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Research Article

Dynamics of a Higher-Order System of Difference Equations

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Consider the following system of difference equations: $\{x_{n+1}^{(i)} = x_{n-m+1}^{(i)}/(A_i\prod_{j=0}^{m-1} x_{n-j}^{(i+j+1)} + \alpha_i), x_{n+1}^{(i+m)} = x_{n+1}^{(i)}, x_{1-l}^{(i+l)} = a_{i,l}, A_{i+m} = A_i, \alpha_{i+m} = \alpha_i\}$, $i, l = 1, 2, \ldots, m$; $n = 0, 1, 2, \ldots$, where m is a positive integer, A_i, α_i, $i = 1, 2, \ldots, m$, and the initial conditions $a_{i,l}$, $i, l = 1, 2, \ldots, m$, are positive real numbers. We obtain the expressions of the positive solutions of the system and then give a precise description of the convergence of the positive solutions. Finally, we give some numerical results.

1. Introduction

Difference equation or system of difference equations is a diverse field which impacts almost every branch of pure and applied mathematics. Not only does it provide us with some simple and useful mathematic models to help elucidate interesting phenomena in applications, but also it can kind of display some surprising complicated dynamics comparing with its analogue differential equations. Hence, the systems of difference equations and difference equations have attracted a lot of attention (see, e.g., the systems of difference equations [1–16] and difference equations [17–29] and the references therein). Among them, symmetric and close to symmetric systems of difference equations have attracted a considerable interest.

Papaschinnopoulos and Schinas [1] studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of the nonlinear difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}},$$
$$y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad (1)$$
$$n = 0, 1, 2, \ldots.$$

In [2], they also investigated the boundedness, persistence, the oscillatory behavior, and the asymptotic behavior of the positive solutions of the system of difference equations:

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{y_{n-i}^{p_i}},$$
$$y_{n+1} = \sum_{i=0}^{k} \frac{B_i}{x_{n-i}^{q_i}}, \quad (2)$$
$$n = 0, 1, 2, \ldots.$$

Clark et al. [3, 4] investigated the global asymptotic stability of the system of difference equations:

$$x_{n+1} = \frac{x_n}{a + cy_n},$$
$$y_{n+1} = \frac{y_n}{b + dx_n}, \quad (3)$$
$$n = 0, 1, 2, \ldots.$$

Camouzis and Papaschinopoulos [5] studied the global asymptotic behavior of positive solutions of the system of rational difference equations:

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}},$$
$$y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad (4)$$
$$n = 0, 1, 2, \ldots.$$

Yang [6] studied the behavior of positive solutions of the system of difference equations:

$$x_n = A + \frac{y_{n-1}}{x_{n-p} y_{n-q}},$$
$$y_n = A + \frac{x_{n-1}}{x_{n-r} y_{n-s}}, \quad (5)$$
$$n = 0, 1, 2, \ldots.$$

Zhang et al. [7] studied the boundedness, the persistence, and global asymptotic stability of the positive solutions of the system of difference equations:

$$x_{n+1} = A + \frac{y_{n-m}}{x_n},$$
$$y_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad (6)$$
$$n = 0, 1, 2, \ldots.$$

Yalçinkaya and Çinar [8] studied the global asymptotic stability of the system of difference equations:

$$z_{n+1} = \frac{t_n + z_{n-1}}{t_n z_{n-1} + a},$$
$$t_{n+1} = \frac{z_n + t_{n-1}}{z_n t_{n-1} + a}, \quad (7)$$
$$n = 0, 1, 2, \ldots.$$

Kurbanlı et al. [9] studied the behavior of the positive solutions of the following system of difference equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1},$$
$$y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \quad (8)$$
$$n = 0, 1, 2, \ldots.$$

Motivated by the above studies, in this note, we consider the following system of difference equations:

$$x_{n+1}^{(i)} = \frac{x_{n-m+1}^{(i)}}{A_i \prod_{j=0}^{m-1} x_{n-j}^{(i+j+1)} + \alpha_i},$$
$$x_{n+1}^{(i+m)} = x_{n+1}^{(i)},$$
$$x_{1-l}^{(i+l)} = a_{i,l}, \quad (9)$$
$$A_{i+m} = A_i,$$
$$\alpha_{i+m} = \alpha_i,$$
$$i, l = 1, 2, \ldots, m; \; n = 0, 1, 2, \ldots,$$

where m is a positive integer, A_i, α_i, $i = 1, 2, \ldots, m$, and the initial conditions $a_{i,l}$, $i, l = 1, 2, \ldots, m$, are positive real numbers. We perfect and generalize the results in related literature.

2. Main Results

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of real numbers, respectively.

