitor will produce in the future a quantity based on the marginal profit obtained in the last period. In this case, system (8) can be rewritten as follows:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k\left(a - c - 3q_{1,t}^2 - dq_{2,t}^2 - 4dq_{1,t}q_{2,t}\right), \\ q_{2,t+1} &= a - c - dq_{1,t}^2 - 3q_{2,t}^2 - 4dq_{1,t}q_{2,t}. \end{aligned} \tag{11}$$

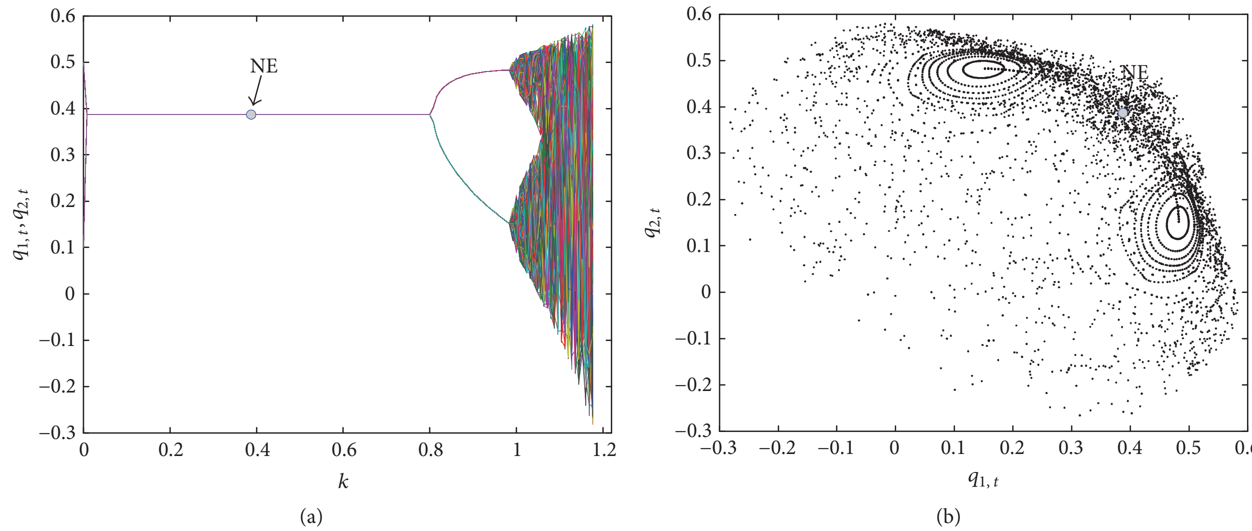


Figure 4: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = -0.2$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = -0.2$.

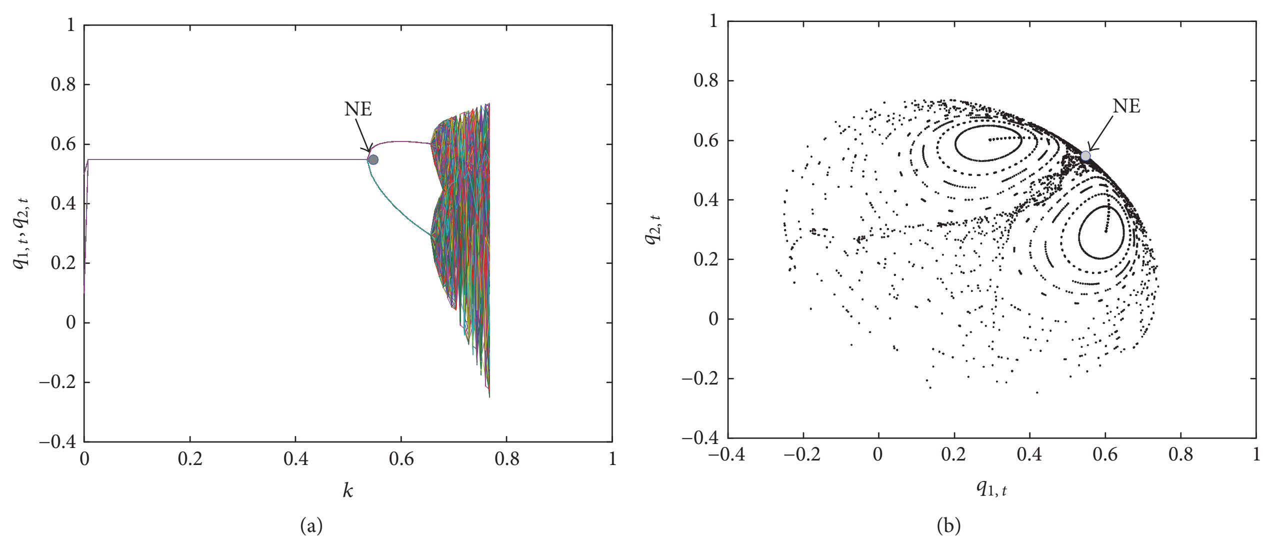


Figure 5: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = -0.4$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = -0.4$.

