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Research Article

Dynamic Consistent NSFD Scheme for a Delayed Viral Infection Model with Immune Response and Nonlinear Incidence

Jinhu Xu[1] and Yan Geng[2]

[1] *Department of Applied Mathematics, Xi'an University of Technology, Xi'an 710048, China*
[2] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China*

Correspondence should be addressed to Jinhu Xu; xujinhu09@163.com

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In this paper, a discrete-time model has been proposed by applying nonstandard finite difference (NSFD) scheme to solve a delayed viral infection model with immune response and general nonlinear incidence. It is shown that the discrete model has equilibria which are exactly the same as those of the original continuous model. Using discrete-time analogue of Lyapunov functionals, the global asymptotic stability of the equilibria of the discrete model is fully determined by the basic reproduction number of the virus and immune response, $\Re_0$ and $\Re_1$, with no restriction on the time step size, which implies that the NSFD scheme preserves the qualitative dynamics of the corresponding continuous model.

1. Introduction

Since samples cannot always be taken frequently from patients, or detection techniques of the virus may not be accurate, testing specific hypotheses based on clinical data is a challengeable task, which justifies the critical role played by mathematical models in describing the dynamics inside the host of various infectious diseases such as HBV, HCV, and HIV. Over the past several decades, many models for studying infectious disease have been proposed and studied. The classical model for within host virus dynamics is a system which includes three ordinary differential equations [1, 2]. However, to take some features into consideration of a real system, like time delay, age structure, and so on, many literatures have been proposed and studied (see [3–8] and references therein). For example, Manna [8] considered a delayed HBV infection model with HBV DNA-containing capsids which takes the following form:

$$\begin{aligned} H' &= s - \mu H(t) - kH(t)V(t), \\ I' &= kH(t-\tau_1)V(t-\tau_1) - \delta I(t) - pI(t)Z(t), \\ D' &= aI(t-\tau_2) - (\beta+\delta)D(t), \\ V' &= \beta D(t) - cV(t), \\ Z' &= qI(t)Z(t) - \sigma Z(t), \end{aligned} \tag{1}$$

where $H(t)$, $I(t)$, $D(t)$, $V(t)$, and $Z(t)$ denote the densities of the uninfected hepatocytes, infected hepatocytes, intracellular HBV DNA-containing capsids, the virions, and CTL cells at time t, respectively. The hepatocytes are assumed to be produced from a source at rate s, have a natural death rate μ, and get infected by the virions at a rate k. δ is the death rate of infected hepatocytes and capsids. a represents the rate of production of HBV DNA-containing capsids. Capsids lead to viral replication at the rate β, and c is the nature death rate of the virions. p is the rate the infected hepatocytes are removed by CTLs while q accounts for the CTL responsiveness and σ represents decay rate for CTLs in absence of stimulation. τ_1 denotes the time needed in the production of productively infected hepatocytes from the uninfected ones; τ_2 means the time spent in the production of matured intracellular HBV DNA-containing capsids which in turn contributes to the production of virions. The global asymptotic stability of the equilibria of model (1) has been investigated in [8] by constructing Lyapunov functionals.

Note that the bilinear incidence rate is a simple description of the infection in model (1). However, as mentioned in [9], a general incidence rate may help us to gain the unification theory by the omission of unessential details. For more details about nonlinear incidence rates, we refer to see [10–12] and references cited in. Hence, inspired by the aforementioned literatures, we consider the following delayed model with general nonlinear incidence:

$$\begin{aligned} H' &= s - \mu H(t) - kH(t) f(V(t)), \\ I' &= kH(t-\tau_1) f(V(t-\tau_1)) - \delta I(t) - pI(t) Z(t), \\ D' &= aI(t-\tau_2) - (\beta+\delta) D(t), \\ V' &= \beta D(t) - cV(t), \\ Z' &= qI(t) Z(t) - \sigma Z(t). \end{aligned} \tag{2}$$

Here, the incidence is assumed to be the nonlinear responses to the concentration of virions taking the form $kHf(V)$, where $f(V)$ denote the force of infection by virus particles and satisfy the following properties [13]:

$$\begin{aligned} f(0) &= 0, \\ f'(V) &> 0, \\ f''(V) &\le 0. \end{aligned} \tag{3}$$

Based on condition (3), it follows from the Mean Value Theorem that

$$f'(V) V \le f(V) \le f'(0) V. \tag{4}$$

Epidemiologically, condition (3) indicates that (i) the disease can not spread if there is no infection; (ii) the incidences $kHf(V)$ become faster as the densities of the virions increase; (iii) the per capita infection rates by virions will slow down due to certain inhibition effect since (4) implies that $(f(V)/V)' \le 0$.

Obviously, the incidence rate with condition (3) contains the bilinear and the saturation incidences.

The initial conditions for model (2) are

$$\begin{aligned} H(\theta) &= \psi_1(\theta), \\ I(\theta) &= \psi_2(\theta), \\ D(\theta) &= \psi_3(\theta), \\ V(\theta) &= \psi_4(\theta), \\ Z(\theta) &= \psi_5(\theta), \\ \psi_i(\theta) \ge 0,\ &\theta \in [-\tau, 0],\ \psi_i(0) > 0\ \ (i = 1,2,3,4,5), \end{aligned} \tag{5}$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $(\psi_1(\theta), \psi_2(\theta), \psi_3(\theta), \psi_4(\theta), \psi_5(\theta)) \in C([-\tau, 0], \mathbb{R}_+^5)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^5$ with $\mathbb{R}_+^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \ge 0,\ i = 1,2,3,4,5\}$.

In order to investigate the dynamics of the solutions for a model like (1), we need to get the exact solution for the model, which is one of the most important tasks of mathematical modelling. However, this is very difficult or even impossible to be determined. Hence, researchers seek numerical ones instead. However, how to select a proper discrete method so that the global properties of solutions of the corresponding continuous models can be efficiently preserved is still an open problem [14]. Recently, Mickens has made an attempt in this regard, by proposing a robust nonstandard finite difference (NSFD) scheme [15, 16], which has been widely employed in the study of different kinds of epidemic models and one important advantage of Mickens' method is that it can be more efficient in preserving the global dynamics to the corresponding continuous epidemic models [10, 17–23]. However, there is no result about discrete viral infection model with time delays and immune response. Therefore, motivated by [15, 16], we obtain from model (2)

$$\begin{aligned} \frac{H_{n+1} - H_n}{\Delta t} &= s - \mu H_{n+1} - kH_{n+1} f(V_n), \\ \frac{I_{n+1} - I_n}{\Delta t} &= kH_{n-m_1+1} f(V_{n-m_1}) - \delta I_{n+1} \\ &\quad - pI_{n+1} Z_{n+1}, \\ \frac{D_{n+1} - D_n}{\Delta t} &= aI_{n-m_2+1} - (\beta+\delta) D_{n+1}, \\ \frac{V_{n+1} - V_n}{\Delta t} &= \beta D_{n+1} - cV_{n+1}, \\ \frac{Z_{n+1} - Z_n}{\Delta t} &= qI_{n+1} Z_{n+1} - \sigma Z_{n+1}, \end{aligned} \tag{6}$$

where $\Delta t > 0$ is the time step size and $(H_n, I_n, D_n, V_n, Z_n)$ are the approximations of the solution $(H(t_n), I(t_n), D(t_n), V(t_n), Z_n)$ of model (2) at the discrete-time point $t_n = \Delta t n$ $(n \in \mathbb{N})$ and $m_1, m_2 \in \mathbb{N}$ are constant integers satisfying $\tau_1 = m_1 \Delta t$, $\tau_2 = m_2 \Delta t$.

The discrete initial condition of model (6) is given as

$$\begin{aligned} H_s &= \phi_s^{(1)} \ge 0, \\ I_s &= \phi_s^{(2)} \ge 0, \\ D_s &= \phi_s^{(3)} \ge 0, \\ V_s &= \phi_s^{(4)} \ge 0, \\ Z_s &= \phi_s^{(5)} \ge 0, \\ &\forall s = -m, -m+1, \ldots, 0, \end{aligned} \tag{7}$$

where $m = \max\{m_1, m_2\}$ and $\phi_0^{(i)} > 0$ $(i = 1,2,3,4,5)$.

In this paper, we will present an affirmative answer that the discrete model (6) which derived by utilizing Mickens' method can efficiently preserve the global properties to the original continuous model (2). The rest of this paper is organized as follows. We first study the global dynamics of the continuous system (2) in Section 2. In Section 3, we investigate the global dynamics of discrete system (6). A brief conclusion ends the paper.

