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Research Article

Decomposition Technique and a Family of Efficient Schemes for Nonlinear Equations

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Various problems of pure and applied sciences can be studied in the unified framework of nonlinear equations. In this paper, a new family of iterative methods for solving nonlinear equations is developed by using a new decomposition technique. The convergence of the new methods is proven. For the implementation and performance of the new methods, some examples are solved and the results are compared with some existing methods.

1. Introduction

The conceptualization and creation of diverse iterative methods for finding efficient and precisely the approximate solution of nonlinear equation, $f(x) = 0$, are a fundamental task in numerical analysis and related areas. This topic has attracted the attention of many researchers, when Abbasbandy [1] initiated the analysis of these methods. Many researchers collate and updated the state of the art of iterative methods. The advantage of multipoint methods is that they do not use higher-order derivatives and have great practical importance because they overcome the theoretical limitations of one-point methods regarding their convergence order and computational efficiency. In this work, a new decomposition technique [2] that is quite different from Adomian decomposition method [3] is applied to some classes of iterative methods for solving the nonlinear equations. We would like to point out that in the implementation of the Adomian decomposition method one has to calculate the derivatives of the so-called Adomian polynomials, which is the drawback of the technique of Adomian method. To overcome the disadvantage, the decomposition of Bhalekar and Daftardar-Gejji [2] is used in this work. The said decomposition technique [2] does not occupy the high-order differentials of the function and is very simple as compared to the Adomian method. Chun et al. [4] has suggested that the nonlinear equations can be written as a coupled system of equations. He et al. [5, 6] have used the idea of coupled system of equations with the decomposition technique of Bhalekar and Daftardar-Gejji [2] to develop several iterative methods to solve the nonlinear equations. In this work, decomposition method [2] sophisticatedly combined with the coupled system of equations is used to develop new family of iterative methods. It is shown that the proposed family of iterative methods contains several well-known iterative techniques as special case. The convergence of the suggested class of method is also proven. Some numerical examples are illustrated to exhibit the efficiency and performance of proposed iterative methods.

2. Iterative Methods

In this section, we are going to present some new iterative methods by the help of quadrature method and basic fundamental law of calculus. Consider the following nonlinear equation:

$$f(x) = 0. \tag{1}$$

By assuming α as a simple zero of (1) and γ as initial guess in the neighbor of α, let $p(x)$ be the auxiliary function such that

$$f(x)\,p(x) = 0. \quad (2)$$

Now we use fundamental law of calculus and quadrature method on $f(x)$ and Taylor's series on $p(x)$; then (2) becomes

$$\begin{aligned} &f(\gamma)\,p(\gamma) + (x-\gamma) \\ &\cdot \left(f(\gamma)\,p'(\gamma) + p(\gamma)\left[\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma))\right]\right) \\ &+ h(x) = 0. \end{aligned} \quad (3)$$

We rewrite (3) as system of coupled equations and then, by combining it with (2), we get

$$\begin{aligned} h(x) &= f(x)\,p(\gamma) - f(\gamma)\,p(\gamma) - (x-\gamma) \\ &\cdot \left[f'(\gamma)\,p(\gamma) + \acute{p}(\gamma)\left(\sum_{i=1}^{p} w_i f'(\gamma + \tau_i (x-\gamma))\right)\right]. \end{aligned} \quad (4)$$

τ_i represent the knots such that $\tau_i \in [0,1]$ and w are weights which are taken in such a way that they satisfy the consistency condition such that

$$\sum_{i=1}^{r} w_i = 1. \quad (5)$$

Now, rearrange (4) as

$$\begin{aligned} x &= \gamma \\ &- \frac{h(x) + f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i (x-\gamma))\right] p(\gamma)}. \end{aligned} \quad (6)$$

Now we let

$$\begin{aligned} x &= \gamma - \frac{h(x) + f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i (x-\gamma))\right] p(\gamma)} \\ &= c + N(x), \end{aligned} \quad (7)$$

where

$$c = \gamma, \quad (8)$$

$$N(x) = -\frac{h(x) + f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i (x-\gamma))\right] p(\gamma)}. \quad (9)$$

Equation (9) shows that $N(x)$ is a nonlinear function. Now, by using the idea of Daftardar-Gejji and Jafari [7], we develop a new iterative scheme. This decomposition technique is developed to solve the nonlinear functions and, by using this decomposition technique, we suggested the iterative method to calculate the root of $f(x) = 0$. The Daftardar-Gejji and Jafari [7] decomposition technique is easier to use than Adomian decomposition method [3] to give the series solution.

