Research Article

Discrete Waveform Relaxation Method for Linear Fractional Delay Differential-Algebraic Equations

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Fractional order delay differential-algebraic equations have the characteristics of time lag and memory and constraint limit. These yield some difficulties in the theoretical analysis and numerical computation. In this paper, we are devoted to solving them by the waveform relaxation method. The corresponding convergence results are obtained, and some numerical examples show the efficiency of the method.

1. Introduction

Fractional delay differential-algebraic equations are composed of fractional delay differential equations and algebraic equations, and they are more accurate in describing some scientific and engineering problems with memory function and algebraic restrictions. This kind of mathematical model is widely applied in many fields of electromagnetism, electrical systems, and biological materials. Nowadays, for the fractional differential equations, the main works focus on obtaining the analytic solution, approximately analytical solution, and numerical solution. For example, the Laplace transform method [1, 2], Fourier transform method [3, 4], and power method [5] are used to obtain the analytic solution. Furthermore, approximate analytical methods have been developed, such as variational iteration method [6, 7], Adomian decomposition method [8], homotopy analysis method [9], and multiple time scale method [10]. As for the studies of numerical methods, scholars discuss the linear multistep method [11], predictor corrector method [12, 13], Adomian method [14–16], spectrum method, and the finite element method [17, 18] for solving the fractional differential equations.

The waveform relaxation (WR) method is proposed by Learsmee and so forth [19]. The main properties of the method are flexibility and good parallelism, which is used to solve differential equations [20, 21], fractional differential equations [22, 23], delay differential equations [24, 25], and differential-algebraic equations [26–28]. The waveform relaxation method based on multisplitting is proposed to improve parallelism [29, 30]. Furthermore, acceleration techniques are used to increase the speed of iteration, such as Krylov subspace acceleration, strange-type preconditioners, and windowing for the WR method [20, 21, 31, 32].

The fractional differential-algebraic equations have received much attention; nevertheless, the numerical methods in this field are still young; a few studies have been considered on the convergence of the numerical methods, such as variational iteration method, Adomian decomposition method, and fractional differential transform method [33, 34]. Ding and Jiang applied the WR method to solve fractional differential-algebraic equations and obtain the convergence results [35]. But it is difficult to verify convergence conditions. The variational iteration method for the fractional delay integrodifferential-algebraic equations has been studied in [36], but it is not suitable for long-time numerical calculation. In order to overcome this limitation, according to the characteristics of the problems, the discrete WR method is employed to solve the linear fractional delay differential-algebraic equations by constructing the efficient iterative method and obtain the convergence results which is much easier to achieve.

2. Convergence

Consider the *Caputo* fractional delay differential-algebraic system

$$M \, {}_a^C D_0^\alpha x(t) + Ax(t) + By(t) + Cx(t-\tau) + Dy(t-\tau)$$

$$= U(t), \quad t \in [0, T],$$

$$Ex(t) + Ny(t) + Fx(t-\tau) + Gy(t-\tau) = V(t), \quad (1)$$

$$t \in [0, T],$$

$$x(0) = y(0) = 0, \quad t \in [0, T],$$

where $D_*^\alpha x(t)$ denotes the *Caputo* derivative of order α, $n_1 - 1 < \alpha \leq n_1$, $n_1 \in N$, $\varphi(t)$, $\psi(t)$ are smooth vector functions, and $\varphi(0) = 0$, $\psi(0) = 0$, $M \in \Re^{n \times n}$, $N \in \Re^{d \times d}$ are nonsingular matrix, $A, C \in \Re^{n \times n}$, $B, D \in \Re^{n \times d}$, $E, F \in \Re^{d \times n}$, $G \in \Re^{d \times d}$, and $x(t) \in C^n[0, T]$, $y(t) \in \Re^d$, $n + d = s$, $U(t) \in \Re^n$, $V(t) \in \Re^d$ are input function.