Let $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)})\}_{n=-m+1}^{\infty}$ be a positive solution of (9). If we set

$$y_{n-m+1}^{(i)} = \frac{1}{x_{n-m+1}^{(i)}}, \quad i = 1, 2, \ldots, m; \; n \in \mathbb{N}, \quad (10)$$

then (9) translates into

$$y_{n+1}^{(i)} = \alpha_i y_{n-m+1}^{(i)} + \frac{A_i}{\prod_{j=0}^{m-2} y_{n-j}^{(i+j+1)}},$$
$$y_{n+1}^{(i+m)} = y_{n+1}^{(i)},$$
$$y_{1-l}^{(i+l)} = b_{i,l}, \quad (11)$$
$$A_{i+m} = A_i,$$
$$\alpha_{i+m} = \alpha_i,$$
$$i, l = 1, 2, \ldots, m; \; n \in \mathbb{N},$$

where $b_{i,l} = 1/a_{i,l}$, $i, l = 1, 2, \ldots, m$.

For convenience, in the following we will investigate (11). Set

$$I_{i,n} = \prod_{l=0}^{m-1} y_{n-l}^{(i+l)},$$
$$\alpha = \prod_{l=0}^{m-1} \alpha_{i+l}, \quad (12)$$
$$i = 1, 2, \ldots, m; \; n \in \mathbb{N},$$

$$Q_{i,j} = \sum_{l=0}^{j-1} \left(\prod_{s=0}^{l-1} \alpha_{i+s} A_{i+l} \right), \quad i, j = 1, 2, \ldots, m, \quad (13)$$

where we appeal to the convention $\prod_{s=0}^{-1} \alpha_{i+s} := 1$.

Combing (12) with (11), we get

$$I_{i+km,n} = I_{i,n}, \quad i = 1,2,\ldots,m;\; k \in \mathbb{N};\; n \in \mathbb{N}. \tag{14}$$

By (11), (12), and (13), we get

$$Q_{i+km,j} = Q_{i,j}, \quad i,j = 1,2,\ldots,m;\; k \in \mathbb{N}, \tag{15}$$

$$Q_{i,j+1} = \alpha_i Q_{i+1,j} + A_i, \qquad i = 1,2,\ldots,m;\; j = 1,2,\ldots,m-1, \tag{16}$$

$$Q_{i,m} + \alpha A_i = \alpha_i Q_{i+1,m} + A_i, \quad i = 1,2,\ldots,m. \tag{17}$$

Lemma 1. *Let* $\{(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)})\}_{n=-m+1}^{\infty}$ *be a positive solution of (11); then*

$$I_{i,n+1} = \alpha_i I_{i+1,n} + A_i, \quad i = 1,2,\ldots,m;\; n \in \mathbb{N}. \tag{18}$$

Proof. From (12) we know

$$I_{i,n+1} = \prod_{l=0}^{m-1} y_{n-l+1}^{(i+l)} = \frac{y_{n+1}^{(i)}}{y_{n-m+1}^{(i+m)}} \prod_{l=0}^{m-1} y_{n-l}^{(i+l+1)} = \frac{y_{n+1}^{(i)}}{y_{n-m+1}^{(i)}} I_{i+1,n}, \tag{19}$$

from (11) we obtain

$$\frac{y_{n+1}^{(i)}}{y_{n-m+1}^{(i)}} = \alpha_i + \frac{A_i}{I_{i+1,n}}, \tag{20}$$

and combining (19) with (20) we get the conclusion.

This completes the proof. □

Lemma 2. *Let* $\{(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)})\}_{n=-m+1}^{\infty}$ *be a positive solution of (11); then*

$$I_{i,(n+1)m+j-1} = \alpha I_{i,nm+j-1} + Q_{i,m}, \qquad i,j = 1,2,\ldots,m;\; n \in \mathbb{N}. \tag{21}$$

Proof. For $i,j = 1,2,\ldots,m$; $n \in \mathbb{N}$, by (18), (13), and (14), we have

$$\begin{aligned} I_{i,(n+1)m+j-1} &= \alpha_i I_{i+1,(n+1)m+j-2} + A_i \\ &= \alpha_i \left(\alpha_{i+1} I_{i+2,(n+1)m+j-3} + A_{i+1}\right) + A_i \\ &= \prod_{l=0}^{1} \alpha_{i+l} I_{i+2,(n+1)m+j-3} + Q_{i,2} \\ &= \prod_{l=0}^{1} \alpha_{i+l} \left(\alpha_{i+2} I_{i+3,(n+1)m+j-4} + A_{i+2}\right) \\ &\quad + Q_{i,2} = \prod_{l=0}^{2} \alpha_{i+l} I_{i+3,(n+1)m+j-4} + Q_{i,3} \\ &= \cdots = \prod_{l=0}^{m-1} \alpha_{i+l} I_{i+m,nm+j-1} + Q_{i,m} \\ &= \alpha I_{i,nm+j-1} + Q_{i,m}. \end{aligned} \tag{22}$$

Hence, (21) holds. □

Lemma 3. *Let* $\{(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)})\}_{n=-m+1}^{\infty}$ *be a positive solution of (11); then*

$$I_{i,nm+j-1} = \alpha^n I_{i,j-1} + \left(\sum_{l=0}^{n-1} \alpha^l\right) Q_{i,m}, \qquad i,j = 1,2,\ldots,m;\; n \in \mathbb{N}, \tag{23}$$

where we appeal to the convention $\sum_{l=0}^{-1} \alpha^l := 0$.