2. Global Dynamics of Model (2)

2.1. Preliminaries. The following result established the positivity and boundedness of solutions of model (2).

Theorem 1. *Let $(H(t), I(t), D(t), V(t), Z(t))$ be the solution of model (2) satisfying conditions (5). Then $H(t), I(t), D(t), V(t)$ and $Z(t)$ are all nonnegative and bounded for all $t > 0$.*

Proof. First, we prove that $H(t) > 0$ for all $t \geq 0$. Assume the contrary and let $T_0 > 0$ such that $H(T_0) = 0$. Then from the first equation of system (2), we have $H'(t) = \lambda > 0$. Therefore, $H(t) < 0$ for $t \in (T_0 - \varepsilon, T_0)$ and $\varepsilon > 0$ is sufficiently small. This contradicts with the fact of $H(t) > 0$ for $t \in [0, T_0)$. It follows that $H(t) > 0$ for $t \geq 0$. Moreover, it follows from (2) that

$$\begin{aligned} I(t) &= I(0) e^{-\int_0^t (\delta + pZ(\theta)) d\theta} \\ &\quad + \int_0^t e^{-\int_\theta^t (\delta + pZ(\theta)) d\theta} kH(\theta - \tau_1) f(V(\theta - \tau_1)) d\theta, \\ D(t) &= D(0) e^{-(\beta+\delta)t} + \int_0^t aI(\theta - \tau_2) e^{(\beta+\delta)(\theta - t)} d\theta, \\ V(t) &= V(0) e^{-ct} + \int_0^t \beta D(\theta) e^{c(\theta - t)} d\theta, \\ Z(t) &= Z(0) e^{-\int_0^t (\sigma - qI(\theta)) d\theta}. \end{aligned} \tag{8}$$

Then the nonnegative immediately follows from the above integral forms and (5).

Next, we show the boundedness of the solution. Define

$$\begin{aligned} G(t) &= H(t) + I(t + \tau_1) + \frac{\delta}{2a} D(t + \tau_1 + \tau_2) \\ &\quad + \frac{\delta}{2a} V(t + \tau_1 + \tau_2) + \frac{p}{q} Z(t + \tau_1), \end{aligned} \tag{9}$$

and $d_0 = \min\{\mu, \delta/2, c, \sigma\}$. It then follows that

$$\begin{aligned} G'(t) &= s - \mu H(t) - \frac{\delta}{2} I(t + \tau_1) - \frac{\delta^2}{2} D(t + \tau_1 + \tau_2) \\ &\quad - \frac{\delta c}{2a} V(t + \tau_1 + \tau_2) - \frac{\sigma p}{q} Z(t + \tau_1) \\ &\leq s - d_0 G(t). \end{aligned} \tag{10}$$

This implies that $G(t)$ is bounded and so are $H(t), I(t), D(t), V(t)$, and $Z(t)$. This completes the proof. □

2.2. Steady States. Obviously, the model (2) always has an infection-free equilibrium $E_0 = (H_0, 0, 0, 0, 0)$ with $H_0 = s/\mu$. This is the only biologically meaningful equilibrium if

$$\mathfrak{R}_0 = \frac{aks\beta f'(0)}{\delta c \mu (\beta + \delta)} < 1. \tag{11}$$

At an equilibrium of model (2) we have

$$\begin{aligned} s &= \mu H + kHf(V), \\ kHf(V) &= \delta I + pIZ, \\ aI &= (\beta + \delta) D, \\ \beta D &= cV, \\ (qI - \sigma) Z &= 0. \end{aligned} \tag{12}$$

It is clear that model (2) only has the following two possible equilibria except the E_0, that is, $E_1 = (H_1, I_1, D_1, V_1, 0)$ and $E_2 = (H_2, I_2, D_2, V_2, Z_2)$, where $H_1, I_1, D_1, V_1, H_2, I_2, D_2, V_2, Z_2$ are all strictly positive.

If $Z = 0$, then the existence of $E_1 = (H_1, I_1, D_1, V_1, 0)$ is equivalent to the existence of positive solution (H_1, I_1, D_1, V_1) of the following equations:

$$\begin{aligned} s &= \mu H + kHf(V), \\ kHf(V) &= \delta I, \\ aI &= (\beta + \delta) D, \\ \beta D &= cV. \end{aligned} \tag{13}$$

From the last three equations of (13) we have

$$\begin{aligned} s - \mu H &= \frac{\delta c (\beta + \delta) V}{a\beta}, \\ I &= \frac{(\beta + \delta) cV}{a\beta}, \\ D &= \frac{cV}{\beta}. \end{aligned} \tag{14}$$

This means that, in order to have $H \geq 0$ and $V > 0$ at an equilibrium, we must have $V \in (0, a\beta s/\delta c(\beta + \delta)]$. Further, substituting I into the second equation of (13) gives

$$H = \frac{\delta c (\beta + \delta) V}{ak\beta f(V)}. \tag{15}$$

Then substituting H into the first equation of (13), direct calculation yields

$$0 = s - \frac{\mu \delta c (\beta + \delta) V}{a\beta k f(V)} - \frac{\delta c (\beta + \delta) V}{a\beta} =: F(V). \tag{16}$$

For $F(V)$ with $V > 0$, it then follows from (4) that

$$\begin{aligned} F'(V) &= -\frac{\delta c (\beta + \delta)}{a\beta} \\ &\quad - \frac{\mu \delta c (\beta + \delta)}{ak\beta} \left(f(V) - f'(V) V \right) < 0. \end{aligned} \tag{17}$$

Moreover, we obtain that

$$\begin{aligned} \lim_{V \to 0^+} F(V) &= \frac{\mu \delta c (\beta + \delta)}{a\beta k f'(0)} (\mathfrak{R}_0 - 1), \\ F\left(\frac{a\beta s}{\delta c (\beta + \delta)} \right) &= -\frac{\mu s}{kf(a\beta s/\delta c(\beta + \delta))} < 0. \end{aligned} \tag{18}$$

Therefore, there exists an infection equilibrium without immunity $E_1 = (H_1, I_1, D_1, V_1, 0)$ when $\mathfrak{R}_0 > 1$.

Define

$$\mathfrak{R}_1 = \frac{qksf\left(\beta a\sigma / qc\left(\beta+\delta\right)\right)}{\delta\sigma\left(\mu + kf\left(\beta a\sigma / qc\left(\beta+\delta\right)\right)\right)} \tag{19}$$

which represents the immune response activation number and determines whether or not a persistent immune response can be established. If $Z \neq 0$, then from (12) we have

$$\begin{aligned} H_2 &= \frac{s}{\mu + kf\left(V_2\right)}, \\ I_2 &= \frac{\sigma}{q}, \\ D_2 &= \frac{a\sigma}{q\left(\beta+\delta\right)}, \\ V_2 &= \frac{a\beta\sigma}{qc\left(\beta+\delta\right)}, \\ Z_2 &= \frac{\delta}{p}\left(\mathfrak{R}_1 - 1\right). \end{aligned} \tag{20}$$

Therefore, the infection equilibrium with immunity $E_2 = (H_2, I_2, D_2, V_2, Z_2)$ exists provided $\mathfrak{R}_1 > 0$. It follows from the properties of function $f(V)$ that $\mathfrak{R}_1 < \mathfrak{R}_0$. Concluding the above analysis we have the following result.

Theorem 2. *For model (2).*

(i) *If $\mathfrak{R}_0 < 1$, then there exists a unique infection-free equilibrium E_0;*

(ii) *If $\mathfrak{R}_1 \leq 1 < \mathfrak{R}_0$, then there exists a unique infection equilibrium without immunity E_1 besides E_0;*

(iii) *If $\mathfrak{R}_1 > 1$, then there exists a unique infection equilibrium with immunity E_2 besides E_0 and E_1.*

2.3. Global Dynamics of Model (2). In this part, we investigate the global asymptotic stability of the equilibria of system (2) by constructing Lyapunov functionals. To this end, we first introduce the following function for the following work, $\varphi(x) = x - 1 - \ln x$ for $x \in (0, +\infty)$, which will be used in Lyapunov functionals. It is easy to show that $\varphi(x) \geq \varphi(1) = 0$.

Theorem 3. *If $R_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable.*

Proof. We construct a Lyapunov functional as follows:

$$\begin{aligned} L_1 &= H_0\varphi\left(\frac{H}{H_0}\right) + I + \frac{\delta}{a}D + \frac{\delta\left(\beta+\delta\right)}{a\beta}V + \frac{p}{q}Z \\ &\quad + \int_{t-\tau_1}^{t} kH\left(\theta\right) f\left(V\left(\theta\right)\right) d\theta + \int_{t-\tau_2}^{t} I\left(\theta\right) d\theta. \end{aligned} \tag{21}$$

To proceed, we denote $u_\tau = u(t-\tau)$ $(u \in \{H, I, V\}$, $\tau \in \{\tau_1, \tau_2\})$, for the sake of convenience. Calculating dL_1/dt along the solutions of system (2) and applying $s = \mu H_0$, together with condition (4), yield

$$\begin{aligned} \frac{dL_1}{dt} &= \left(1 - \frac{H_0}{H}\right)\left(s - \mu H - kHf\left(V\right)\right) \\ &\quad + kH_{\tau_1} f\left(V_{\tau_1}\right) - \delta I - pIZ \\ &\quad + \frac{\delta}{a}\left(aI_{\tau_2} - \left(\beta+\delta\right)D\right) \\ &\quad + \frac{\delta\left(\beta+\delta\right)}{a\beta}\left(\beta D - cV\right) \\ &\quad + k\left(Hf\left(V\right) - H_{\tau_1} f\left(V_{\tau_1}\right)\right) + \delta\left(I - I_{\tau_2}\right) \\ &= \mu H_0\left(1 - \frac{H_0}{H}\right)\left(1 - \frac{H}{H_0}\right) + kH_0 f\left(V\right) \\ &\quad - \frac{\delta\left(\beta+\delta\right)cV}{a\beta} - \frac{p\sigma}{q}Z \\ &\leq \mu H_0\left(1 - \frac{H_0}{H}\right)\left(1 - \frac{H}{H_0}\right) \\ &\quad + \frac{\delta c\left(\beta+\delta\right)}{a\beta}\left(\mathfrak{R}_0 - 1\right)V - \frac{p\sigma}{q}Z. \end{aligned} \tag{22}$$

Therefore, if $\mathfrak{R}_0 \leq 1$, then $dL_1/dt \leq 0$. Furthermore, it can be shown that the largest invariant subset of $\{dL_1/dt = 0\}$ is the singleton $\{E_0\}$. Thus, the infection-free equilibrium E_0 is globally asymptotically stable. This completes the proof. □

To establish the global stability of the infection equilibrium without immunity E_1 when $\mathfrak{R}_1 \leq 1 < \mathfrak{R}_0$, we first give the following results.