$$x = \sum_{i=0}^{\infty} x_i. \quad (10)$$

We decompose $N(x)$ as

$$N(x) = N(x_0) + \sum_{i=1}^{\infty}\left\{ N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right\}. \quad (11)$$

From (7), (10), and (11), we get

$$\begin{aligned} x_0 &= c \\ x_1 &= N(x_0) \\ x_2 &= N(x_1) - N(x_0) \\ &\vdots \\ x_m &= N\left(\sum_{j=0}^{m-1} x_j\right) + N\left(\sum_{j=0}^{m-2} x_j\right). \end{aligned} \quad (12)$$

From (12), we have

$$x_1 + x_2 + x_3 + \cdots + x_m = N(x_0 + x_1 + \cdots + x_{m-1}). \quad (13)$$

By approximating (13), we get

$$X_m = x_0 + x_1 + x_2 + x_3 + \cdots + x_{m+1}. \quad (14)$$

So

$$\lim_{m\to\infty} X_{m+1} = x \quad (15)$$

where $m = 0$.

From (8), (12), and (15), we get

$$x \approx X_0 = x_0 = c = \gamma. \quad (16)$$

where $m = 1$.

From (12), (14), and (15), we get

$$x \approx X_1 = x_0 + x_1 = \gamma + N(x_0). \quad (17)$$

From (9), we have

$$\begin{aligned} x_1 &= N(x_0) \\ &= -\frac{(h(x_0) + f(\gamma)\,p(\gamma))}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i (x_0-\gamma))\right] p(\gamma)}. \end{aligned} \quad (18)$$

From (2), (3), and (4), we get

$$h(x_0) = 0. \quad (19)$$

By putting (19) in (18) and combination with (17), we have

$$x = x_0 + x_1 = \gamma - \frac{f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 - \gamma))\right] p(\gamma)}, \tag{20}$$

$$x = \gamma - \frac{f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 - \gamma))\right] p(\gamma)}. \tag{21}$$

Equation (21) allows us to suggest iterative method (one-step) for the approximate root of (1).

Algorithm 1. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$x_{n+1} = x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \quad n = 0, 1, 2, \ldots \tag{22}$$

Algorithm 1 is suggested by M. A. Noor and K. I. Noor [8, 9] by the help of variational iterative techniques.

From (21), we have

$$x_0 + x_1 - \gamma = -\frac{f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + f'(\gamma)\,p(\gamma)}. \tag{23}$$

Also from (4) and using the idea of Yun [10], we have

$$h(x_0 + x_1) = f(x_0 + x_1)\,p(\gamma) - f(\gamma)\,p(\gamma) - (x_0 + x_1 - \gamma)\left[p'(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] p(\gamma)\right]. \tag{24}$$

By combining (23) and (24), we get

$$h(x_0 + x_1) = f(x_0 + x_1)\,p(\gamma) - f(\gamma)\,p(\gamma) + \left(\frac{f(\gamma)\,p(\gamma)}{\acute{p}(\gamma)\,f(\gamma) + f'(\gamma)\,p(\gamma)}\right)\left[p'(\gamma)\,f'(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] p(\gamma)\right]. \tag{25}$$

From (9) and (25), we have

$$\begin{aligned} x_1 + x_2 &= N(x_0 + x_1) \\ &= -\frac{h(x_0 + x_1) + f(\gamma)\,p(\gamma)}{g'(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] p(\gamma)} \\ &= -\frac{f(\gamma)\,p(\gamma)}{p'(\gamma)\,f(\gamma) + f'(\gamma)\,p(\gamma)} - \frac{f(x_0 + x_1)\,p(\gamma)}{p'(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] p(\gamma)}. \end{aligned} \tag{26}$$

Also from (2) and (4) and by Yun's idea [10],

$$h(x_0 + x_1) = 0. \tag{27}$$

Now for $m = 2$, (8), (12), and (15) give the following equation:

$$x \approx X_2 = x_0 + x_1 + x_2 = \gamma + N(x_0 + x_1). \tag{28}$$

Now by combining (7), (26), (27), and (28), we get

$$x = \gamma - \frac{f(\gamma)\,p(\gamma)}{p'(\gamma)\,f(\gamma) + f'(\gamma)\,p(\gamma)} - \frac{f(x_0 + x_1)\,p(\gamma)}{p'(\gamma)\,f(\gamma) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_0 + x_1 - \gamma))\right] p(\gamma)}. \tag{29}$$

Using the above relation, we suggested the iterative method (two-step) for finding the approximate root of (1).