Proof. We will prove the conclusion by induction. For $i,j = 1,2,\ldots,m$, $n = 0$, it is obvious that (23) holds. For $i,j = 1,2,\ldots,m$, $n = 1$, from Lemma 2, we know that (23) holds.

Suppose that (23) holds for $n = k$, then for $n = k+1$, by Lemma 2 we have

$$\begin{aligned} I_{i,(k+1)m+j-1} &= \alpha I_{i,km+j-1} + Q_{i,m} \\ &= \alpha^{k+1} I_{i,j-1} + \alpha \left(\sum_{l=0}^{k-1} \alpha^l\right) Q_{i,m} + Q_{i,m} \\ &= \alpha^{k+1} I_{i,j-1} + \left(\sum_{l=0}^{k} \alpha^l\right) Q_{i,m}. \end{aligned} \tag{24}$$

Hence, (23) holds for $n = k+1$, from which we get the conclusion. □

In the following, set

$$r(n) = \begin{cases} 0, & \text{when } n+1 \bmod m = 0; \\ (n+1 \bmod m) - m, & \text{when } n+1 \bmod m \neq 0, \end{cases}$$

$$p_n = \begin{cases} \left\lfloor \dfrac{n+1}{m} \right\rfloor - 1, & \text{when } n+1 \bmod m = 0; \\ \left\lfloor \dfrac{n+1}{m} \right\rfloor, & \text{when } n+1 \bmod m \neq 0, \end{cases} \tag{25}$$

where $\lfloor \cdot \rfloor$ is floor function.

Lemma 4. *Let* $\{(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)})\}_{n=-m+1}^{\infty}$ *be a positive solution of (11); then*

$$y_{n+1}^{(i)} = y_{r(n)}^{(i)} \prod_{l=0}^{p_n} \left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,m+r(n)-1} + \left(\sum_{s=0}^{l-1} \alpha^s\right) Q_{i+1,m}}\right), \qquad i = 1,2,\ldots,m;\; n \in \mathbb{N}. \tag{26}$$

Proof. In fact, for $i = 1, 2, \ldots, m$; $n \in \mathbb{N}$ by (11) and Lemma 3 we have

$$y_{n+1}^{(i)} = y_{n-m+1}^{(i)}\left(\alpha_i + \frac{A_i}{I_{i+1,n}}\right) = y_{n-2m+1}^{(i)}\left(\alpha_i + \frac{A_i}{I_{i+1,n-m}}\right)\left(\alpha_i + \frac{A_i}{I_{i+1,n}}\right) = \cdots = y_{r(n)}^{(i)}\prod_{l=0}^{p_n}\left(\alpha_i + \frac{A_i}{I_{i+1,n-lm}}\right) = y_{r(n)}^{(i)}\prod_{l=0}^{p_n}\left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,m+r(n)-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right). \quad (27)$$

Hence, (26) holds. □

In the following, set

$$\sum_{l=0}^{\infty}\ln\left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,j-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) := \eta_{i,m-j}, \quad i, j = 1, 2, \ldots, m. \quad (28)$$

It is obvious that $\eta_{i,m-j} \in \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$.

Lemma 5. *For $i = 1, 2, \ldots, m$, the following statements are true.*

(1) *Suppose that $\alpha = 1$, $\alpha_i = 1$ or $\alpha \ge 1$, $\alpha_i > 1$ or $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i > 0$; then $\eta_{i,m-j} = +\infty$.*

(2) *Suppose that $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i = 0$ or $\alpha > 1$, $\alpha_i = 1$; then $\eta_{i,m-j} \in \mathbb{R}$.*

(3) *Suppose that $\alpha \ge 1$, $\alpha_i < 1$ or $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i < 0$; then $\eta_{i,m-j} = -\infty$.*

Proof. (1) Case 1. $\alpha = 1$, $\alpha_i = 1$. Note that

$$\ln\left(1 + \frac{A_i}{I_{i+1,j-1} + lQ_{i+1,m}}\right) \sim \frac{A_i}{I_{i+1,j-1} + lQ_{i+1,m}}, \quad \text{as } l \longrightarrow \infty, \quad (29)$$
$$\sum_{l=0}^{\infty}\frac{A_i}{I_{i+1,j-1} + lQ_{i+1,m}} = +\infty.$$

It follows that $\eta_{i,m-j} = +\infty$.