Applying the WR method for the coefficient matrix of (1), we get the iteration form

$$M_1 \, {}_a^C D_0^\alpha x^{(k+1)}(t) + A_1 x^{(k+1)}(t) + B_1 y^{(k+1)}(t)$$

$$+ C_1 x^{(k+1)}(t-\tau) + D_1 y^{(k+1)}(t-\tau)$$

$$= M_2 \, {}_a^C D_0^\alpha x^{(k)}(t) + A_2 x^{(k)}(t) + B_2 y^{(k)}(t)$$

$$+ C_2 x^{(k)}(t-\tau) + D_2 y^{(k)}(t-\tau) + U(t),$$

$$E_1 x^{(k+1)}(t) + N_1 y^{(k+1)}(t) + F_1 x^{(k+1)}(t-\tau)$$

$$+ G_1 y^{(k+1)}(t-\tau) = E_2 x^{(k)}(t) + N_2 y^{(k)}(t) \quad (2)$$

$$+ F_2 x^{(k)}(t-\tau) + G_2 y^{(k)}(t-\tau) + V(t),$$

$$x^{(k+1)}(t) = x^{(k)}(t) = \cdots = x^{(0)}(t) = \varphi(t),$$

$$t \in [-\tau, 0],$$

$$y^{(k+1)}(t) = y^{(k)}(t) = \cdots = y^{(0)}(t) = \psi(t),$$

$$t \in [-\tau, 0],$$

$$k = 0, 1, \ldots,$$

where $A = A_1 - A_2$, $B = B_1 - B_2$, $C = C_1 - C_2$, $D = D_1 - D_2$, $E = E_1 - E_2$, $F = F_1 - F_2$, $G = G_1 - G_2$, $M = M_1 - M_2$, $N = N_1 - N_2$, and M_1, N_1 are nonsingular matrix. We assume that the step is h, $\tau = mh$, $m \geq 1$, $t_{p-j} = t_p - jh$, $p \geq 1$, $x_{p-j}^{(k)}$ is approximating value of $x^{(k)}(t_{p-j})$, and $y_{p-j}^{(k)}$ is approximating value of $y^{(k)}(t_{p-j})$, $j = 0, 1, \ldots, p$.

Based on the fractional *Caputo* derivative of Grünwald-Letnikov discrete format

$$_a^C D_0^\alpha x(t) \doteq \frac{1}{h^\alpha} \sum_{j=0}^{p} \omega_j x_{p-j}, \quad (3)$$

where $\omega_j = (-1)^j \binom{\alpha}{j} = (-1)^j (\Gamma(j-\alpha)/\Gamma(-\alpha)\Gamma(j+1))$ and $\binom{\alpha}{0} = \Gamma(-\alpha)/\Gamma(-\alpha)\Gamma(1) = 1$, we obtain the iterative format

$$\left(M_1 + h^\alpha A_1\right) x_p^{(k+1)} + \sum_{j=1}^{p} \omega_j M_1 x_{p-j}^{(k+1)} + h^\alpha \overline{v}_n$$

$$= \left(M_2 + h^\alpha A_2\right) x_p^{(k)} + \sum_{j=1}^{p} \omega_j M_2 x_{p-j}^{(k)} + h^\alpha \overline{u}_n,$$

$$E_1 x_p^{(k+1)} + N_1 y_p^{(k+1)} + F_1 x_{p-m}^{(k+1)} + G_1 y_{p-m}^{(k+1)}$$

$$= E_2 x_p^{(k)} + N_2 y_p^{(k)} + F_2 x_{p-m}^{(k)} + G_2 y_{p-m}^{(k)} + V\left(t_p\right), \quad (4)$$

$$x^{(k+1)}(t) = x^{(k)}(t) = \cdots = x^{(0)}(t) = \varphi(t),$$

$$t \in [-\tau, 0],$$

$$y^{(k+1)}(t) = y^{(k)}(t) = \cdots = y^{(0)}(t) = \psi(t),$$

$$t \in [-\tau, 0],$$

$$p = 1, 2, \ldots, N, \quad k = 0, 1, \ldots,$$

where

$$\overline{v}_n = B_1 y_p^{(k+1)} + C_1 x_{p-m}^{(k+1)} + D_1 y_{p-m}^{(k+1)},$$

$$\overline{u}_n = B_2 y_p^{(k)} + C_2 x_{p-m}^{(k)} + D_2 y_{p-m}^{(k)} + U\left(t_p\right), \quad (5)$$

where N is the number of the time steps and $Nh = T$, the initial value $x_0^{(0)} = 0$. Introduce

$$X^{(k)} = \left[\left(X_1^{(k)}\right)^T, \left(X_2^{(k)}\right)^T, \ldots, \left(X_p^{(k)}\right)^T\right]^T,$$

$$Y^{(k)} = \left[\left(Y_1^{(k)}\right)^T, \left(Y_2^{(k)}\right)^T, \ldots, \left(Y_p^{(k)}\right)^T\right]^T. \quad (6)$$

Next, we will discuss two cases.