Algorithm 2. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= x_n - \frac{f(y_n)\,p(x_n)}{p'(x_n)\,f(x_n) + \left[\sum_{i=1}^{r} w_i f'(\gamma + \tau_i(x_n - \gamma))\right] p(x_n)}. \end{aligned} \tag{30}$$

Some Special Case of Algorithm 2. Now we have to consider value of Algorithm 2. Take $p = 1$, $w_1 = 1$, and $\tau_1 = 0$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 3. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}. \end{aligned} \tag{31}$$

Take $p = 1$, $w_1 = 1$, and $\tau_1 = 1$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 4. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(y_n)\,p(x_n)}. \end{aligned} \tag{32}$$

Take $p = 1$, $w_1 = 1$, and $\tau_1 = 1/2$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 5. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= y_n \\ &\quad - \frac{f(y_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'((y_n + x_n)/2)\,p(x_n)}. \end{aligned} \tag{33}$$

Take $p = 2$, $w_1 = 1/4$, $w_2 = 3/4$, $\tau_1 = 0$, and $\tau_2 = 2/3$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 6. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= y_n \\ &\quad - \frac{4f(y_n)\,p(x_n)}{4p'(x_n)\,f(x_n) + (f'(x_n) + 3f'((2y_n + x_n)/3))\,p(x_n)}. \end{aligned} \tag{34}$$

Take $p = 3$, $w_1 = 1/6$, $w_2 = 2/3$, $w_3 = 1/6$, $\tau_1 = 0$, $\tau_2 = 1/2$, and $\tau_3 = 1$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 7. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)}, \\ x_{n+1} &= y_n - \frac{6f(y_n)\,p(x_n)}{6p'(x_n)\,f(x_n) + (f'(x_n) + 4f'((y_n + x_n)/2) + f'(y_n))\,p(x_n)}. \end{aligned} \tag{35}$$

Algorithm 7 is suggested by Shah and Noor [11].

Take $p = 3$, $w_1 = 1/4$, $w_2 = 1/2$, $w_3 = 1/4$, $\tau_1 = 0$, $\tau_2 = 1/2$, and $\tau_3 = 1$ in Algorithm 2 and it reduces to the following algorithm.

Algorithm 8. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)\,p(x_n)}{p'(x_n)\,f(x_n) + f'(x_n)\,p(x_n)} \\ x_{n+1} &= y_n - \frac{4f(y_n)\,p(x_n)}{4p'(x_n)\,f(x_n) + (f'(x_n) + 2f'((y_n + x_n)/2) + f'(y_n))\,p(x_n)}. \end{aligned} \tag{36}$$

Algorithms 3–8 give main iterative schemes which are helpful to generate several higher-order iterative methods for different values of auxiliary function $p(x_n)$. So proper selection of auxiliary function manages to help us to diversify the problem for best possible implementation for obtaining the solution of nonlinear equation (1). So we consider auxiliary function $p(x) = e^{(-\alpha x)}$ to obtain the result by putting it in Algorithms 3–8.

After putting the value of $p(x) = e^{(-\alpha x)}$ in Algorithms 3–8, we get Algorithms 9–14.

Algorithm 9. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n) - \alpha f(x_n)}. \end{aligned} \tag{37}$$

Algorithm 10. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n) - \alpha f(x_n)}. \end{aligned} \tag{38}$$

Algorithm 11. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'((y_n + x_n)/2) - \alpha f(x_n)}. \end{aligned} \tag{39}$$

Algorithm 12. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)},$$
$$x_{n+1} = y_n - \frac{4f(y_n)}{f'(x_n) + 3f'((2y_n + x_n)/3) - 4\alpha f(x_n)}. \quad (40)$$

Algorithm 13. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)},$$
$$x_{n+1} = y_n - \frac{6f(y_n)}{f'(x_n) + 4f'((y_n + x_n)/2) + f'(y_n) - 6\alpha f(x_n)}. \quad (41)$$

Algorithm 14. The iterative scheme which will compute the approximate root x_{n+1} for initial guess x_0 is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n) - \alpha f(x_n)},$$
$$x_{n+1} = y_n - \frac{4f(y_n)}{f'(x_n) + 2f'((y_n + x_n)/2) + f'(y_n) - 4\alpha f(x_n)}. \quad (42)$$

Shah and Noor [11] suggest that Algorithm 14 has cubic order of convergence.