Case 2. $\alpha \ge 1$, $\alpha_i > 1$. Note that

$$\lim_{l\to+\infty}\ln\left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,j-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) = \ln(\alpha_i) > 0. \quad (30)$$

Hence, $\eta_{i,m-j} = +\infty$.

Case 3. $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i > 0$. By (16) and (17), we have

$$\alpha_i + \frac{A_i(1-\alpha)}{Q_{i+1,m}} = \frac{\alpha_i Q_{i+1,m} + A_i - \alpha A_i}{Q_{i+1,m}} = \frac{Q_{i,m}}{Q_{i+1,m}}. \quad (31)$$

$$\begin{aligned}Q_{i,m} - Q_{i+1,m} &= \sum_{l=0}^{m-1}\left(\prod_{p=0}^{l-1}\alpha_{i+p}A_{i+l}\right) - \sum_{l=0}^{m-1}\left(\prod_{p=0}^{l-1}\alpha_{i+p+1}A_{i+l+1}\right) \\ &= \left(1 - \frac{\alpha}{\alpha_i}\right)A_i + (\alpha_i - 1)\sum_{l=0}^{m-2}\left(\prod_{p=0}^{l-1}\alpha_{i+p+1}A_{i+l+1}\right) \\ &= \left(1 - \frac{\alpha}{\alpha_i}\right)A_i + (\alpha_i - 1)Q_{i+1,m-1} \\ &= \frac{1}{\alpha_i}\left[(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i\right],\end{aligned} \quad (32)$$

That is,

$$Q_{i,m} - Q_{i+1,m} = \frac{1}{\alpha_i}\left[(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i\right]. \quad (33)$$

When $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i > 0$, combining (31) with (33) we get

$$\lim_{l\to+\infty}\ln\left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,j-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) = \ln\left(\alpha_i + \frac{A_i(1-\alpha)}{Q_{i+1,m}}\right) > 0. \quad (34)$$

Hence, $\eta_{i,m-j} = +\infty$.

(2) Case 1. $\alpha < 1$, $(\alpha_i - 1)Q_{i,m} + (1-\alpha)A_i = 0$. In this case, by (33) we know $Q_{i,m} = Q_{i+1,m}$ and

$$\alpha_i + \frac{A_i}{\alpha^l I_{i+1,j-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}} = 1 + \frac{(\alpha_i - 1)\alpha^l\left((1-\alpha)I_{i+1,j-1} + Q_{i+1,m}\right)}{(1-\alpha)\alpha^l I_{i+1,j-1} + (1-\alpha^l)Q_{i+1,m}}. \quad (35)$$

Hence,

$$\ln\left(\alpha_i + \frac{A_i}{\alpha^l I_{i+1,j-1} + \left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) \sim \frac{(\alpha_i - 1)\alpha^l\left((1-\alpha)I_{i+1,j-1} - Q_{i+1,m}\right)}{(1-\alpha)\alpha^l I_{i+1,j-1} + (1-\alpha^l)Q_{i+1,m}} \quad \text{as } l \longrightarrow \infty. \quad (36)$$

Note that the series $\sum_{l=0}^{\infty}((\alpha_i-1)\alpha^l((1-\alpha)I_{i+1,j-1}-Q_{i+1,m})/((1-\alpha)\alpha^l I_{i+1,j-1}+(1-\alpha^l)Q_{i+1,m}))$ is convergent, and we have $\eta_{i,m-j}\in\mathbb{R}$.

Case 2. $\alpha>1$, $\alpha_i=1$. Since

$$\ln\left(1+\frac{A_i}{\alpha^l I_{i+1,j-1}+\left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) \sim \frac{A_i}{\alpha^l I_{i+1,j-1}+\left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}, \quad \text{as } l\longrightarrow\infty. \tag{37}$$

The series $\sum_{l=0}^{\infty}(A_i/(\alpha^l I_{i+1,j-1}+(\sum_{s=0}^{l-1}\alpha^s)Q_{i+1,m}))$ is convergent, and we know that $\eta_{i,m-j}\in\mathbb{R}$.

(3) Case 1. $\alpha\geq1$, $\alpha_i<1$. Note that

$$\lim_{l\to+\infty}\ln\left(\alpha_i+\frac{A_i}{\alpha^l I_{i+1,j-1}+\left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) = \ln(\alpha_i)<0. \tag{38}$$

Hence, $\eta_{i,m-j}=-\infty$.