Lemma 4. *Under condition (3), for $V_i > 0$, $i = 1, 2$, it holds that*

$$\left(f\left(V_1\right) - f\left(V_2\right)\right)\left(V_1 - V_2\right) > 0, \tag{23}$$

$$\left(\frac{f\left(V_2\right)}{V_2} - \frac{f\left(V_1\right)}{V_1}\right)\left(V_1 - V_2\right) > 0, \tag{24}$$

$$\left(\frac{f\left(V\right)}{f\left(V_i\right)} - \frac{V}{V_i}\right)\left(1 - \frac{f\left(V_i\right)}{f\left(V\right)}\right) \leq 0, \quad i = 1, 2. \tag{25}$$

This Lemma can be easily obtained from the properties of function $f(V)$. Hence, we omit the proof. Based on Lemma 4, we present the following Lemma.

Lemma 5. *Suppose the condition (3) is satisfied and $\mathfrak{R}_0 > 1$. Then H_i, I_i, D_i, V_i $(i = 1, 2)$ exist satisfying*

$$\begin{aligned} \operatorname{sgn}\left(H_2 - H_1\right) &= \operatorname{sgn}\left(I_1 - I_2\right) = \operatorname{sgn}\left(D_1 - D_2\right) \\ &= \operatorname{sgn}\left(V_1 - V_2\right) = \operatorname{sgn}\left(\mathfrak{R}_1 - 1\right). \end{aligned} \tag{26}$$

Proof. First, we claim that $\operatorname{sgn}(H_2 - H_1) = \operatorname{sgn}(I_1 - I_2)$. On the contrary, we have $\operatorname{sgn}(H_1 - H_2) = \operatorname{sgn}(I_1 - I_2)$. It follows from the conditions of the equilibria E_1 and E_2 that $\operatorname{sgn}(I_1 - I_2) = \operatorname{sgn}(D_1 - D_2) = \operatorname{sgn}(V_1 - V_2)$; furthermore,

$$\begin{aligned}\mu(H_2 - H_1) &= k(H_1 f(V_1) - H_2 f(V_2)) \\ &= k(H_1 - H_2) f(V_1) \\ &\quad + kH_2 (f(V_1) - f(V_2)).\end{aligned} \tag{27}$$

Thus, from (23) in Lemma 4 we get $\operatorname{sgn}(H_2 - H_1) = \operatorname{sign}(H_1 - H_2)$ which leads to a contradiction. Thus, $\operatorname{sign}(H_2 - H_1) = \operatorname{sgn}(I_1 - I_2)$.

Next, we will prove $\operatorname{sgn}(\mathfrak{R}_1 - 1) = \operatorname{sgn}(I_1 - I_2)$. To this end, let

$$\widetilde{\mathfrak{R}}_1 = \frac{kH_2 f(V_2)}{\delta I_2}; \tag{28}$$

then it is easy to see that $\operatorname{sgn}(\mathfrak{R}_1 - 1) = \operatorname{sgn}(\widetilde{\mathfrak{R}}_1 - 1)$. Thus, we just need to prove $\operatorname{sgn}(\widetilde{\mathfrak{R}}_1 - 1) = \operatorname{sgn}(I_1 - I_2)$. Use the equilibria conditions of E_1 and E_2, which gives

$$\begin{aligned}\widetilde{\mathfrak{R}} - 1 &= \frac{kH_2 f(V_2)}{\delta I_2} - \frac{kH_1 f(V_1)}{\delta I_1} \\ &= \frac{ak\beta}{\delta c(\beta+\delta)} \left[\frac{H_2 f(V_2)}{V_2} - \frac{H_1 f(V_1)}{V_1}\right] \\ &= \frac{ak\beta}{\delta c(\beta+\delta)} \left[(H_2 - H_1)\frac{f(V_2)}{V_2}\right. \\ &\quad \left. + H_1 \left(\frac{f(V_2)}{V_2} - \frac{f(V_1)}{V_1}\right)\right].\end{aligned} \tag{29}$$

Thus, it follows from (24) that $\operatorname{sgn}(\widetilde{\mathfrak{R}}_1 - 1) = \operatorname{sgn}(I_1 - I_2)$. This completes the proof. □

Theorem 6. *If $\mathfrak{R}_1 \le 1 < \mathfrak{R}_0$, then the infection equilibrium without immunity E_1 is globally asymptotically stable.*

Proof. Constructing a Lyapunov functional L_2 as follows

$$\begin{aligned}L_2 &= H_1 \varphi\left(\frac{H}{H_1}\right) + I_1 \varphi\left(\frac{I}{I_1}\right) + \frac{\delta}{a} D_1 \varphi\left(\frac{D}{D_1}\right) \\ &\quad + \frac{\delta(\beta+\delta)V_1}{a\beta}\varphi\left(\frac{V}{V_1}\right) + \frac{p}{q}Z \\ &\quad + kH_1 f(V_1)\int_{t-\tau_1}^{t} \varphi\left(\frac{H(\theta) f(V(\theta))}{H_1 f(V_1)}\right) d\theta \\ &\quad + \delta I_1 \int_{t-\tau_2}^{t} \varphi\left(\frac{I(\theta)}{I_1}\right) d\theta.\end{aligned} \tag{30}$$

Take the derivative of L_2 along solutions of model (2) and recall that the equilibrium conditions of E_1 are

$$\begin{aligned}s &= \mu H_1 + kH_1 f(V_1), \\ kH_1 f(V_1) = \delta I_1 &= \frac{\delta(\beta+\delta) D_1}{a} = \frac{\delta c(\beta+\delta) V_1}{a\beta}.\end{aligned} \tag{31}$$

Then, we obtain that

$$\begin{aligned}\frac{dL_2}{dt} &= \left(1 - \frac{H_1}{H}\right)(s - \mu H - kHf(V)) + \left(1 - \frac{I_1}{I}\right) \\ &\quad \cdot \left(kH_{\tau_1} f(V_{\tau_1}) - \delta I - pIZ\right) + \frac{\delta}{a}\left(1 - \frac{D_1}{D}\right)\left(aI_{\tau_2}\right. \\ &\quad \left. - (\beta+\delta) D\right) + \frac{\delta(\beta+\delta)}{a\beta}\left(1 - \frac{V_1}{V}\right)(\beta D - cV) \\ &\quad + \frac{p}{q}(qIZ - \sigma Z) + kH_1 f(V_1)\left(\frac{Hf(V)}{H_1 f(V_1)}\right. \\ &\quad \left. - \frac{H_{\tau_1} f(V_{\tau_1})}{H_1 f(V_1)} + \ln\frac{H_{\tau_1} f(V_{\tau_1})}{Hf(V)}\right) + \delta I_1 \left(\frac{I}{I_1} - \frac{I_{\tau_2}}{I_1}\right. \\ &\quad \left. + \ln\frac{I_{\tau_2}}{I}\right) = \mu H_1 \left(1 - \frac{H_1}{H}\right)\left(1 - \frac{H}{H_1}\right) + \left(1\right. \\ &\quad \left. - \frac{H_1}{H}\right)(kH_1 f(V_1) - kHf(V)) + \left(1 - \frac{I_1}{I}\right) \\ &\quad \cdot \left(kH_{\tau_1} f(V_{\tau_1}) - \delta I_1 \frac{I}{I_1}\right) + \frac{\delta}{a}\left(1 - \frac{D_1}{D}\right)\left(aI_1 \frac{I_{\tau_2}}{I_1}\right. \\ &\quad \left. - \frac{(\beta+\delta)\delta D_1}{a}\frac{D}{D_1}\right) + \frac{\delta(\beta+\delta)}{a\beta}\left(1 - \frac{V_1}{V}\right) \\ &\quad \cdot \left(\beta D_1 \frac{D}{D_1} - cV_1 \frac{V}{V_1}\right) + pZ\left(I_1 - \frac{\sigma}{q}\right) \\ &\quad + kH_1 f(V_1)\left(\frac{Hf(V)}{H_1 f(V_1)} - \frac{H_{\tau_1} f(V_{\tau_1})}{H_1 f(V_1)}\right. \\ &\quad \left. + \ln\frac{H_{\tau_1} f(V_{\tau_1})}{Hf(V)}\right) + \delta I_1 \left(\frac{I}{I_1} - \frac{I_{\tau_2}}{I_1} + \ln\frac{I_{\tau_2}}{I}\right) \\ &= \mu H_1 \left(1 - \frac{H_1}{H}\right)\left(1 - \frac{H}{H_1}\right) + kH_1 f(V_1)\left[4 - \frac{H_1}{H}\right. \\ &\quad - \frac{H_{\tau_1} f(V_{\tau_1}) I_1}{H_1 f(V_1) I} - \frac{I_{\tau_2} D_1}{ID} - \frac{DV_1}{D_1 V} + \frac{f(V)}{f(V_1)} - \frac{V}{V_1} \\ &\quad \left. + \ln\frac{H_{\tau_1} f(V_{\tau_1}) I_{\tau_2}}{Hf(V) I}\right] + pZ(I_1 - I_2) = \mu H_1 \left(1\right. \\ &\quad \left. - \frac{H_1}{H}\right)\left(1 - \frac{H}{H_1}\right) + kH_1 f(V_1)\left[-\varphi\left(\frac{H_1}{H}\right)\right. \\ &\quad - \varphi\left(\frac{H_{\tau_1} f(V_{\tau_1}) I_1}{H_1 f(V_1) I}\right) - \varphi\left(\frac{I_{\tau_2} D_1}{ID}\right) - \varphi\left(\frac{DV_1}{D_1 V}\right) \\ &\quad \left. + \frac{f(V)}{f(V_1)} - \frac{V}{V_1} + \ln\frac{f(V_1) V}{f(V) V_1}\right] + pZ(I_1 - I_2) \\ &= \mu H_1 \left(1 - \frac{H_1}{H}\right)\left(1 - \frac{H}{H_1}\right) + kH_1 f(V_1)\end{aligned}$$