First Case. When $p < m$, (4) can be written as

$$\begin{bmatrix} H_1 & h^\alpha \left(I \otimes B_1\right) \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix} \begin{bmatrix} X^{(k+1)} \\ Y^{(k+1)} \end{bmatrix}$$

$$= \begin{bmatrix} H_2 & h^\alpha \left(I \otimes B_2\right) \\ I \otimes E_2 & I \otimes N_2 \end{bmatrix} \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix} + \begin{bmatrix} \widehat{Z} \\ \widehat{Q} \end{bmatrix}, \quad (7)$$

where $H_i = M_\alpha \otimes M_i + h^\alpha(I \otimes A_i)$, $i = 1, 2$, and

$$M_\alpha = \begin{bmatrix} w_0 & & & \\ w_1 & w_0 & & \\ \vdots & \ddots & \ddots & \\ w_{p-1} & \cdots & w_1 & w_0 \end{bmatrix}; \quad (8)$$

introduce $\widehat{Z} = [\hat{z}_1^T, \hat{z}_2^T, \ldots, \hat{z}_i^T, \ldots, \hat{z}_p^T]^T$, $\widehat{Q} = [\hat{q}_1^T, \hat{q}_2^T, \ldots,$ $\hat{q}_i^T, \ldots, \hat{q}_p^T]^T$, where

$$\hat{z}_i = h^\alpha \left(U\left(t_i\right) - C\varphi\left(t_{i-m}\right) - D\psi\left(t_{i-m}\right)\right),$$
$$i = 1, 2, \ldots, p, \quad (9)$$
$$\hat{q}_i = V\left(t_i\right) - F\varphi\left(t_{i-m}\right) - G\psi\left(t_{i-m}\right), \quad i = 1, 2, \ldots, p.$$

Let $\delta_1^{(k+1)} = [(X^{(k+1)} - X^{(k)})^T, (Y^{(k+1)} - Y^{(k)})^T]^T$, $k = 0, 1, \ldots$.

From (7), we have

$$\delta_1^{(k+1)} = \begin{bmatrix} H_1 & h^\alpha \left(I \otimes B_1\right) \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} H_2 & h^\alpha \left(I \otimes B_2\right) \\ I \otimes E_2 & I \otimes N_2 \end{bmatrix} \delta_1^{(k)}. \quad (10)$$

Let the iterative matrix be R_1 in (10); the convergence condition of the iterative process (4) is $\rho(R_1) < 1$. Moreover, we have the following theorem.

Theorem 1. *For the small step h and $p < m$, if*

$$\rho\left(M_1^{-1}M_2\right) < 1,$$
$$\rho\left(N_1^{-1}N_2\right) < 1, \quad (11)$$

then the discrete waveform relaxation iteration method (4) is convergent.

Proof. The iteration process (4) convergence is decided by $\overline{R}_1$. Firstly, the left coefficient matrix of (7) can be written as

$$\begin{bmatrix} H_1 & h^\alpha \left(I \otimes B_1\right) \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix}$$
$$= \begin{bmatrix} M_\alpha \otimes M_1 & 0 \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix} + h^\alpha \begin{bmatrix} I \otimes A_1 & I \otimes B_1 \\ 0 & 0 \end{bmatrix}, \quad (12)$$

because of the following equation:

$$\begin{bmatrix} M_\alpha \otimes M_1 & 0 \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \left(M_\alpha \otimes M_1\right)^{-1} & 0 \\ W_1 & \left(I \otimes N_1\right)^{-1} \end{bmatrix}, \quad (13)$$

where $W_1 = -(I \otimes N_1)^{-1}(I \otimes E_1)(M_\alpha \otimes M_1)^{-1}$; then,