Case 2. $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i<0$. Combining (31) with (33) we get

$$\lim_{l\to+\infty}\ln\left(\alpha_i+\frac{A_i}{\alpha^l I_{i+1,j-1}+\left(\sum_{s=0}^{l-1}\alpha^s\right)Q_{i+1,m}}\right) = \ln\left(\alpha_i+\frac{A_i(1-\alpha)}{Q_{i+1,m}}\right)<0. \tag{39}$$

Hence, $\eta_{i,m-j}=-\infty$. □

Theorem 6. *Let $\{(y_n^{(1)},y_n^{(2)},\ldots,y_n^{(m)})\}_{n=-m+1}^{\infty}$ be a positive solution of (11). The following statements are true.*

(1) *Suppose that $\alpha=1$, $\alpha_i=1$ or $\alpha\geq1$, $\alpha_i>1$ or $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i>0$; then $\lim_{n\to\infty}y_{n+1}^{(i)}=+\infty$, $i=1,2,\ldots,m$.*

(2) *Suppose that $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i=0$ or $\alpha>1$, $\alpha_i=1$; then $\lim_{k\to\infty}y_{km-j+1}^{(i)}=y_{-j+1}^{(i)}\exp(\eta_{i,j-1})$, $i,j=1,2,\ldots,m$.*

(3) *Suppose that $\alpha\geq1$, $\alpha_i<1$ or $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i<0$; then $\lim_{n\to\infty}y_{n+1}^{(i)}=0$, $i=1,2,\ldots,m$.*

Proof. By Lemma 4 and (28) we know

$$\lim_{k\to\infty}\ln\left(y_{km-j+1}^{(i)}\right)=\ln\left(y_{-j+1}^{(i)}\right)+\eta_{i,j-1}. \tag{40}$$

The conclusion follows by Lemma 5. □

Theorem 7. *Let $\{(x_n^{(1)},x_n^{(2)},\ldots,x_n^{(m)})\}_{n=-m+1}^{\infty}$ be a positive solution of (9). The following statements are true.*

(1) *Suppose that $\alpha=1$, $\alpha_i=1$ or $\alpha\geq1$, $\alpha_i>1$ or $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i>0$; then $\lim_{n\to\infty}x_{n+1}^{(i)}=0$, $i=1,2,\ldots,m$.*

(2) *Suppose that $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i=0$ or $\alpha>1$, $\alpha_i=1$; then $\lim_{k\to\infty}x_{km-j+1}^{(i)}=x_{-j+1}^{(i)}\exp(-\eta_{i,j-1})$, $i,j=1,2,\ldots,m$.*

(3) *Suppose that $\alpha\geq1$, $\alpha_i<1$ or $\alpha<1$, $(\alpha_i-1)Q_{i,m}+(1-\alpha)A_i<0$; then $\lim_{n\to\infty}x_{n+1}^{(i)}=+\infty$, $i=1,2,\ldots,m$.*

Proof. The proof follows by Theorem 6 and (10). □

3. Numerical Results

In this section, we give some numerical simulations to illustrate our results. Consider the following system of difference equations:

$$x_{n+1}^{(i)}=\frac{x_{n-2}^{(i)}}{A_i\prod_{j=0}^{2}x_{n-j}^{(i+j+1)}+\alpha_i}, \quad A_{i+3}=A_i, \quad \alpha_{i+3}=\alpha_i, \quad i=1,2,3;\ n=3,4,5,\ldots. \tag{41}$$

For convenience, set $\Theta=(\alpha_1,\alpha_2,\alpha_3)$, $\Xi=(A_1,A_2,A_3)$ and $\Lambda=(x_1^{(1)},x_1^{(2)},x_1^{(3)},x_2^{(1)},x_2^{(2)},x_2^{(3)},x_3^{(1)},x_3^{(2)},x_3^{(3)})$.

Example 1. In (41), we take $\Theta=(1.5,1.3,1.8)$, $\Xi=(3,4,2)$, $\Lambda=(1,4,7,3,6,9,2,5,8)$. From Table 1 and Figure 1(a) we see that

$$\lim_{n\to\infty}x_{n+1}^{(1)}=0, \quad \lim_{n\to\infty}x_{n+1}^{(2)}=0, \quad \lim_{n\to\infty}x_{n+1}^{(3)}=0. \tag{42}$$

Example 2. In (41), we take $\Theta=(1.2,1,1.1)$, $\Xi=(2,5,3)$, $\Lambda=(0.5,0.3,0.9,0.7,1,0.2,0.9,0.7,0.3)$. From Table 2 and Figure 1(b) we see that

$$\operatorname{Lim}_{n\to\infty}x_{n+1}^{(1)}=0, \quad \lim_{k\to\infty}x_{3k+j}^{(2)}=x_j^{(2)}\exp\left(-\eta_{2,3-j}\right), \quad j=1,2,3, \quad \lim_{n\to\infty}x_{n+1}^{(3)}=0. \tag{43}$$