$$\cdot\left[-\varphi\left(\frac{H_1}{H}\right)-\varphi\left(\frac{H_{\tau_1}f\left(V_{\tau_1}\right)I_1}{H_1f\left(V_1\right)I}\right)-\varphi\left(\frac{I_{\tau_2}D_1}{ID}\right)\right.$$
$$-\varphi\left(\frac{DV_1}{D_1V}\right)-\varphi\left(\frac{f\left(V_1\right)V}{f\left(V\right)V_1}\right)$$
$$\left.+\left(1-\frac{f\left(V_1\right)}{f\left(V\right)}\right)\left(\frac{f\left(V\right)}{f\left(V_1\right)}-\frac{V}{V_1}\right)\right]+pZ\left(I_1-I_2\right). \quad (32)$$

From Lemma 5 we can obtain that $\mathfrak{R}_1 \leq 1$ is equivalent to $I_1 - I_2 \leq 0$. Combined with (25) in Lemma 4, it follows that $dL_2/dt \leq 0$ for all $H, I, D, V, Z > 0$. Furthermore, it can be shown that the largest invariant subset of $\{dL_2/dt = 0\}$ is the singleton $\{E_1\}$. Hence, the global asymptotic stability of the infection equilibrium without immunity E_1 follows from LaSalle's invariant principle. This completes the proof. □

Theorem 7. *If $\mathfrak{R}_1 > 1$, then the infection equilibrium with immunity E_2 is globally asymptotically stable.*

Proof. Constructing a Lyapunov functional L_3 as follows:

$$L_3 = H_2\varphi\left(\frac{H}{H_2}\right)+I_2\varphi\left(\frac{I}{I_2}\right)+\frac{\left(\delta+pZ_2\right)}{a}D_2\varphi\left(\frac{D}{D_2}\right)+\frac{\left(\delta+pZ_2\right)\left(\beta+\delta\right)V_2}{a\beta}\varphi\left(\frac{V}{V_2}\right)+\frac{p}{q}\varphi\left(\frac{Z}{Z_2}\right)+kH_2f\left(V_2\right)\int_{t-\tau_1}^{t}\varphi\left(\frac{H\left(\theta\right)f\left(V\left(\theta\right)\right)}{H_2f\left(V_2\right)}\right)d\theta+\left(\delta+pZ_2\right)I_2\int_{t-\tau_2}^{t}\varphi\left(\frac{I\left(\theta\right)}{I_2}\right)d\theta. \quad (33)$$

Take the derivative of L_3 along solutions of model (2) and recall that the equilibrium conditions of E_2 are

$$s = \left(\mu+kf\left(V_2\right)\right)H_2,$$
$$I_2 = \frac{\sigma}{q},$$
$$kH_2f\left(V_2\right) = \left(\delta+pZ_2\right)I_2 = \frac{\left(\delta+pZ_2\right)\left(\beta+\delta\right)D_2}{a} = \frac{\left(\delta+pZ_2\right)\left(\beta+\delta\right)cV_2}{a\beta}. \quad (34)$$

Then, we obtain that

$$\frac{dL_3}{dt} = \left(1-\frac{H_2}{H}\right)\left(s-\mu H-kHf\left(V\right)\right)+\left(1-\frac{I_2}{I}\right)\cdot\left(kH_{\tau_1}f\left(V_{\tau_1}\right)-\delta I-pIZ\right)+\frac{\left(\delta+pZ_2\right)}{a}\left(1-\frac{D_2}{D}\right)\left(aI_{\tau_2}-\left(\beta+\delta\right)D\right)+\frac{\left(\delta+pZ_2\right)\left(\beta+\delta\right)}{a\beta}\left(1-\frac{V_2}{V}\right)\left(\beta D-cV\right)+\frac{p}{q}\left(1-\frac{Z_2}{Z}\right)\left(qIZ-\sigma Z\right)$$
$$+kH_2f\left(V_2\right)\left(\frac{Hf\left(V\right)}{H_2f\left(V_2\right)}-\frac{H_{\tau_1}f\left(V_{\tau_1}\right)}{H_2f\left(V_2\right)}+\ln\frac{H_{\tau_1}f\left(V_{\tau_1}\right)}{Hf\left(V\right)}\right)+\left(\delta+pZ_2\right)I_2\left(\frac{I}{I_2}-\frac{I_{\tau_2}}{I_2}+\ln\frac{I_{\tau_2}}{I}\right)=\mu H_2\left(1-\frac{H_2}{H}\right)\left(1-\frac{H}{H_2}\right)+\left(1-\frac{H_2}{H}\right)\left(kH_2f\left(V_2\right)-kHf\left(V\right)\right)+\left(1-\frac{I_2}{I}\right)\cdot\left(kH_{\tau_1}f\left(V_{\tau_1}\right)-kH_2f\left(V_2\right)\frac{I}{I_2}+pIZ_2-pIZ\right)$$
$$+\frac{\left(\delta+pZ_2\right)}{a}\left(1-\frac{D_2}{D}\right)\left(aI_2\frac{I_{\tau_2}}{I_2}-\frac{\left(\beta+\delta\right)\delta D_2}{a}\frac{D}{D_2}\right)+\frac{\left(\delta+pZ_2\right)\left(\beta+\delta\right)}{a\beta}\left(1-\frac{V_2}{V}\right)\left(\beta D_1\frac{D}{D_1}-cV_1\frac{V}{V_1}\right)+\frac{p}{q}\left(1-\frac{Z_2}{Z}\right)\left(qIZ-\sigma Z\right)+kH_2f\left(V_2\right)\left(\frac{Hf\left(V\right)}{H_2f\left(V_2\right)}-\frac{H_{\tau_1}f\left(V_{\tau_1}\right)}{H_2f\left(V_2\right)}+\ln\frac{H_{\tau_1}f\left(V_{\tau_1}\right)}{Hf\left(V\right)}\right)+\left(\delta+pZ_2\right)I_2\left(\frac{I}{I_2}-\frac{I_{\tau_2}}{I_2}+\ln\frac{I_{\tau_2}}{I}\right)$$
$$=\mu H_2\left(1-\frac{H_2}{H}\right)\left(1-\frac{H}{H_2}\right)+kH_2f\left(V_2\right)\left[4-\frac{H_2}{H}-\frac{H_{\tau_1}f\left(V_{\tau_1}\right)I_2}{H_2f\left(V_2\right)I}-\frac{I_{\tau_2}D_2}{ID}-\frac{DV_2}{D_2V}+\frac{f\left(V\right)}{f\left(V_2\right)}-\frac{V}{V_2}+\ln\frac{H_{\tau_1}f\left(V_{\tau_1}\right)I_{\tau_2}}{Hf\left(V\right)I}\right]$$
$$=\mu H_2\left(1-\frac{H_2}{H}\right)\left(1-\frac{H}{H_2}\right)+kH_2f\left(V_2\right)\cdot\left[-\varphi\left(\frac{H_2}{H}\right)-\varphi\left(\frac{H_{\tau_1}f\left(V_{\tau_1}\right)I_2}{H_2f\left(V_2\right)I}\right)-\varphi\left(\frac{I_{\tau_2}D_2}{ID}\right)-\varphi\left(\frac{DV_2}{D_2V}\right)+\frac{f\left(V\right)}{f\left(V_2\right)}-\frac{V}{V_2}+\ln\frac{f\left(V_2\right)V}{f\left(V\right)V_2}\right]$$
$$=\mu H_2\left(1-\frac{H_2}{H}\right)\left(1-\frac{H}{H_2}\right)+kH_2f\left(V_2\right)$$

$$\cdot\left[-\varphi\left(\frac{H_2}{H}\right)-\varphi\left(\frac{H_{\tau_1}f\left(V_{\tau_1}\right)I_2}{H_2f\left(V_2\right)I}\right)-\varphi\left(\frac{I_{\tau_2}D_2}{ID}\right)\right.$$
$$-\varphi\left(\frac{DV_2}{D_2V}\right)-\varphi\left(\frac{f\left(V_2\right)V}{f\left(V\right)V_2}\right)$$
$$\left.+\left(1-\frac{f\left(V_2\right)}{f\left(V\right)}\right)\left(\frac{f\left(V\right)}{f\left(V_2\right)}-\frac{V}{V_2}\right)\right]. \tag{35}$$

Similar to the proof of Theorem 6, we have $dL_3/dt \le 0$ for all $H, I, D, V, Z > 0$. Furthermore, it can be shown that the largest invariant subset of $\{dL_3/dt = 0\}$ is the singleton $\{E_2\}$. Hence, the global asymptotic stability of the infection equilibrium with immunity E_2 follows from LaSalle's invariant principle. This completes the proof. □

3. Global Dynamics of the Discrete Model (6)

In this section, we will show that the discrete model (6) can efficiently preserve the global asymptotic stability of the equilibria for corresponding continuous model (2).