3. Convergence Analysis

This section helps us to find out the convergence of Algorithm 2 by using Taylor's series.

Theorem 15. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be sufficiently differentiable function that has a simple zero at $\alpha \in I$, where I is an open interval. If x_0 is at neighbor of α, then Algorithm 2 has at least third-order convergence.*

Proof. Let α be a simple zero of f and then, expanding $f(x_n)$ and $f'(x_n)$ in Taylor's series about α, we have

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right],$$
$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)\right], \quad (43)$$

where $c_k = (1/k!)(f^{(k)}/f'(\alpha))$, $k = 2, 3, \ldots$, and $e_n = x - \alpha$. Now we expand $f(x_n)p(x_n)$, $f(x_n)p'(x_n)$, and $f'(x_n)p(x_n)$ by Taylor's series:

$$\begin{aligned} f(x_n)p(x_n) &= f'(\alpha)\left[p(\alpha)e_n + (p(\alpha) + c_2p'(\alpha))e_n^2 + g_1e_n^3 + O(e_n^4)\right], \\ f(x_n)p'(x_n) &= f'(\alpha)\left[g'(\alpha)e_n + (g'(\alpha) + c_2p''(\alpha))e_n^2 + g_2e_n^3 + O(e_n^4)\right], \\ f'(x_n)p(x_n) &= f'(\alpha)\left[p(\alpha) + (p'(\alpha) + 2c_2p(\alpha))e_n + g_1e_n^2 + g_3e_n^3 + O(e_n^4)\right], \end{aligned} \quad (44)$$

where

$$\begin{aligned} g_1 &= \frac{g''(\alpha)}{2} + c_3p(\alpha) + c_2p'(\alpha), \\ g_2 &= \frac{p'''(\alpha)}{2} + c_3p'(\alpha) + c_2p''(\alpha), \\ g_3 &= \frac{p'''(\alpha)}{6} + 4c_4p(\alpha) + c_2p''(\alpha) + 3c_3p'(\alpha). \end{aligned} \quad (45)$$

From (44), we get

$$\frac{f(x_n)p(x_n)}{f'(x_n)p(x_n) + f(x_n)p'(x_n)} = e_n - \left(\frac{p'(\alpha)}{p(\alpha)} + c_2\right)e_n^2 + \left(\frac{p''(\alpha)}{p(\alpha)} + 2c_3 - 2\frac{p'(\alpha)}{p(\alpha)}c_2 - 2\left(\frac{p'(\alpha)}{p(\alpha)}\right)^2 - 2c_2^2\right)e_n^3 + O(e_n^4); \quad (46)$$

by using (46), we get

$$y := \alpha + \left(\frac{p'(\alpha)}{p(\alpha)} + c_2\right)e_n^2 + \left(\frac{p''(\alpha)}{p(\alpha)} + 2c_3 - 2\frac{p'(\alpha)}{p(\alpha)}c_2 - 2\left(\frac{p'(\alpha)}{p(\alpha)}\right)^2 - 2c_2^2\right)e_n^3 + O(e_n^4). \quad (47)$$

Now by Taylor's series we expand $f(y)$ and $f'(y)$ about α and we get

$$f(y) \coloneqq f'(\alpha)\left(\frac{p'(\alpha)}{p(\alpha)}+c_2\right)e_n^2+f'(\alpha)\left(\frac{p''(\alpha)}{p(\alpha)}+2c_3-2\frac{p'(\alpha)}{p(\alpha)}c_2-2\left(\frac{p'(\alpha)}{p(\alpha)}\right)^2-2c_2^2\right)e_n^3+O\left(e_n^4\right), \tag{48}$$

$$f'(y) \coloneqq f'(\alpha)\left[\left(2c_2\frac{p'(\alpha)}{p(\alpha)}+c_2\right)e_n^2+O\left(e_n^3\right)\right]. \tag{49}$$