Example 3. In (41), we take $\Theta=(1,1,2)$, $\Xi=(3,4,6)$, $\Lambda=(1,4,5.5,2,5,8,3,6,7)$. From Table 3 and Figure 1(c) we see that

$$\lim_{k\to\infty}x_{3k+j}^{(1)}=x_j^{(1)}\exp\left(-\eta_{1,3-j}\right), \quad \lim_{k\to\infty}x_{3k+j}^{(2)}=x_j^{(2)}\exp\left(-\eta_{2,3-j}\right), \quad j=1,2,3, \quad \lim_{n\to\infty}x_{n+1}^{(3)}=0. \tag{44}$$

Table 1

n	95	96	97	98	99	100
$x_n^{(1)}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_n^{(2)}$	0.0006	0.0009	0.0000	0.0005	0.0007	0.0000
$x_n^{(3)}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2

n	95	96	97	98	99	100
$x_n^{(1)}$	0.0020	0.0022	0.0011	0.0017	0.0018	0.0009
$x_n^{(2)}$	0.2396	0.4081	0.1478	0.2396	0.4081	0.1478
$x_n^{(3)}$	0.0073	0.0118	0.0085	0.0066	0.0108	0.0077

Table 3

n	95	96	97	98	99	100
$x_n^{(1)}$	0.9344	2.0000	0.0052	0.9344	2.0000	0.0052
$x_n^{(2)}$	2.4481	3.8536	0.0132	2.4481	3.8536	0.0132
$x_n^{(3)}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example 4. In (41), we take $\Theta = (1, 0.9, 1.2)$, $\Xi = (4, 0.4, 0.3)$, $\Lambda = (1, 0.5, 0.6, 0.8, 0.8, 0.3, 0.6, 0.2, 0.5)$. From Table 4 and Figure 1(d) we see that

$$\lim_{k\to\infty} x_{3k+j}^{(1)} = x_j^{(1)} \exp\left(-\eta_{1,3-j}\right), \quad j = 1, 2, 3,$$
$$\lim_{n\to\infty} x_{n+1}^{(2)} = +\infty, \tag{45}$$
$$\lim_{n\to\infty} x_{n+1}^{(3)} = 0.$$

Example 5. In (41), we take $\Theta = (10/9, 0.8, 9/8)$, $\Xi = (5, 1, 7)$, $\Lambda = (7, 0.4, 10, 2, 0.9, 5, 5, 0.6, 9)$. From Table 5 and Figure 2(a) we see that

$$\lim_{k\to\infty} x_{3k+j}^{(1)} = x_j^{(1)} \exp\left(-\eta_{1,3-j}\right), \quad j = 1, 2, 3,$$
$$\lim_{n\to\infty} x_{n+1}^{(2)} = +\infty, \tag{46}$$
$$\lim_{n\to\infty} x_{n+1}^{(3)} = 0.$$

Example 6. In (41), we take $\Theta = (1, 0.8, 1.25)$, $\Xi = (2, 0.1, 4)$, $\Lambda = (20, 0.8, 30, 15, 0.5, 18, 10, 0.2, 25)$. From Table 6 and Figure 2(b) we see that

$$\lim_{n\to\infty} x_{n+1}^{(1)} = 0,$$
$$\lim_{n\to\infty} x_{n+1}^{(2)} = +\infty, \tag{47}$$
$$\lim_{n\to\infty} x_{n+1}^{(3)} = 0.$$

Table 4

n	195	196	197	198	199	200
$x_n^{(1)}$	0.0551	0.2586	0.0801	0.0551	0.2586	0.0801
$x_n^{(2)}$	150.6878	359.6003	566.9402	167.4252	399.5398	629.9081
$x_n^{(3)}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5

n	95	96	97	98	99	100
$x_n^{(1)}$	0.0035	0.0315	0.0006	0.0031	0.0280	0.0005
$x_n^{(2)}$	570.3772	418.8940	35.3935	710.9386	522.1435	44.1177
$x_n^{(3)}$	0.0112	0.0233	0.0001	0.0098	0.0204	0.0001

Table 6

n	95	96	97	98	99	100
$x_n^{(1)}$	0.1805	1.8961	0.0333	0.1783	1.8737	0.0329
$x_n^{(2)}$	460.0534	187.1326	24.3153	574.7237	233.7778	30.3762
$x_n^{(3)}$	0.0010	0.0014	0.0000	0.0008	0.0011	0.0000

Table 7

n	75	76	77	78	79	80
$x_n^{(1)}$	0.1590	0.0000	0.9893	0.1387	0.0000	0.8631
$x_n^{(2)}$	422.5695	0.0031	195.9576	491.4949	0.0036	227.9814
$x_n^{(3)}$	6.6679	0.0007	5.4747	6.5355	0.0007	5.3663

Table 8

n	45	46	47	48	49	50
$x_n^{(1)}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$x_n^{(2)}$	0.6839	0.1089	1.0899	0.6839	0.1089	1.0899
$x_n^{(3)}$	33410	4620	33100	69610	9620	68960