Proposition 2. *The system given by (11) admits the fixed point $E = (\bar{q}_1, \bar{q}_2)$ that satisfies the condition $(3 - d)(\bar{q}_1^2 - \bar{q}_2^2) = \bar{q}_2$.*

The Jacobian matrix of system (11) at this point is given by

$$J : \begin{bmatrix} 1 - 2k\left(3\bar{q}_1 + 2d\bar{q}_2\right) & -2kd\left(2\bar{q}_1 + \bar{q}_2\right) \\ -2d\left(\bar{q}_1 + 2\bar{q}_2\right) & -2\left(2d\bar{q}_1 + 3\bar{q}_2\right) \end{bmatrix} \tag{12}$$

whose trace and determinant are, respectively,

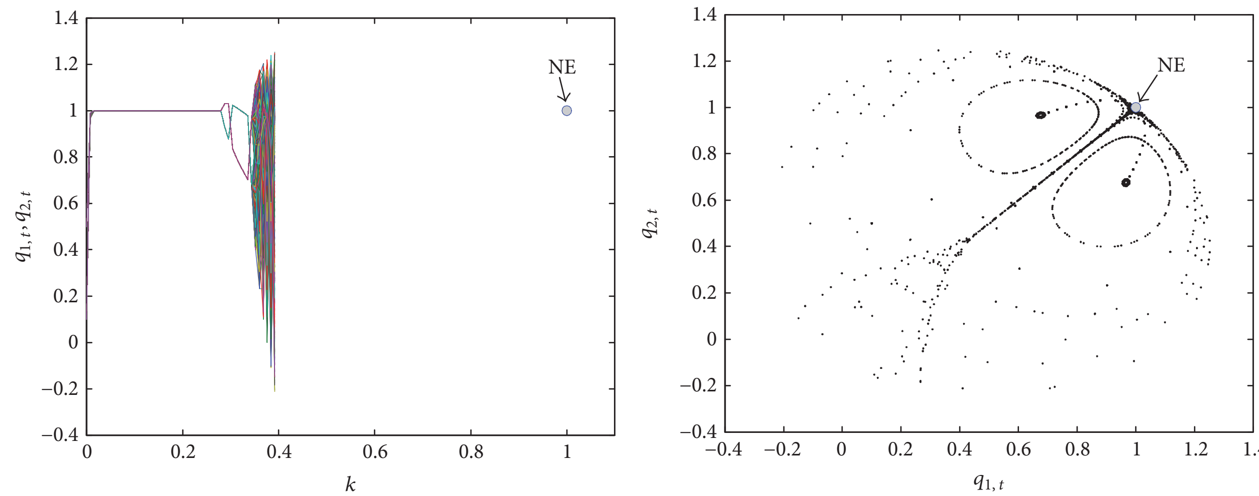
$$\begin{aligned} \mathrm{Tr} &= 1 - 2k\left(2d\bar{q}_2 + 3\bar{q}_1\right) - 2\left(2d\bar{q}_1 + 3\bar{q}_2\right), \\ \mathrm{Det} &= -2\left[\left(d\bar{q}_1 + 3\bar{q}_2\right)\right. \\ &\quad \left.- 2k\left(3 - d\right)\left(2d\bar{q}_1^2 + d\bar{q}_1\bar{q}_2 + 2d\bar{q}_2^2 + 3\bar{q}_1\bar{q}_2\right)\right]. \end{aligned} \tag{13}$$

To study the stability of the fixed point E, we recall the well-known stability conditions [7] which are generally given by

$$\begin{aligned} &\text{(i)} \quad F := 1 + \mathrm{Tr} + \mathrm{Det} > 0 \\ &\text{(ii)} \quad TC := 1 - \mathrm{Tr} + \mathrm{Det} > 0 \\ &\text{(iii)} \quad H := 1 - \mathrm{Det} > 0. \end{aligned} \tag{14}$$

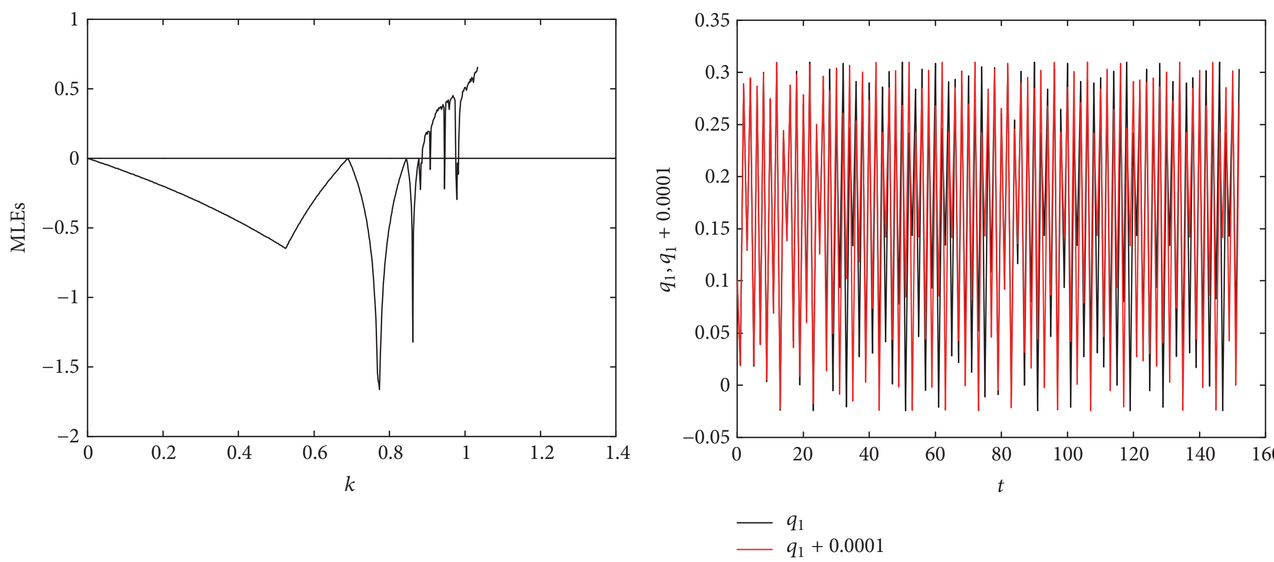
It is known that the violation of any conditions of the above with the other two being simultaneously satisfied leads to different types of bifurcation. Using $E = (\bar{q}_1, \bar{q}_2)$, the conditions can be rewritten in the following form:

$$\begin{aligned} F &:= 2 - 12\bar{q}_2 + 6k\left(6\bar{q}_1\bar{q}_2 - \bar{q}_1\right) \\ &\quad + d\left[4k\left(6\bar{q}_1^2 + 6\bar{q}_2^2 - \bar{q}_2\right) - 8\bar{q}_1\right] \end{aligned}$$



(a) (b)

Figure 6: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = -0.54$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = -0.54$.



(a) (b)

Figure 7: (a) The maximum Lyapunov exponent with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = 0.2$. (b) Sensitivity to initial conditions at $a = 1,\ c = 0.7,\ d = 0.2,\ k = 0.9$ and $q_{1,0} = 0.1, q_{2,0} = 0.5$.

$$\begin{aligned}
&\quad - 4kd^2\left(2\overline{q}_1^2 + \overline{q}_1\overline{q}_2 + 2\overline{q}_2^2\right),\\
TC &:= 6k\overline{q}_1\left(1 + 6\overline{q}_2\right) + 4kd\left(6\overline{q}_1^2 + 6\overline{q}_2^2 + \overline{q}_2\right)\\
&\quad - 4kd^2\left(2\overline{q}_1^2 + \overline{q}_1\overline{q}_2 + 2\overline{q}_2^2\right),\\
H &:= 1 + 2\left(d\overline{q}_1 + 3\overline{q}_2\right)\\
&\quad - 4d\left(3 - d\right)\left(3d\overline{q}_1^2 + d\overline{q}_1\overline{q}_2 + 2d\overline{q}_2^2 + 3\overline{q}_1\overline{q}_2\right).
\end{aligned} \tag{15}$$

The above conditions do not give any information about the stability of the fixed point of system (11). Instead, we use some numerical simulations to get some insights about the stability behavior of the fixed point. We choose the system's parameters as follows: $a = 1$, $c = 0.7$, and $d = -0.55$. It is shown in Figure 8 that both firms are unstable since bifurcations start to appear from the initial quantities chosen. In addition, simulation experiments have shown that for any values of the system's parameters both firms will be entirely unstable and this

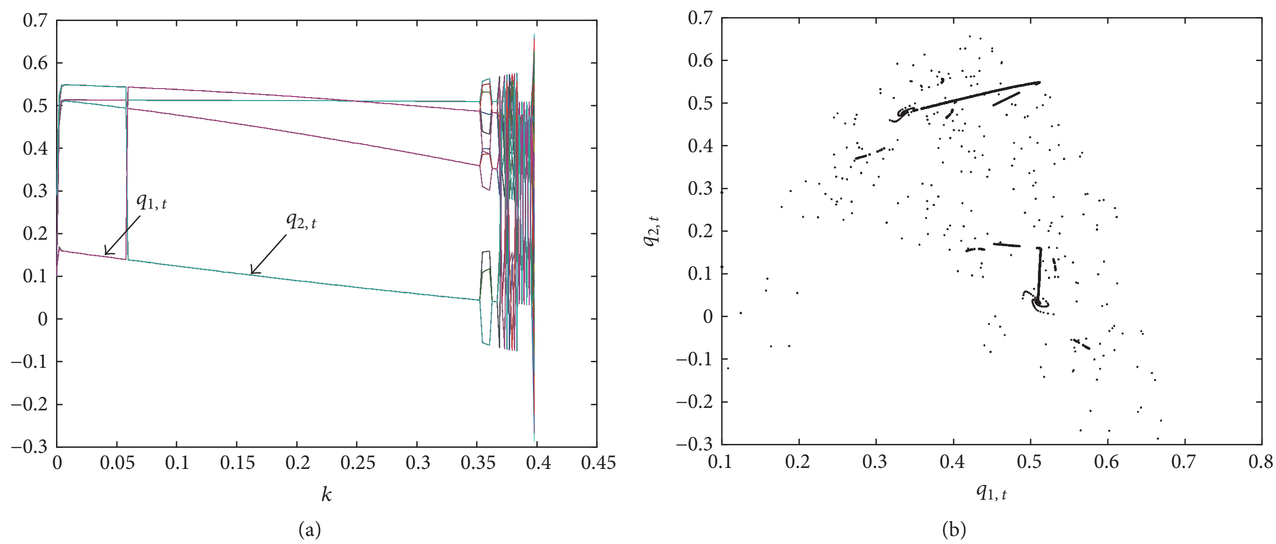


FIGURE 8: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = -0.55$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = -0.55$.