It is easy to validate that model (6) has the same equilibria as model (2). We also denote the equilibria as $E_0 = (x_0, 0, 0, 0)$ and $E_1 = (H_1, I_1, D_1, V_1, 0)$ and $E_2 = (H_2, I_2, D_2, V_2, Z_2)$.

The following equations can be easily obtained by rearranging the formulations in equations of (6):

$$\begin{aligned} H_{n+1} &= \frac{s\Delta t + H_n}{1+\Delta t\left(\mu + kf\left(V_n\right)\right)}, \\ I_{n+1} &= \frac{I_n + \Delta t k H_{n-m_1+1} f\left(V_{n-m_1}\right)}{1+\Delta t\left(\delta + pZ_{n+1}\right)}, \\ D_{n+1} &= \frac{D_n + \Delta t a I_{n-m_2+1}}{1+\Delta t\left(\beta+\delta\right)}, \\ V_{n+1} &= \frac{V_n + \Delta t \beta D_{n+1}}{1+\Delta t c}, \\ Z_{n+1} &= \frac{Z_n}{1+\Delta t\left(\sigma - qI_{n+1}\right)}. \end{aligned} \tag{36}$$

Lemma 8. *The solution $(H_n, I_n, D_n, V_n, Z_n)$ of system (6) subject to condition (7) exists uniquely and is positive and bounded for all $n \in \mathbb{N}$. In addition, $0 < I_n < (1+\Delta t\sigma)/\Delta t q$ for $n = 1, 2, \ldots$.*

Proof. We first validate that $(H_1, I_1, D_1, V_1, Z_1)$ exists uniquely and is positive. According to the first equation (36), we know that $H_1 > 0$. Next, we consider Z_1. It follows from the second and fifth equations of (36) that

$$Z_1 = Z_0 + \Delta t\left[q\frac{I_0 + \Delta t k H_{-m_1+1} f\left(V_{-m_1}\right)}{1+\Delta t\left(\delta + pZ_1\right)}Z_1 - \sigma Z_1\right]. \tag{37}$$

Define

$$\begin{aligned} h\left(Z_1\right) &= \Delta t p\left(1+\Delta t\sigma\right)Z_1^2 + \left[\left(1+\Delta t\delta\right)\left(1+\Delta t\sigma\right)\right. \\ &\quad \left.- \Delta t p Z_0 - \Delta t q\left(I_0 + \Delta t k H_{-m_1+1} f\left(V_{-m_1}\right)\right)\right]Z_1 \\ &\quad - \left(1+\Delta t\delta\right)Z_0, \end{aligned} \tag{38}$$

which is a quadratic function. Note that $h(0) = -(1+\Delta t\delta)Z_0 < 0$ and $\lim_{Z_1\to\infty}h(Z_1) = \infty$; there is a unique $Z_1 > 0$ such that $h(Z_1) = 0$. That is, (37) holds.

Furthermore, we consider I_1. Combining with the second and last equation of (36), we have

$$\begin{aligned} I_1 = I_0 + \Delta t &\left[kH_{-m_1+1} f\left(V_{-m_1}\right) - \delta I_1\right. \\ &\left. - pI_1\frac{Z_0}{1+\Delta t\left(\sigma - qI_1\right)}\right]. \end{aligned} \tag{39}$$

Define

$$\begin{aligned} h\left(I_1\right) &= \Delta t q\left(1+\Delta t\delta\right)I_1^2 - \left[\left(1+\Delta t\delta\right)\left(1+\Delta t\sigma\right)\right. \\ &\quad \left.+ \Delta t p Z_0 + \Delta t q\left(I_0 + \Delta t k H_{-m_1+1} f\left(V_{-m_1}\right)\right)\right]I_1 \\ &\quad + \left(1+\Delta t\sigma\right)\left[I_0 + \Delta t k H_{-m_1+1} f\left(V_{-m_1}\right)\right]. \end{aligned} \tag{40}$$

Since $Z_1 > 0$, it follows from the last equation of system (36) that $I_1 < (1+\Delta t\sigma)/\Delta t q$. Then we have

$$\begin{aligned} h(0) &= \left(1+\Delta t\sigma\right)\left[I_0 + \Delta t k H_{-m_1+1} f\left(V_{-m_1}\right)\right] \\ &> 0, \\ h\left(\frac{1+\Delta t\sigma}{\Delta t q}\right) &= -\frac{pZ_0\left(1+\Delta t\sigma\right)}{q} < 0. \end{aligned} \tag{41}$$

Due to $h(I_1)$ being a quadratic function, there exists a unique $I_1 \in (0, (1+\Delta t\sigma)/\Delta t q)$ such that $h(I_1) = 0$. Therefore, (39) holds. Finally, we consider D_1 and V_1. It follows from the third and fourth equations of (36) that $D_1 = (D_0 + \Delta t a I_{-m_2+1})/(1+\Delta t(\beta+\sigma))$, $V_1 = (V_0 + \Delta t\beta D_1)/(1+\Delta t c)$. Thus, D_1 and V_1 uniquely exist and are positive. Therefore, $(H_1, I_{,} D_1, V_1, Z_1)$ exists uniquely and is positive.

For $n = 1$, repeat the above process; we can show that $(H_2, I_2, D_2, V_2, Z_2)$ exists uniquely and is positive. Owing to $Z_2 > 0$, we also have $I_2 < (1+\Delta t\sigma)/\Delta t q$. Therefore, by using the mathematical induction, for all $n \ge 0$, we know that $(H_n, I_n, D_n, V_n, Z_n)$ exists uniquely and is positive with $I_n < (1+\Delta t\sigma)/\Delta t q$. This completes the proof.

In order to prove the boundedness of solutions, we define

$$\begin{aligned} U_n = H_n + I_{n+m_1} &+ \frac{\delta}{2a}\left(D_{n+m_1+m_2} + V_{n+m_1+m_2}\right) \\ &+ \frac{p}{q}Z_{n+m_1}. \end{aligned} \tag{42}$$

Then we have

$$U_{n+1}-U_n=\Delta t\left(s-\mu H_{n+1}-\frac{\delta}{2a}I_{n+m_1+1}-\frac{\delta^2}{2a}D_{n+m_1+m_2+1}-\frac{\delta c}{2a}V_{n+m_1+m_2+1}-\frac{p\sigma}{q}Z_{n+m_1}\right)\le \Delta t\left(s-d_0U_{n+1}\right), \tag{43}$$

where $d_0=\min\{\mu,\delta/2,c,\sigma\}$. Thus, we obtain

$$U_{n+1}\le \frac{1}{1+\Delta t d_0}U_n+\frac{\Delta t s}{1+\Delta t d_0}. \tag{44}$$

The following inequality can be easily reduced from the induction

$$U_n\le\left(\frac{1}{1+\Delta t d_0}\right)^n U_0+\frac{s}{d_0}\left[1-\left(\frac{1}{1+\Delta d_0}\right)^n\right], \tag{45}$$

which implies that

$$\limsup_{n\to\infty} U_n\le\frac{s}{d_0}. \tag{46}$$

This means that $\{U_n\}$ is bounded. Therefore, $\{H_n\},\{I_n\}$, $\{D_n\},\{V_n\}$ and $\{Z_n\}$ are bounded. This completes the proof. □

3.1. Global Stability of Equilibria

Theorem 9. *If $\mathfrak{R}_0\le 1$, then the infection-free equilibrium E_0 of system (6) is globally asymptotically stable.*

Proof. Define a discrete Lyapunov function

$$W_n^{(1)}=\frac{1}{\Delta t}\left[H_0\varphi\left(\frac{H_n}{H_0}\right)+I_n+\frac{\delta}{a}D_n+\frac{\delta(\beta+\delta)}{a\beta}(1+\Delta tc)V_n+\frac{p}{q}(1+\Delta t\sigma)Z_n\right]+\sum_{j=n-m_1}^{n-1}kH_{j+1}f\left(V_j\right)+\delta\sum_{j=n-m_2}^{n-1}I_{j+1}. \tag{47}$$