Example 7. In (41), we take $\Theta = (1.1, 0.8, 1)$, $\Xi = (2, 3, 1)$, $\Lambda = (20, 50, 20, 80, 58, 18, 10, 60, 16)$. From Table 7 and Figure 3(a) we see that

$$\lim_{n\to\infty} x_{n+1}^{(1)} = 0,$$
$$\lim_{n\to\infty} x_{n+1}^{(2)} = +\infty, \tag{48}$$
$$\lim_{n\to\infty} x_{n+1}^{(3)} = 0.$$

Example 8. In (41), we take $\Theta = (2, 0.5, 0.4)$, $\Xi = (1, 6, 2)$, $\Lambda = (0.3, 3, 1, 0.7, 5, 1.2, 0.5, 2, 1.4)$. From Table 8 and Figure 3(b) we see that

$$\lim_{n\to\infty} x_{n+1}^{(1)} = 0,$$
$$\lim_{k\to\infty} x_{3k+j}^{(2)} = x_j^{(2)} \exp\left(-\eta_{2,3-j}\right), \quad j = 1, 2, 3, \tag{49}$$
$$\lim_{n\to\infty} x_{n+1}^{(3)} = +\infty.$$

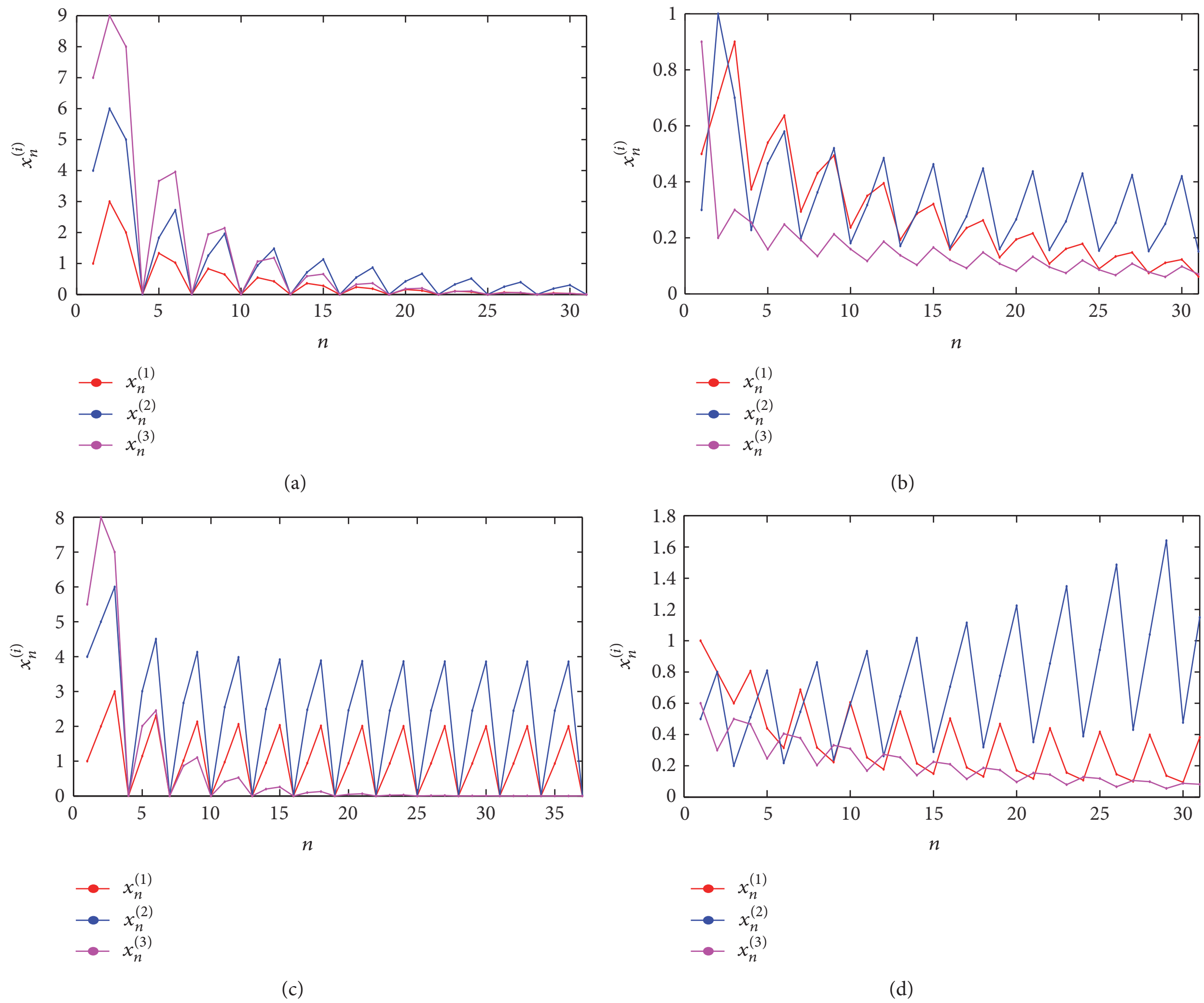


Figure 1: $\alpha > 1$.