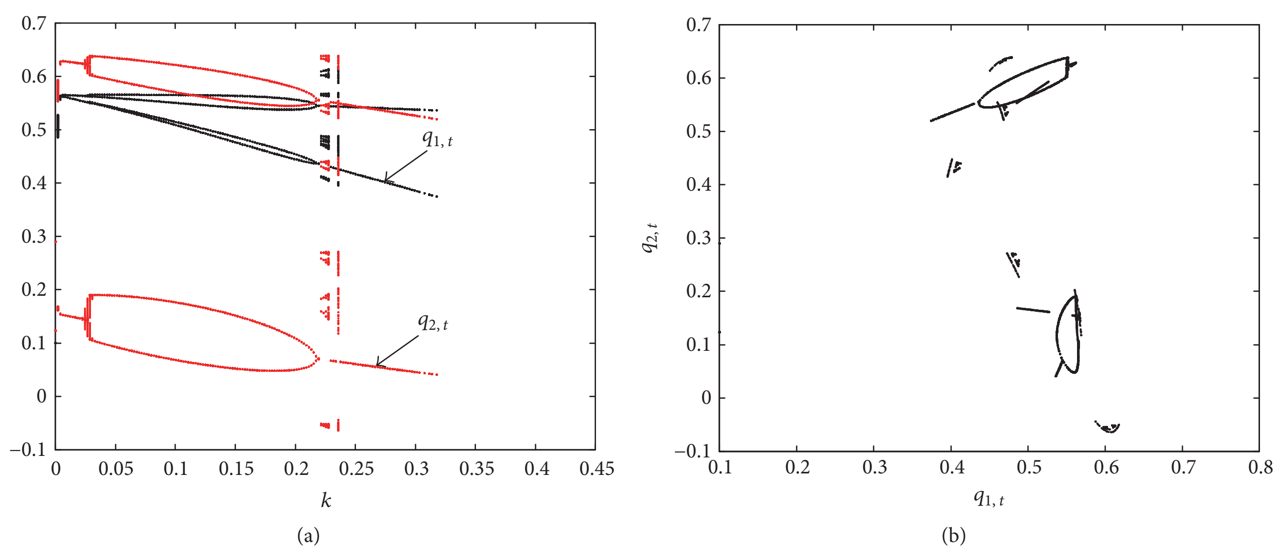


FIGURE 9: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = -0.6$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = -0.6$.

is clear from Figures 9, 10, and 11. The reason for that may be the naive expectation adopted by the second firm.

3. Puu's Games and Main Results

Puu's incomplete information approach is an alternative to the bounded rationality approach. It has a main advantage that it is realistic since a firm does not need to know the form of the profit function to get an estimate of the quantity (Cournot game) or price (Bertrand game). Instead all it needs is its profits and the quantities (or prices in the case of Bertrand) in the past two time steps. However it has a serious problem; that is, the system $q_{i,t+1} = q_{i,t} + k(q_{i,t})((\pi_{i,t} - \pi_{i,t-1})/(q_{i,t} - q_{i,t-1}))$, $i = 1, 2$, is numerically unstable as it approaches equilibrium ($q_{i,t+1} = q_{i,t} = q_{i,t-1}$, $i = 1, 2$). There is no guarantee that the rate of convergence of the profits will be faster than or equal to that of the quantities (or prices). Now, using (5), the following system is obtained:

$$
\begin{aligned}
q_{1,t+1} = q_{1,t} + k\left(q_{1,t}\right)\Bigg[&a - c \\
&- \left(q_{1,t}^2 + q_{1,t}q_{1,t-1} + q_{1,t-1}^2\right) \\
&- \frac{d\left(q_{1,t}q_{2,t}^2 - q_{2,t-1}^2 q_{1,t-1}\right)}{q_{1,t} - q_{1,t-1}} \\
&- \frac{2d\left(q_{1,t}^2 q_{2,t} - q_{1,t-1}^2 q_{2,t-1}\right)}{q_{1,t} - q_{1,t-1}}\Bigg],
\end{aligned}
$$