$$\begin{bmatrix} H_1 & h^\alpha \left(I \otimes B_1\right) \\ I \otimes E_1 & I \otimes N_1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \left(M_\alpha \otimes M_1\right)^{-1} & 0 \\ W_1 & \left(I \otimes N_1\right)^{-1} \end{bmatrix}^{-1} + O\left(h^\alpha\right), \quad (14)$$

and we can get

$$R_1$$
$$= \left(\begin{bmatrix} \left(M_\alpha \otimes M_1\right)^{-1} & 0 \\ W_1 & \left(I \otimes N_1\right)^{-1} \end{bmatrix} + O\left(h^\alpha\right)\right)$$
$$\cdot \left(\begin{bmatrix} M_\alpha \otimes M_2 & 0 \\ I \otimes E_2 & I \otimes N_2 \end{bmatrix} + h^\alpha \begin{bmatrix} I \otimes A_2 & I \otimes B_2 \\ 0 & 0 \end{bmatrix}\right) \quad (15)$$
$$= \begin{bmatrix} \left(M_\alpha \otimes M_1^{-1}\right)\left(M_\alpha \otimes M_2\right) & 0 \\ \widetilde{W}_1 & \left(I \otimes N_1\right)^{-1}\left(I \otimes N_2\right) \end{bmatrix}$$
$$+ O\left(h^\alpha\right),$$

where $\widetilde{W}_1 = W_1(M_\alpha \otimes M_2) + (I \otimes N_1)^{-1}(I \otimes N_2)$; we have

$$\rho\left(\left(M_\alpha \otimes M_1\right)^{-1}\left(M_\alpha \otimes M_2\right)\right) = \rho\left(M_1^{-1}M_2\right). \quad (16)$$

Based on the matrix spectral radius definition of R_1 expression in (15), if

$$\rho\left(M_1^{-1}M_2\right) < 1,$$
$$\rho\left(N_1^{-1}N_2\right) < 1, \quad (17)$$

and h is small enough, we have $\rho(R_1) < 1$.

Therefore, when $p < m$, the convergence is proved. □

Second Case. When $p > m$, we assume $p = m + c$; then, system (4) can be written as

$$\begin{bmatrix} \overline{H}_1 & \overline{N}_1 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix} \begin{bmatrix} X^{(k+1)} \\ Y^{(k+1)} \end{bmatrix} = \begin{bmatrix} \overline{H}_2 & \overline{N}_2 \\ \overline{K}_2 & \overline{L}_2 \end{bmatrix} \begin{bmatrix} X^{(k)} \\ Y^{(k)} \end{bmatrix} + \begin{bmatrix} \widehat{S} \\ \widehat{T} \end{bmatrix}, \quad (18)$$

where

$$\overline{H}_i = M_\alpha \otimes M_i + h^\alpha \left(I \otimes A_i + \overline{P}_\xi \otimes C_i\right), \quad i = 1, 2,$$
$$\overline{N}_i = h^\alpha \left(I \otimes B_i + \overline{P}_\xi \otimes D_i\right), \quad i = 1, 2,$$
$$\overline{K}_i = I \otimes E_i + \overline{P}_\xi \otimes F_i, \quad i = 1, 2,$$
$$\overline{L}_i = I \otimes N_i + \overline{P}_\xi \otimes G_i, \quad i = 1, 2,$$

$$
\overline{P}_\xi = \begin{pmatrix} 0 & & & & & & \\ 0 & 0 & & & & & \\ \vdots & \ddots & \ddots & & & & \\ 1 & & 0 & 0 & & & \\ \vdots & \ddots & & \ddots & \ddots & & \\ 0 & \cdots & 1 & \cdots & 0 & 0 \end{pmatrix},
$$

$$(19)$$

where

$$
\begin{aligned}
\overline{P}_\xi &= (\xi_{ij}) \\
&= \begin{cases} 1, & i = m, m+1, \ldots, N, \ j = i - m + 1, \\ 0, & \text{others}; \end{cases}
\end{aligned}
$$

$$(20)$$

introduce $\widehat{S} = [\widehat{s}_1^T, \widehat{s}_2^T, \ldots, \widehat{s}_i^T, \ldots, \widehat{s}_p^T]^T$, $\widehat{T} = [\widehat{t}_1^T, \widehat{t}_2^T, \ldots, \widehat{t}_i^T, \ldots, \widehat{t}_p^T]^T$, where $\widehat{s}_i$, $\widehat{t}_i$ are