Recall that $s=\mu H_0$; then we have

$$\begin{aligned}W_{n+1}^{(1)}-W_n^{(1)}&=\frac{1}{\Delta t}\left[H_{n+1}-H_n+H_0\ln\frac{H_n}{H_{n+1}}+I_{n+1}-I_n+\frac{\delta}{a}(D_{n+1}-D_n)+\frac{\delta(\beta+\delta)}{a\beta}(1+\Delta tc)(V_{n+1}-V_n)\right.\\&\left.+\frac{p}{q}(1+\Delta t\sigma)(Z_{n+1}-Z_n)\right]+kH_{n+1}f(V_n)-kH_{n-m_1+1}f\left(V_{n-m_1}\right)+\delta I_{n+1}-\delta I_{n-m_2+1}\\&\le\frac{1}{\Delta t}\left[\left(1-\frac{H_0}{H_{n+1}}\right)(H_{n+1}-H_n)+I_{n+1}-I_n+\frac{\delta}{a}(D_{n+1}-D_n)+\frac{\delta(\beta+\delta)}{a\beta}(1+\Delta tc)(V_{n+1}-V_n)\right.\\&\left.+\frac{p}{q}(1+\Delta t\sigma)(Z_{n+1}-Z_n)\right]+kH_{n+1}f(V_n)-kH_{n-m_1+1}f\left(V_{n-m_1}\right)+\delta I_{n+1}-\delta I_{n-m_2+1}\\&=\left(1-\frac{H_0}{H_{n+1}}\right)\left(\mu H_0-\mu H_{n+1}-kH_{n+1}f(V_n)\right)+kH_{n-m_1+1}f\left(V_{n-m_1}\right)-\delta I_{n+1}+\frac{\delta}{a}\left(aI_{n-m_2+1}-(\beta+\delta)D_{n+1}\right)\\&+\frac{\delta(\beta+\delta)}{a\beta}(\beta D_{n+1}-cV_{n+1})+\frac{\delta c(\beta+\delta)}{a\beta}(V_{n+1}-V_n)+\frac{p}{q}(qI_{n+1}Z_{n+1}-\sigma Z_{n+1})+\frac{p\sigma}{q}(Z_{n+1}-Z_n)+kH_{n+1}f(V_n)\\&-kH_{n-m_1+1}f\left(V_{n-m_1}\right)+\delta I_{n+1}-\delta I_{n-m_2+1}\\&=\mu H_0\left(1-\frac{H_0}{H_{n+1}}\right)\left(1-\frac{H_{n+1}}{H_0}\right)+kH_0f(V_n)-\frac{\delta c(\beta+\delta)}{a\beta}V_n-\frac{p\sigma}{q}Z_n\le\mu H_0\left(1-\frac{H_0}{H_{n+1}}\right)\left(1-\frac{H_{n+1}}{H_0}\right)\\&+\frac{\delta c(\beta+\delta)}{a\beta}V_n(\mathfrak{R}_0-1)-\frac{p\sigma}{q}Z_n.\end{aligned} \tag{48}$$

It then follows that if $\mathfrak{R}_0\le 1$, $W_{n+1}^{(1)}-W_n^{(1)}\le 0$, for all $n\in\mathbb{N}$. Thus, $W_n^{(1)}$ is monotone decreasing sequence. Due to $W_n^{(1)}\ge 0$, there is a limit $\lim_{n\to\infty}W_n^{(1)}\ge 0$ which implies that $\lim_{n\to\infty}(W_{n+1}^{(1)}-W_n^{(1)})=0$. Moreover, it can be shown that $\lim_{n\to\infty}H_n=H_0$, $\lim_{n\to\infty}I_n=0$, $\lim_{n\to\infty}D_n=0$, $\lim_{n\to\infty}V_n=0$, $\lim_{n\to\infty}Z_n=0$. Hence, E_0 is globally asymptotically. This completes the proof. □

Theorem 10. *If $\mathfrak{R}_1\le 1<\mathfrak{R}_0$, then the infection equilibrium without immunity E_1 of system (6) is globally asymptotically stable.*

Proof. Define a discrete Lyapunov functional as follows:

$$W_n^{(2)} = \frac{1}{\Delta t}\left[H_1\varphi\left(\frac{H_n}{H_1}\right) + I_1\varphi\left(\frac{I_n}{I_1}\right) + \frac{\delta}{a}D_1\varphi\left(\frac{D_n}{D_1}\right) + \frac{\delta(\beta+\delta)}{a\beta}V_1\varphi\left(\frac{V_n}{V_1}\right) + \frac{p}{q}Z_n\right] + kH_1 f(V_1) \cdot \sum_{j=n-m_1}^{n-1}\varphi\left(\frac{H_{j+1}f(V_j)}{H_1 f(V_1)}\right) + \delta I_1 \sum_{j=n-m_2}^{n-1}\varphi\left(\frac{I_{j+1}}{I_1}\right). \quad (49)$$

Combine with the model (6) and the equilibrium conditions (31) for E_1. Then we get the difference of $W_n^{(2)}$

$$\begin{aligned}
W_{n+1}^{(2)} - W_n^{(2)} &= \frac{1}{\Delta t}\left[H_{n+1} - H_n + H_1 \ln\frac{H_n}{H_{n+1}} + I_{n+1} - I_n + I_1 \ln\frac{I_n}{I_{n+1}} + \frac{\delta}{a}\left(D_{n+1} - D_n + D_1 \ln\frac{D_n}{D_{n+1}}\right)\right.\\
&\quad + \frac{\delta(\beta+\delta)}{a\beta}\left(V_{n+1} - V_n + V_1 \ln\frac{V_n}{V_{n+1}}\right) \left. + \frac{p}{q}(Z_{n+1} - Z_n)\right] + kH_1 f(V_1)\\
&\quad \cdot\left(\sum_{j=n-m_1+1}^{n}\varphi\left(\frac{H_{j+1}f(V_j)}{H_1 f(V_1)}\right) - \sum_{j=n-m_1}^{n-1}\varphi\left(\frac{H_{j+1}f(V_j)}{H_1 f(V_1)}\right)\right)\\
&\quad + \delta I_1\left(\sum_{j=n-m_2+1}^{n}\varphi\left(\frac{I_{j+1}}{I_1}\right) - \sum_{j=n-m_2}^{n-1}\varphi\left(\frac{I_{j+1}}{I_1}\right)\right) + kH_1 f(V_1)\left(\varphi\left(\frac{f(V_{n+1})}{f(V_1)}\right) - \varphi\left(\frac{f(V_n)}{f(V_1)}\right)\right)\\
&\le \frac{1}{\Delta t}\left[\left(1 - \frac{H_1}{H_{n+1}}\right)(H_{n+1} - H_n) + \left(1 - \frac{I_1}{I_{n+1}}\right)(I_{n+1} - I_n) + \frac{\delta}{a}\left(1 - \frac{D_1}{D_{n+1}}\right) \times (D_{n+1} - D_n)\right.\\
&\quad \left. + \frac{\delta(\beta+\delta)}{a\beta}\left(1 - \frac{V_1}{V_{n+1}}\right)(V_{n+1} - V_n) + \frac{p}{q}(Z_{n+1} - Z_n)\right] + kH_1 f(V_1)\\
&\quad \cdot\left(\sum_{j=n-m_1+1}^{n}\varphi\left(\frac{H_{j+1}f(V_j)}{H_1 f(V_1)}\right) - \sum_{j=n-m_1}^{n-1}\varphi\left(\frac{H_{j+1}f(V_j)}{H_1 f(V_1)}\right)\right)\\
&\quad + \delta I_1\left(\sum_{j=n-m_2+1}^{n}\varphi\left(\frac{I_{j+1}}{I_1}\right) - \sum_{j=n-m_2}^{n-1}\varphi\left(\frac{I_{j+1}}{I_1}\right)\right) + kH_1 f(V_1)\left(\varphi\left(\frac{f(V_{n+1})}{f(V_1)}\right) - \varphi\left(\frac{f(V_n)}{f(V_1)}\right)\right)\\
&= \left(1 - \frac{H_1}{H_{n+1}}\right)(s - \mu H_{n+1} - kH_{n+1} f(V_n)) + \left(1 - \frac{I_1}{I_{n+1}}\right)\left(kH_{n-m_1+1} f(V_{n-m_1}) - \delta I_{n+1} - pI_{n+1}Z_{n+1}\right)\\
&\quad + \frac{\delta}{a}\left(1 - \frac{D_1}{D_{n+1}}\right)\left(aI_{n-m_2+1} - (\beta+\delta)D_{n+1}\right) + \frac{\delta(\beta+\delta)}{a\beta}\left(1 - \frac{V_1}{V_{n+1}}\right) \times (\beta D_{n+1} - cV_{n+1}) + \frac{p}{q}(Z_{n+1} - Z_n) + kH_1 f(V_1)\\
&\quad \cdot\left[\varphi\left(\frac{H_{n+1}f(V_n)}{H_1 f(V_1)}\right) - \varphi\left(\frac{H_{n-m_1+1}f(V_{n-m_1})}{H_1 f(V_1)}\right)\right] + \delta I_1\left(\varphi\left(\frac{I_{n+1}}{I_1}\right) - \varphi\left(\frac{I_{n-m_2+1}}{I_1}\right)\right) + kH_1 f(V_1)\\
&\quad \cdot\left(\varphi\left(\frac{f(V_{n+1})}{f(V_1)}\right) - \varphi\left(\frac{f(V_n)}{f(V_1)}\right)\right) = \mu H_1\left(1 - \frac{H_1}{H_{n+1}}\right)\left(1 - \frac{H_{n+1}}{H_1}\right) + \left(1 - \frac{H_1}{H_{n+1}}\right)(kH_1 f(V_1)\\
&\quad - kH_{n+1} f(V_n)) + \left(1 - \frac{I_1}{I_{n+1}}\right) \cdot\left(kH_{n-m_1+1} f(V_{n-m_1}) - \delta I_1 \frac{I_{n+1}}{I_1}\right) + \frac{\delta}{a}\left(1 - \frac{D_1}{D_{n+1}}\right) \times \left(aI_1\frac{I_{n-m_2+1}}{I_1} - (\beta+\delta)D_1\frac{D_{n+1}}{D_1}\right)\\
&\quad + \frac{\delta(\beta+\delta)}{a\beta}\left(1 - \frac{V_1}{V_{n+1}}\right) \times \left(\beta D_1 \frac{D_{n+1}}{D_1} - cV_1\frac{V_{n+1}}{V_1}\right) + kH_1 f(V_1)\left[\frac{H_{n+1}f(V_n)}{H_1 f(V_1)} - \frac{H_{n-m_1+1}f(V_{n-m_1})}{H_1 f(V_1)} + \ln\frac{H_{n-m_1+1}f(V_{n-m_1})}{H_{n+1}f(V_n)}\right]\\
&\quad + \delta I_1\left(\frac{I_{n+1}}{I_1} - \frac{I_{n-m_2+1}}{I_1} + \ln\frac{I_{n-m_2+1}}{I_1}\right) + kH_1 f(V_1)
\end{aligned}$$