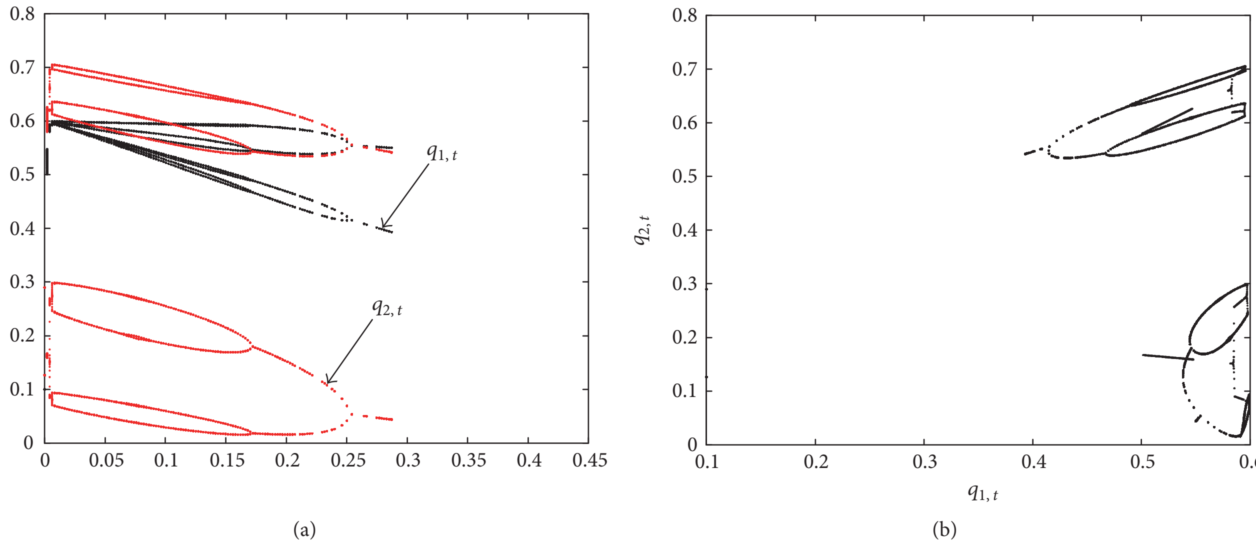


FIGURE 10: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = -0.62$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = -0.62$.

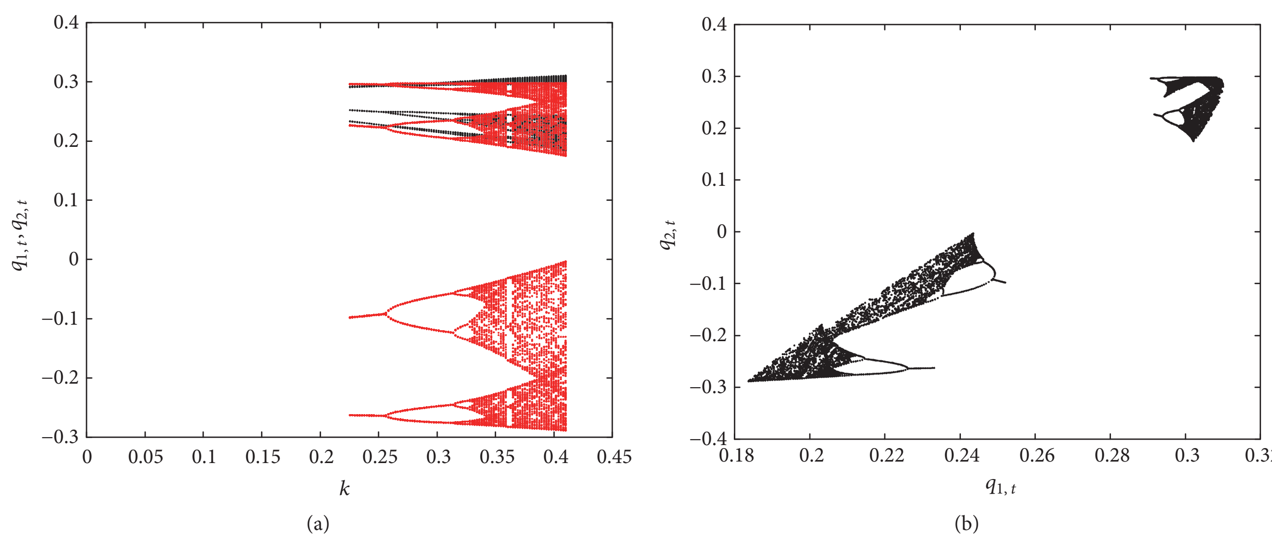
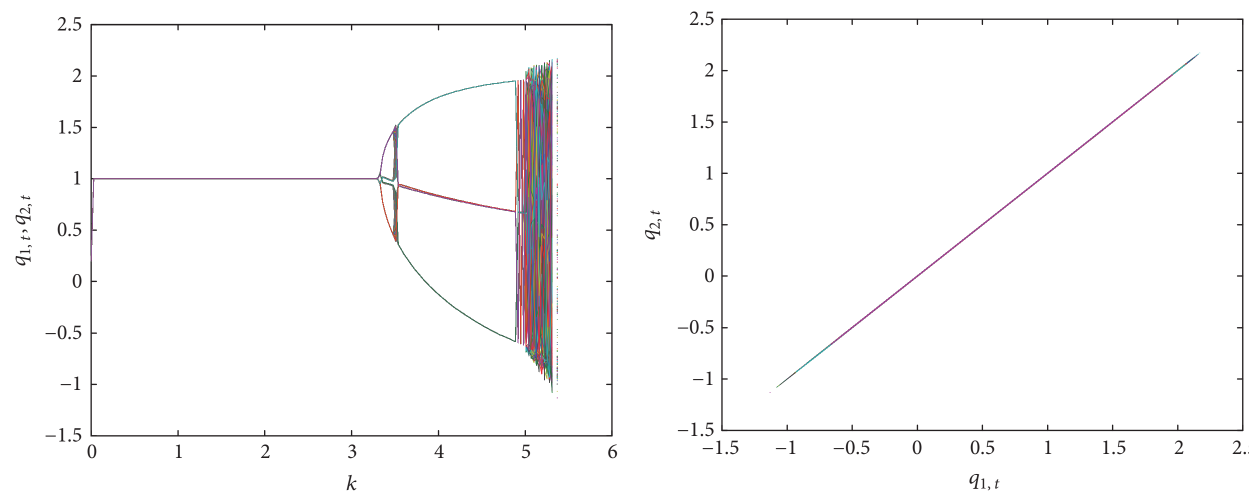