Now, expanding $f(y)p'(x)$ and $\sum_{i=1}^{r} w_i f'(x+\tau_i(y-x))$ in Taylor's series, we get

$$f(y)g'(x)=f'(\alpha)\left[\frac{p'^2}{p(\alpha)}+c_2\right]e_n^2-f'(\alpha)\cdot\left[2\frac{p'^2p''(\alpha)}{p(\alpha)}+c_2p''(\alpha)-2c_3p'(\alpha)+2c_2\frac{p'^2}{p(\alpha)}+2\frac{p'^3}{p(\alpha)^2}+2c_2p'(\alpha)\right]e_n^3+O\left(e_n^4\right), \tag{50}$$

$$\sum_{i=1}^{r} w_i f'(x+\tau_i(y-x))=f'(\alpha)\left[1-2c_2\sum_{i=1}^{r} w_i(-1+t_i)e_n+O\left(e_n^2\right)\right].$$

Using (47), (48), and (50), we obtain

$$x_{n+1}=\alpha+\left(c_2\frac{p'(\alpha)}{p(\alpha)}+\left(\frac{p'(\alpha)}{p(\alpha)}\right)^2-2\frac{c_2\sum_{i=1}^{r} w_i(-1+t_i)}{p(\alpha)}-2{c_2}^2\sum_{i=1}^{r} w_i(-1+t_i)\right){e_n}^3+O\left(e_n^3\right); \tag{51}$$

so, from (51), we have the error equation as follows:

$$e_{n+1}=\left(c_2\frac{p'(\alpha)}{p(\alpha)}+\left(\frac{p'(\alpha)}{p(\alpha)}\right)^2-2\frac{c_2\sum_{i=1}^{r} w_i(-1+t_i)}{p(\alpha)}-2{c_2}^2\sum_{i=1}^{r} w_i(-1+t_i)\right){e_n}^3+O\left(e_n^3\right). \tag{52}$$

In a similar way, we can find the convergence of other methods. □

4. Numerical Results

We use Intel Core i5 computer with 4 GB RAM and operating system is Windows 10 (32-bit). We use Maple 13 for computation and Matlab to plot the graphs. In our computation, we use the following stopping criteria:

$$\left|x_{n+1}-x_n\right|+\left|f(x)\right|<\epsilon, \tag{53}$$

where $\epsilon = 10^{-15}$.

Example 16. Consider the growth of population over short periods of time by assuming that population grows continuously with time at a rate proportional to the number present at that time. Let $N(t)$ denote the number at time t and let λ denote the constant birth rate of the population. If immigration is permitted at ta constant rate ν, then the population satisfies the differential equation

$$\frac{dN(t)}{dt}=\lambda N(t)+\nu \tag{54}$$

whose solution is given by

$$N(t)=N_0e^{\lambda t}+\frac{\nu}{\lambda}\left(e^{\lambda t}-1\right). \tag{55}$$

Suppose that a certain population contains $N(0) = 1{,}000{,}000$ individuals initially, that 435,000 individuals immigrate into the community in the first year, and that $N(1) = 1{,}564{,}000$ individuals are present at the end of one year. To determine this population, we have to find λ in the equation

$$1,564,000=1,000,000e^{\lambda}+\frac{435,000}{\lambda}\left(e^{\lambda}-1\right). \tag{56}$$

We use $x_0 = 1$ as initial guess and approximated answer up to 15 decimal digits is 0.10099792985750.

Example 17. In this example, we consider a model in which a particle is moved on an inclined plane making angle ϕ change its with constant rate, say ν. At start, the particle is at rest.

$$\frac{d\phi}{dt}=\nu<0. \tag{57}$$

After time t, the position of the particle is calculated by the following equation:

$$x(t)=-\frac{g}{2\nu^2}\left(\frac{e^{\nu t}-e^{-\nu t}}{2}-\sin\nu t\right). \tag{58}$$

In 1 second, the particle covered 1.7 ft. We want to calculate the rate with which the particle changes its position. As g is known as gravitational constant, its value is equal to 32.17 ft/sec^2. The solution of problem approximated to the 15 decimals digits is −0.317061774531088. We us $x_0 = 1$ as initial guess for this example.

Table 1: Numerical results for Example 16 when $p(x) = e^{-\alpha x}$.

Method	α	n	x_n	$\lvert f(x)\rvert$	$\lvert x_{i+1} - x_i\rvert$
NM	—	13	0.100997929685740	0.00	1.8000000000$e-14$
Algorithm 9	0.5	4	0.100997929685740	0.00	1.3000000000$e-14$
Algorithm 10	0.4	6	0.100997929685740	0.00	1.8000000000$e-14$
Algorithm 11	0.2	3	0.100997929685740	0.00	7.0523515910$e-06$
Algorithm 12	0.3	4	0.100997929685740	0.00	1.8000000000$e-14$
Algorithm 13	0.2	5	0.100997929685740	0.00	1.3000000000$e-14$
Algorithm 14 (Shah and Noor [11])	0.13	3	0.100997929685740	0.00	4.0339206898$e-05$

Table 2: Numerical results for Example 17 when $p(x) = e^{-\alpha x}$.