$$\begin{aligned}
&\cdot\left(\frac{f(V_{n+1})}{f(V_1)}-\frac{f(V_n)}{f(V_1)}+\ln\frac{f(V_n)}{f(V_{n+1})}\right)\\
&+pZ_{n+1}\left(I_1-\frac{\sigma}{q}\right)=\mu H_1\left(1-\frac{H_1}{H_{n+1}}\right)\left(1\right.\\
&\left.-\frac{H_{n+1}}{H_1}\right)+kH_1f(V_1)\left[4-\frac{H_1}{H_{n+1}}\right.\\
&-\frac{I_1H_{n-m_1+1}f(V_{n-m_1})}{I_{n+1}H_1f(V_1)}-\frac{D_1I_{n-m_2+1}}{D_{n+1}I_1}-\frac{V_1D_{n+1}}{V_{n+1}D_1}\\
&\left.-\frac{V_{n+1}}{V_1}+\frac{f(V_n)}{f(V_1)}+\ln\frac{H_{n-m_1+1}f(V_{n-m_1})I_{n-m_2+1}}{H_{n+1}I_{n+1}f(V_{n+1})}\right]\\
&+pZ_{n+1}(I_1-I_2)=\mu H_1\left(1-\frac{H_1}{H_{n+1}}\right)\left(1-\frac{H_{n+1}}{H_1}\right)\\
&+kH_1f(V_1)\left[-\varphi\left(\frac{H_1}{H_{n+1}}\right)-\varphi\left(\frac{D_1I_{n-m_2+1}}{D_{n+1}I_1}\right)\right.\\
&-\varphi\left(\frac{I_1H_{n-m_1+1}f(V_{n-m_1})}{I_{n+1}H_1f(V_1)}\right)-\varphi\left(\frac{V_1D_{n+1}}{V_{n+1}D_1}\right)\\
&\left.+\frac{f(V_{n+1})}{f(V_1)}-\frac{V_{n+1}}{V_1}+\ln\frac{f(V_1)V_{n+1}}{f(V_{n+1})V_1}\right]+pZ_{n+1}\left(I_1\right.\\
&\left.-I_2\right)=\mu H_1\left(1-\frac{H_1}{H_{n+1}}\right)\left(1-\frac{H_{n+1}}{H_1}\right)\\
&+kH_1f(V_1)\left[-\varphi\left(\frac{H_1}{H_{n+1}}\right)-\varphi\left(\frac{D_1I_{n-m_2+1}}{D_{n+1}I_1}\right)\right.\\
&-\varphi\left(\frac{I_1H_{n-m_1+1}f(V_{n-m_1})}{I_{n+1}H_1f(V_1)}\right)-\varphi\left(\frac{V_1D_{n+1}}{V_{n+1}D_1}\right)\\
&-\varphi\left(\frac{f(V_1)V_{n+1}}{V_1f(V_{n+1})}\right)\\
&\left.+\left(1-\frac{f(V_1)}{f(V_{n+1})}\right)\left(\frac{f(V_{n+1})}{f(V_1)}-\frac{V_{n+1}}{V_1}\right)\right]\\
&+pZ_{n+1}(I_1-I_2).
\end{aligned}\tag{50}$$

Similar to the proof of Theorem 6, we get $W_{n+1}^{(2)}-W_n^{(2)}\le 0$, for all $n\in\mathbb{N}$. That is, $W_n^{(2)}$ is monotone decreasing sequence. Furthermore, since $W_n^{(2)}>0$, there is a limit $\lim_{n\to\infty}W_n^{(2)}\ge 0$. Hence, $\lim_{n\to\infty}(W_{n+1}^{(2)}-W_n^{(2)})=0$. Furthermore, from model (6), it is easy to show that $\lim_{n\to\infty}H_n=H_1$, $\lim_{n\to\infty}I_n=H_1$, $\lim_{n\to\infty}D_n=D_1$, $\lim_{n\to\infty}V_n=V_1$, $\lim_{n\to\infty}Z_n=0$, which implies that E_1 is globally asymptotically stable. This completes the proof. □

Theorem 11. *If $\mathfrak{R}_1>1$, then the infection equilibrium with immunity E_2 is globally asymptotically stable.*

Proof. Define a discrete Lyapunov functional as follows:

$$\begin{aligned}
W_n^{(3)}&=\frac{1}{\Delta t}\left[H_2\varphi\left(\frac{H_n}{H_2}\right)+I_2\varphi\left(\frac{I_n}{I_2}\right)+\frac{\delta}{a}D_2\varphi\left(\frac{D_n}{D_2}\right)\right.\\
&\left.+\frac{\delta(\beta+\delta)}{a\beta}V_2\varphi\left(\frac{V_n}{V_2}\right)+\frac{p}{q}\varphi\left(\frac{Z_n}{Z_2}\right)\right]+kH_2f(V_2)\\
&\cdot\sum_{j=n-m_1}^{n-1}\varphi\left(\frac{H_{j+1}f(V_j)}{H_2f(V_2)}\right)+\delta I_2\sum_{j=n-m_2}^{n-1}\varphi\left(\frac{I_{j+1}}{I_2}\right).
\end{aligned}\tag{51}$$

Combine with the model (6) and the equilibrium conditions (34) for E_2; then we obtain the difference of $W_n^{(3)}$