$$
\begin{aligned}
\widehat{s}_i &= \begin{cases} h^\alpha \left(U(t_i) - C\varphi(t_{i-m}) - D\psi(t_{i-m}) \right), & i = 1, 2, \ldots, m-1, \\ h^\alpha U(t_i), & i = m, m+1, \ldots, p, \end{cases} \\
\widehat{t}_i &= \begin{cases} V(t_i) - F\varphi(t_{i-m}) - G\psi(t_{i-m}), & i = 0, \ldots, m-1, \\ V(t_i), & i = m, m+1, \ldots, p. \end{cases}
\end{aligned}
$$

$$(21)$$

Let $\widetilde{\delta}_1^{(k+1)} = [(X^{(k+1)} - X^{(k)})^T, (Y^{(k+1)} - Y^{(k)})^T]^T$, $k = 0, 1, \ldots$, and from (18) we have

$$
\widetilde{\delta}_1^{(k+1)} = \begin{bmatrix} \overline{H}_1 & \overline{N}_1 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \overline{H}_2 & \overline{N}_2 \\ \overline{K}_2 & \overline{L}_2 \end{bmatrix} \widetilde{\delta}_1^{(k)}.
$$

$$(22)$$

Let the iterative matrix be R_2 in (22); then, the convergence condition of the iterative process (22) is $\rho(R_2) < 1$; we have the following theorem.

Theorem 2. *For the small step h, if it satisfies*

$$
\begin{aligned}
\rho\left(M_1^{-1} M_2\right) &< 1, \\
\rho\left(N_1^{-1} N_2\right) &< 1,
\end{aligned}
$$

$$(23)$$

then the discrete waveform relaxation iteration method (18) is convergent.

Proof. The iteration process (18) convergence is decided by R_2. Firstly, the left coefficient matrix of (18) can be written as

$$
\begin{aligned}
&\begin{bmatrix} \overline{H}_1 & \overline{N}_1 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix} \\
&= \begin{bmatrix} M_\alpha \otimes M_1 & 0 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix} \\
&\quad + h^\alpha \begin{bmatrix} I \otimes A_1 + \overline{P}_\xi \otimes C_1 & I \otimes B_1 + \overline{P}_\xi \otimes D_1 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

$$(24)$$

and because

$$
\begin{aligned}
&\overline{L}_1^{-1} \\
&= \begin{bmatrix} N_1^{-1} & & & & & \\ 0 & N_1^{-1} & & & & \\ \vdots & & \ddots & \ddots & & \\ -N_1^{-1} G_1 N_1^{-1} & & 0 & N_1^{-1} & & \\ & \ddots & & & \ddots & \ddots \\ 0 & & -N_1^{-1} G_1 N_1^{-1} & \cdots & 0 & N_1^{-1} \end{bmatrix} \\
&= (I \otimes N_1)^{-1} + \begin{pmatrix} 0 & 0 \\ \widetilde{L}_1 & 0 \end{pmatrix},
\end{aligned}
$$

$$(25)$$

where $\widetilde{L}_1 = -\overline{P}_\xi \otimes (-N_1^{-1} G_1 N_1^{-1})$, hence

$$
\begin{bmatrix} M_\alpha \otimes M_1 & 0 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix}^{-1} = \begin{bmatrix} (M_\alpha \otimes M_1)^{-1} & 0 \\ W_2 & \overline{L}_1^{-1} \end{bmatrix},
$$

$$(26)$$

where $W_2 = -L_1^{-1}(I \otimes E_1 + \overline{P}_\xi \otimes F_1)(M_\alpha \otimes M_1)^{-1}$; then,

$$
\begin{aligned}
\begin{bmatrix} \overline{H}_1 & \overline{N}_1 \\ \overline{K}_1 & \overline{L}_1 \end{bmatrix}^{-1} &= \begin{bmatrix} (M_\alpha \otimes M_1)^{-1} & 0 \\ \widehat{W}_2 & (I \otimes N_1)^{-1} \end{bmatrix} \\
&\quad + O(h^\alpha),
\end{aligned}
$$