Method	α	n	x_n	$\lvert f(x)\rvert$	$\lvert x_{i+1} - x_i\rvert$
NM	—	10	−0.317061774531075	1$e-14$	1.900000000000000$e-14$
Algorithm 9	−0.2	4	−0.317061774531089	0.00	8.000000000000000$e-14$
Algorithm 10	−0.1	3	−0.317061774531081	8$e-14$	6.237130600000000$e-08$
Algorithm 11	−0.4	4	−0.317061774531021	8$e-14$	2.760300000000000$e-11$
Algorithm 12	−0.2	4	−0.317061774531021	8$e-14$	5.000000000000000$e-14$
Algorithm 13	0.0019	2	−0.317061774531097	7$e-14$	1.423721031620000$e-04$
Algorithm 14 (Shah and Noor [11])	−0.23	4	−0.317061774531126	8$e-14$	1.700000000000000$e-14$

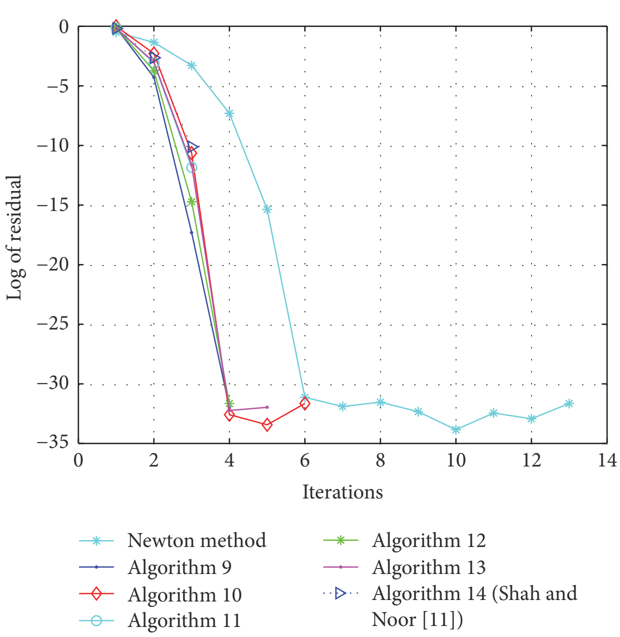

Figure 1

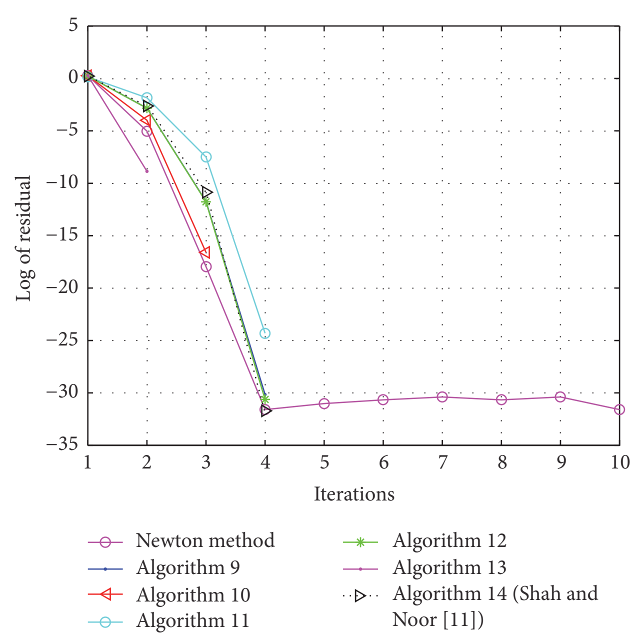

Figure 2

5. Conclusion

In this paper, the coupled system of equations with the new decomposition technique has been used to develop a family of iterative methods for solving nonlinear equations, which includes several well-known and new methods. Technique of derivation of the iterative methods is very simple as compared to the Adomian decomposition method. This is another aspect of the simplicity. The convergence analysis of the new iterative methods has been proven. We have solved some examples and the methods are compared, which are exhibited in Tables 1 and 2 and Figures 1 and 2.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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