FIGURE 11: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = 0.7$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = 0.7$.

$$
\begin{aligned}
q_{2,t+1} = q_{2,t} + k\left(q_{2,t}\right) \Bigg[& a - c \\
& - \left(q_{2,t}^2 + q_{2,t}q_{2,t-1} + q_{2,t-1}^2\right) \\
& - \frac{d\left(q_{2,t}q_{1,t}^2 - q_{1,t-1}^2 q_{2,t-1}\right)}{q_{2,t} - q_{2,t-1}} \\
& - \frac{2d\left(q_{2,t}^2 q_{1,t} - q_{2,t-1}^2 q_{1,t-1}\right)}{q_{2,t} - q_{2,t-1}} \Bigg].
\end{aligned} \tag{16}
$$

The singularity of the above system at the equilibrium point causes instability. To overcome this disadvantage, there are two possibilities. The first is that the two firms are different enough such that eventually only one firm persists while the other goes bankrupt. In this case, one regains the case of monopoly and, therefore, the market is dominated by this firm. The other possibility is that the two firms are close enough that they all persist. This means that when the game is with incomplete information, firms may or may not know some information about the other firms, for example, their "type," their strategies, their payoffs, or their preferences. Here, we assume that both firms know their amount of



(a) (b)

FIGURE 12: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = -0.3$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = -0.3$.

quantities at time t and, in this case, the above system can be approximated by setting $q_{1,t} \approx q_{2,t}$:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + k\left(q_{1,t}\right) \\ &\quad \cdot \left[a - c - (1+3d)\left(q_{1,t}^2 + q_{1,t}q_{1,t-1} + q_{1,t-1}^2\right)\right], \\ q_{2,t+1} &= q_{2,t} + k\left(q_{2,t}\right) \\ &\quad \cdot \left[a - c - (1+3d)\left(q_{2,t}^2 + q_{2,t}q_{2,t-1} + q_{2,t-1}^2\right)\right]. \end{aligned} \tag{17}$$

Now, two different cases for $k(q_{1,t})$ are studied.

Case 1. Let $k(q_{1,t}) = k$ be a constant; then the following proposition is given.

Proposition 3. *The system given by (17) admits the positive fixed point* $E^* = (\sqrt{(a-c)/3(1+3d)}, \sqrt{(a-c)/3(1+3d)})$, $a > c$, *and* $d \in (-1/3, 1)$.

The Jacobian matrix of system (17) at this point is given by

$$J : \begin{bmatrix} 1 - 6k(1+3d)\sqrt{\dfrac{a-c}{3(1+3d)}} & 0 \\ 0 & 1 - 6k(1+3d)\sqrt{\dfrac{a-c}{3(1+3d)}} \end{bmatrix} \tag{18}$$

whose eigenvalues are $\lambda_1 = \lambda_2 = 1 - 6k(1 + 3d)\sqrt{(a-c)/3(1+3d)}$ *and therefore* E^* *is stable under the condition* $k < 1/\sqrt{3(a-c)(1+3d)}$. *We use some simulations to confirm these obtained results. We start with* $a = 1$, $c = 0.7$, $d = -0.3$. *Figure 12 shows that this fixed point of system (17) is asymptotically stable for some values of the parameter* k. *Moreover, when* d *increases above the interval* $(-1/3, 0)$, *the region of stability decreases and the fixed point becomes unstable due to chaos. Figures 13 and 14 show the behavior of system (17) for different values of* d.

4. Controlling Chaos via Parameters Adjustment

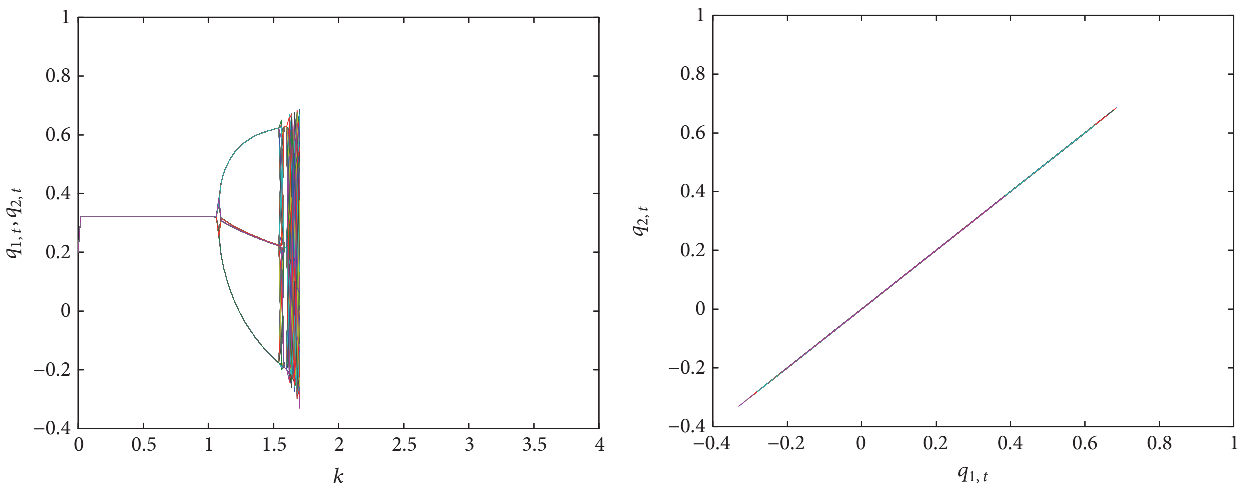
Since chaos is an undesirable phenomena in economic systems, then it needs to be controlled. In this section, we develop a control method to suppress chaos in the systems presented in the previous section. To start the numerical simulation of the controlled system, we should rewrite system (17) in the following form:

$$\begin{aligned} q_{1,t+1} &= \left(1 - \frac{1}{\alpha}\right) q_{1,t} + \left(\frac{1}{\alpha}\right) \\ &\quad \cdot \left[a - c - (1+3d)\left(q_{1,t}^2 + q_{1,t}q_{1,t-1} + q_{1,t-1}^2\right)\right], \\ q_{2,t+1} &= \left(1 - \frac{1}{\alpha}\right) q_{2,t} + \left(\frac{1}{\alpha}\right) \\ &\quad \cdot \left[a - c - (1+3d)\left(q_{2,t}^2 + q_{2,t}q_{2,t-1} + q_{2,t-1}^2\right)\right]. \end{aligned} \tag{19}$$

Figure 15 shows that for the parameters $a = 1$, $c = 0.7$, $d = -0.03$, and $\alpha \leq 0.5$ the system behaves chaotically and the fixed point of the system is unstable. In addition, for any value for the parameter of product differentiation less than -0.03 and the other parameters are fixed, the system gets involved in the chaotic region and hence the fixed point is unstable. Figure 16 presents another controllable case of the system at $a = 1$, $c = 0.7$, $d = 0.6$. Once $\alpha \geq 1.2$ with the other parameters being fixed and $d \in [-0.3, 1]$ the fixed point becomes asymptotically stable and this is clear in Figure 16.

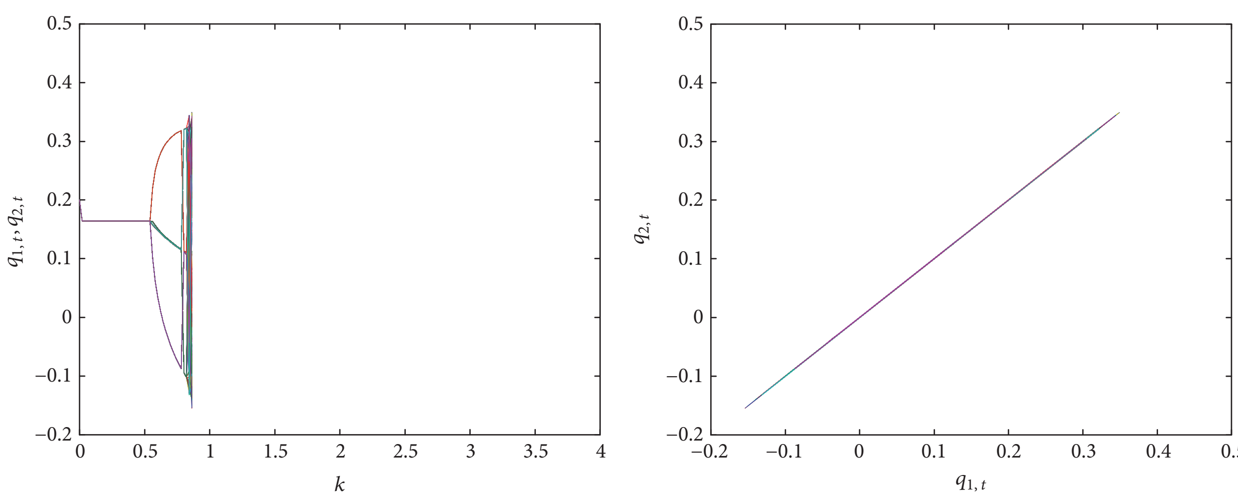
5. Conclusion

In this paper, we have generalized and extended results in literature for the Cournot duopoly games with product differentiation. Based on a proposed cubic utility function that is derived from a constant elasticity of substitution



(a) (b)

FIGURE 13: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = -0.01$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = -0.01$.