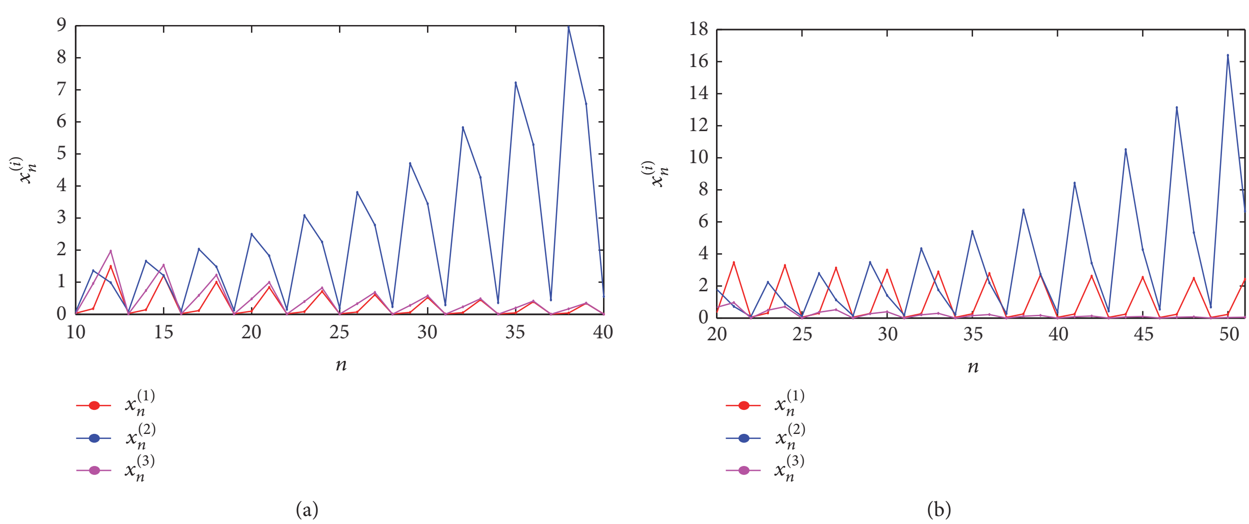
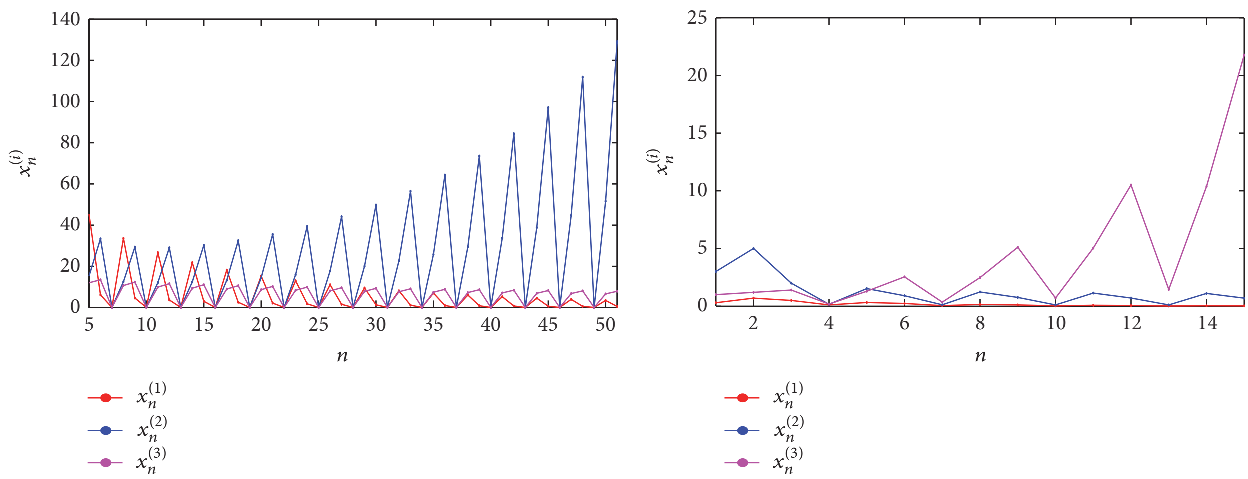


Figure 2: $\alpha = 1$.



(a) (b)

FIGURE 3: $\alpha < 1$.

Competing Interests

The authors declare that they have no competing interests.

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