$$\begin{aligned}
W_{n+1}^{(3)}-W_n^{(3)}&=\frac{1}{\Delta t}\left[H_{n+1}-H_n+H_2\ln\frac{H_n}{H_{n+1}}+I_{n+1}\right.\\
&-I_n+I_2\ln\frac{I_n}{I_{n+1}}+\frac{\delta}{a}\left(D_{n+1}-D_n+D_2\ln\frac{D_n}{D_{n+1}}\right)\\
&+\frac{\delta(\beta+\delta)}{a\beta}\left(V_{n+1}-V_n+V_2\ln\frac{V_n}{V_{n+1}}\right)\\
&\left.+\frac{p}{q}\left(Z_{n+1}-Z_n+\ln\frac{Z_n}{Z_{n+1}}\right)\right]+kH_2f(V_2)\\
&\cdot\left(\sum_{j=n-m_1+1}^{n}\varphi\left(\frac{H_{j+1}f(V_j)}{H_2f(V_2)}\right)\right.\\
&\left.-\sum_{j=n-m_1}^{n-1}\varphi\left(\frac{H_{j+1}f(V_j)}{H_2f(V_2)}\right)\right)\\
&+\delta I_2\left(\sum_{j=n-m_2+1}^{n}\varphi\left(\frac{I_{j+1}}{I_2}\right)-\sum_{j=n-m_2}^{n-1}\varphi\left(\frac{I_{j+1}}{I_2}\right)\right)\\
&+kH_2f(V_2)\left(\varphi\left(\frac{f(V_{n+1})}{f(V_2)}\right)-\varphi\left(\frac{f(V_n)}{f(V_2)}\right)\right)\\
&\le\frac{1}{\Delta t}\left[\left(1-\frac{H_2}{H_{n+1}}\right)(H_{n+1}-H_n)\right.\\
&+\left(1-\frac{I_2}{I_{n+1}}\right)(I_{n+1}-I_n)+\frac{\delta}{a}\left(1-\frac{D_2}{D_{n+1}}\right)\\
&\times(D_{n+1}-D_n)\\
&+\frac{\delta(\beta+\delta)}{a\beta}\left(1-\frac{V_2}{V_{n+1}}\right)(V_{n+1}-V_n)\\
&\left.+\frac{p}{q}\left(1-\frac{Z_2}{Z_{n+1}}\right)(Z_{n+1}-Z_n)\right]+kH_2f(V_2)\\
&\cdot\left(\sum_{j=n-m_1+1}^{n}\varphi\left(\frac{H_{j+1}f(V_j)}{H_2f(V_2)}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=n-m_1}^{n-1} \varphi\left(\frac{H_{j+1} f\left(V_j\right)}{H_2 f\left(V_2\right)}\right)\Bigg) \\
&+ \delta I_2 \left(\sum_{j=n-m_2+1}^{n} \varphi\left(\frac{I_{j+1}}{I_2}\right) - \sum_{j=n-m_2}^{n-1} \varphi\left(\frac{I_{j+1}}{I_2}\right)\right) \\
&+ kH_2 f\left(V_2\right)\left(\varphi\left(\frac{f\left(V_{n+1}\right)}{f\left(V_2\right)}\right) - \varphi\left(\frac{f\left(V_n\right)}{f\left(V_2\right)}\right)\right) \\
&= \left(1 - \frac{H_2}{H_{n+1}}\right)\left(s - \mu H_{n+1} - kH_{n+1} f\left(V_n\right)\right) + \Bigg(1 \\
&- \frac{I_2}{I_{n+1}}\Bigg)\Big(kH_{n-m_1+1} f\left(V_{n-m_1}\right) - \delta I_{n+1} \\
&- pI_{n+1}Z_{n+1}\Big) + \frac{\delta}{a}\left(1 - \frac{D_2}{D_{n+1}}\right)\Big(aI_{n-m_2+1} \\
&- \left(\beta + \delta\right) D_{n+1}\Big) + \frac{\delta\left(\beta + \delta\right)}{a\beta}\left(1 - \frac{V_2}{V_{n+1}}\right)\Big(\beta D_{n+1} \\
&- cV_{n+1}\Big) + \frac{p}{q}\left(1 - \frac{Z_2}{Z_{n+1}}\right)\left(qI_{n+1}Z_{n+1} - \sigma Z_n\right) \\
&+ kH_2 f\left(V_2\right)\Bigg[\varphi\left(\frac{H_{n+1} f\left(V_n\right)}{H_2 f\left(V_2\right)}\right) \\
&- \varphi\left(\frac{H_{n-m_1+1} f\left(V_{n-m_1}\right)}{H_2 f\left(V_2\right)}\right)\Bigg] + \delta I_2 \Bigg(\varphi\left(\frac{I_{n+1}}{I_2}\right) \\
&- \varphi\left(\frac{I_{n-m_2+1}}{I_2}\right)\Bigg) + kH_2 f\left(V_2\right)\Bigg(\varphi\left(\frac{f\left(V_{n+1}\right)}{f\left(V_2\right)}\right) \\
&- \varphi\left(\frac{f\left(V_n\right)}{f\left(V_2\right)}\right)\Bigg) = \mu H_2 \left(1 - \frac{H_2}{H_{n+1}}\right)\left(1 - \frac{H_{n+1}}{H_2}\right) \\
&+ \left(1 - \frac{H_2}{H_{n+1}}\right)\left(kH_2 f\left(V_2\right) - kH_{n+1} f\left(V_n\right)\right) + \Bigg(1 \\
&- \frac{I_2}{I_{n+1}}\Bigg)\Bigg(kH_{n-m_1+1} f\left(V_{n-m_1}\right) - kH_2 f\left(V_2\right)\frac{I_{n+1}}{I_2} \\
&+ pI_{n+1}Z_2 - pI_{n+1}Z_{n+1}\Bigg) + \frac{\delta}{a}\left(1 - \frac{D_2}{D_{n+1}}\right) \\
&\cdot \left(aI_2 \frac{I_{n-m_2+1}}{I_2} - \left(\beta + \delta\right) D_2 \frac{D_{n+1}}{D_2}\right) + \frac{\delta\left(\beta + \delta\right)}{a\beta}\Bigg(1 \\
&- \frac{V_2}{V_{n+1}}\Bigg) \times \left(\beta D_2 \frac{D_{n+1}}{D_2} - cV_2 \frac{V_{n+1}}{V_2}\right) + kH_2 f\left(V_2\right) \\
&\cdot \Bigg[\frac{H_{n+1} f\left(V_n\right)}{H_2 f\left(V_2\right)} - \frac{H_{n-m_1+1} f\left(V_{n-m_1}\right)}{H_2 f\left(V_2\right)} \\
&+ \ln \frac{H_{n-m_1+1} f\left(V_{n-m_1}\right)}{H_{n+1} f\left(V_n\right)}\Bigg] + \delta I_2 \Bigg(\frac{I_{n+1}}{I_2} - \frac{I_{n-m_2+1}}{I_2} \\
&+ \ln \frac{I_{n-m_2+1}}{I_2}\Bigg) + kH_2 f\left(V_2\right)\Bigg(\frac{f\left(V_{n+1}\right)}{f\left(V_2\right)} - \frac{f\left(V_n\right)}{f\left(V_2\right)} \\
&+ \ln \frac{f\left(V_n\right)}{f\left(V_{n+1}\right)}\Bigg) = \mu H_2 \left(1 - \frac{H_2}{H_{n+1}}\right)\left(1 - \frac{H_{n+1}}{H_2}\right) \\
&+ kH_2 f\left(V_2\right)\Bigg[4 - \frac{H_2}{H_{n+1}} - \frac{I_2 H_{n-m_1+1} f\left(V_{n-m_1}\right)}{I_{n+1} H_2 f\left(V_2\right)} \\
&- \frac{D_2 I_{n-m_2+1}}{D_{n+1} I_2} - \frac{V_2 D_{n+1}}{V_{n+1} D_2} - \frac{V_{n+1}}{V_2} + \frac{f\left(V_n\right)}{f\left(V_2\right)} \\
&+ \ln \frac{H_{n-m_1+1} f\left(V_{n-m_1}\right) I_{n-m_2+1}}{H_{n+1} I_{n+1} f\left(V_{n+1}\right)}\Bigg] = \mu H_2 \Bigg(1 \\
&- \frac{H_2}{H_{n+1}}\Bigg)\left(1 - \frac{H_{n+1}}{H_2}\right) + kH_2 f\left(V_2\right)\Bigg[-\varphi\left(\frac{H_2}{H_{n+1}}\right) \\
&- \varphi\left(\frac{D_2 I_{n-m_2+1}}{D_{n+1} I_2}\right) - \varphi\left(\frac{I_2 H_{n-m_1+1} f\left(V_{n-m_1}\right)}{I_{n+1} H_2 f\left(V_2\right)}\right) \\
&- \varphi\left(\frac{V_2 D_{n+1}}{V_{n+1} D_2}\right) + \frac{f\left(V_{n+1}\right)}{f\left(V_2\right)} - \frac{V_{n+1}}{V_2} \\
&+ \ln \frac{f\left(V_2\right) V_{n+1}}{f\left(V_{n+1}\right) V_2}\Bigg] = \mu H_2 \left(1 - \frac{H_2}{H_{n+1}}\right)\Bigg(1 \\
&- \frac{H_{n+1}}{H_2}\Bigg) + kH_2 f\left(V_2\right)\Bigg[-\varphi\left(\frac{H_2}{H_{n+1}}\right) \\
&- \varphi\left(\frac{D_2 I_{n-m_2+1}}{D_{n+1} I_2}\right) - \varphi\left(\frac{I_2 H_{n-m_1+1} f\left(V_{n-m_1}\right)}{I_{n+1} H_1 f\left(V_2\right)}\right) \\
&- \varphi\left(\frac{V_2 D_{n+1}}{V_{n+1} D_2}\right) - \varphi\left(\frac{f\left(V_2\right) V_{n+1}}{V_2 f\left(V_{n+1}\right)}\right) \\
&+ \left(1 - \frac{f\left(V_2\right)}{f\left(V_{n+1}\right)}\right)\left(\frac{f\left(V_{n+1}\right)}{f\left(V_2\right)} - \frac{V_{n+1}}{V_2}\right)\Bigg].
\end{aligned}
\tag{52}
$$

Similar to the proof of Theorem 10, if $\mathfrak{R}_1 > 1$, then we have $W_{n+1}^{(3)} - W_n^{(3)} \le 0$, for all $n \in \mathbb{N}$. That is, $W_n^{(3)}$ is monotone decreasing sequence. Furthermore, since $W_n^{(3)} > 0$, there is a limit $\lim_{n\to\infty} W_n^{(3)} \ge 0$. Hence, $\lim_{n\to\infty}(W_{n+1}^{(3)} - W_n^{(3)}) = 0$. Combined with model (6), it can be shown that $\lim_{n\to\infty} H_n = H_2$, $\lim_{n\to\infty} I_n = I_2$, $\lim_{n\to\infty} D_n = D_2$, $\lim_{n\to\infty} V_n = V_2$, $\lim_{n\to\infty} Z_n = Z_2$, which implies that E_2 is globally asymptotically stable. This completes the proof. □

4. Conclusions

In this paper, we proposed and investigated a delayed virus infection model with immune response and general nonlinear incidence. To come up with the efficient numerical method for the proposed delayed model, we then consider the discretization of the original continuous model by utilizing NSFD scheme. The advantage of the NSFD scheme is that the global properties of the solutions for the corresponding continuous model can be preserved. A crucial observation regarding the advantage of the NSFD scheme is that the discrete model has equilibria which are exactly the same as those of the original continuous model and the conditions for their stability are identical in case of both the continuous and discrete models. Specifically, if $\mathfrak{R}_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable; if $\mathfrak{R}_1 \leq 1 < \mathfrak{R}_0$, then the infection equilibrium without immunity E_1 is globally asymptotically stable; if $\mathfrak{R}_1 > 1$, then the infection equilibrium with immunity E_2 is globally asymptotically stable. The results imply that the NSFD scheme can efficiently preserve the global properties of solutions for original continuous model [8]. Applying this method to the other types of delayed epidemic models is our future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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