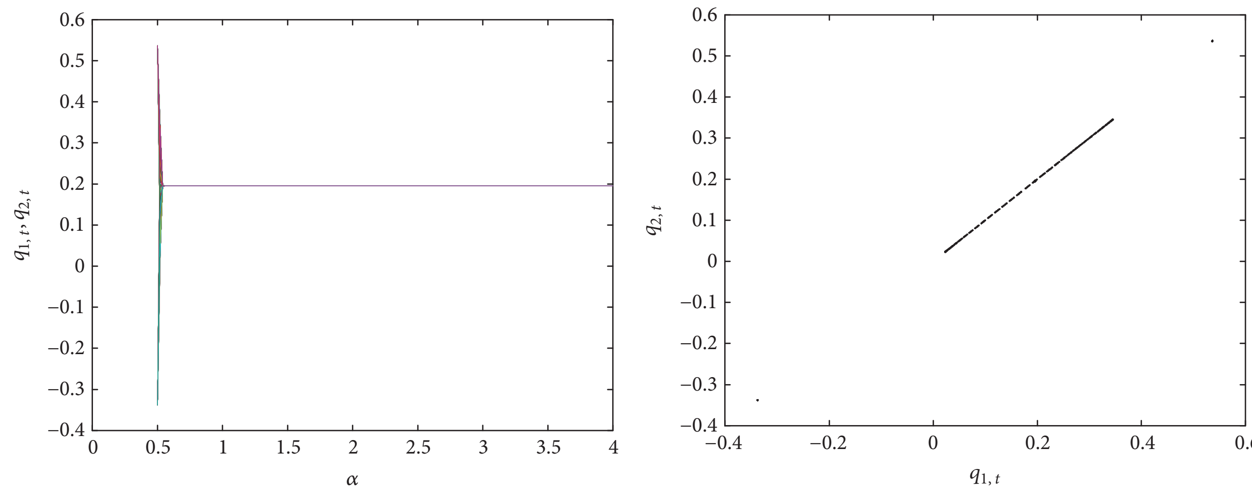


(a) (b)

FIGURE 14: (a) Bifurcation diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1,\ c = 0.7,\ d = 0.9$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1,\ c = 0.7,\ d = 0.9$.

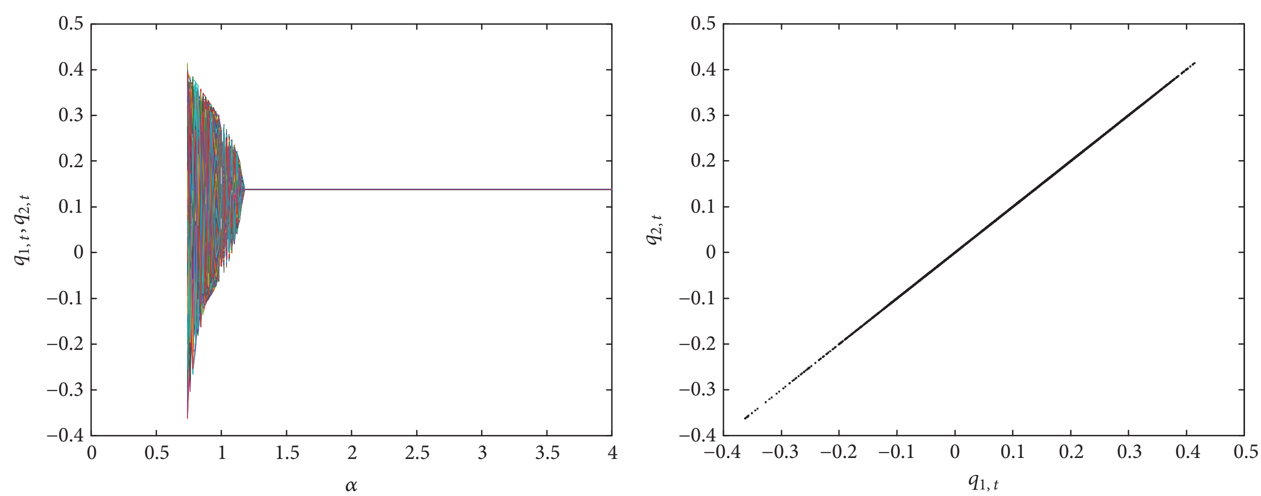
production function (CES), firms have made some computational skills required to make their decisions. In particular, the time evolution of those games has been modeled by a discrete dynamic system obtained by the iteration of a two-dimensional map. A rational Cournot duopoly and a Puu duopoly with quantity competition have been proposed. For each game, the Nash equilibrium of the game has been computed. Complete analytical and numerical studies of the stability conditions for the Nash point have been investigated. The analysis of bifurcation which causes qualitative changes in the behavior of games and causes loss of stability of Nash equilibrium has been discussed through numerical explorations. We conclude according to the obtained results that it is not directed to say which one of the two games is better than the other since both games are sensitive to their parameters. Finally, a developed control technique has been applied to Puu's game.

For a long time, although the quadratic utility function has got more attention in economic studies, but in decision analysis many applications have some critical steps about the estimation of a suitable utility function. The choice of a utility function may be critical as it may have an influence on the decision-makers. We have adopted in this paper the cubic utility function because it has certain properties which are preferred to those of quadratic one. One of these properties includes the fact that the utility function should be monotonically increasing under certain restrictions on its coefficients. We believe that since utility function reflects the individual's preferences, it is not likely that one form of utility will be used to correctly predict individual's behavior.



(a) (b)

FIGURE 15: (a) Behavior diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = -0.03$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = -0.03$.



(a) (b)

FIGURE 16: (a) Behavior diagram of $q_{1,t}$ and $q_{2,t}$ with respect to k at the system parameters: $a = 1$, $c = 0.7$, $d = 0.6$. (b) Attractor of $q_{1,t}$ and $q_{2,t}$ at the system parameters: $a = 1$, $c = 0.7$, $d = 0.6$.

Therefore, our future studies will focus on exploring types of utility function including the exponential one and study its impact on dynamic economic games.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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