$$(27)$$

where $\overline{W}_2 = W_2 + \widetilde{L}_1$; we have

$$
\begin{aligned}
R_2 &= \left(\begin{bmatrix} (M_\alpha \otimes M_1)^{-1} & 0 \\ W_2 & \overline{L}_1^{-1} \end{bmatrix} + O(h^\alpha) \right) \\
&\quad \cdot \left(\begin{bmatrix} M_\alpha \otimes M_2 & 0 \\ \overline{K}_2 & \overline{L}_2 \end{bmatrix} \right. \\
&\quad \left. + h^\alpha \begin{bmatrix} I \otimes A_2 + \overline{P}_\xi \otimes C_2 & I \otimes B_2 + \overline{P}_\xi \otimes D_2 \\ 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} (M_\alpha \otimes M_1^{-1})(M_\alpha \otimes M_2) & 0 \\ \widetilde{W}_2 & \overline{L}_1^{-1} \overline{L}_2 \end{bmatrix} + O(h^\alpha),
\end{aligned}
$$

$$(28)$$

where $\widetilde{W}_2 = \overline{W}_2(M_\alpha \otimes M_2) + \overline{L}_1^{-1} \overline{L}_2$, and because

$$\overline{L}_1^{-1}\overline{L}_2 = \begin{bmatrix} N_1^{-1}N_2 & & & & & \\ 0 & N_1^{-1}N_2 & & & & \\ \vdots & & \ddots & & \ddots & \\ -N_1^{-1}G_1 N_1^{-1}G_2 & & & 0 & & N_1^{-1}N_2 \\ & \ddots & & & \ddots & \ddots \\ 0 & & -N_1^{-1}G_1 N_1^{-1}G_2 & \cdots & 0 & N_1^{-1}N_2 \end{bmatrix} = (I \otimes N_1)^{-1}(I \otimes N_2) + \begin{pmatrix} 0 & 0 \\ \widetilde{L}_2 & 0 \end{pmatrix}, \quad (29)$$

where $\widetilde{L}_2 = -\overline{P}_\xi \otimes (N_1^{-1}G_1 N_2^{-1}G_2)$, therefore, we have

$$\rho\left(\overline{L}_1^{-1}\overline{L}_2\right) = \rho\left(N_1^{-1}N_2\right); \quad (30)$$

from (8), we have

$$\rho\left((M_\alpha \otimes M_1)^{-1}(M_\alpha \otimes M_2)\right) = \rho\left(M_1^{-1}M_2\right). \quad (31)$$

Based on the matrix spectral radius definition and combining R_2 expression of (28) and if

$$\begin{aligned} \rho\left(M_1^{-1}M_2\right) &< 1, \\ \rho\left(N_1^{-1}N_2\right) &< 1, \end{aligned} \quad (32)$$

and h is enough small, we have $\rho(R_2) < 1$.
Therefore, the convergence is proved. $\qquad\square$

3. Illustrative Examples

In this section, some illustrative examples are given to show the efficiency of the WR method for solving the linear fractional delay differential-algebraic equations.

Example 1. Consider the initial value problem of the linear fractional delay differential-algebraic equations

$$\begin{bmatrix} -5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} {}^C_a D_0^\alpha x(t) + \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0.3 & 0.5 \\ 0.001 & 0 & -0.5 \end{bmatrix} x(t)$$

$$+ \begin{bmatrix} 0.12 & 0.2 & 0 \\ 0 & 0.1 & -0.5 \\ 0 & 0 & 0.8 \end{bmatrix} x(t-0.5) = \begin{bmatrix} \sin(t) \\ \sin(t) \\ 0 \end{bmatrix}, \quad (33)$$

$$t \in [0,T],$$

$$x(t) = (0,0,0)^T, \quad t \in [-0.5,0],$$

where $\alpha = 0.5$, $x(t) = (x_1(t), x_2(t), x_3(t))^T$; introduce $\widetilde{x}(t) = (x_1(t), x_2(t))^T$, $y(t) = x_3(t)$; system (33) can be written as

$$M\,{}^C_a D_0^\alpha \widetilde{x}(t) + A\widetilde{x}(t) + By(t) + C\widetilde{x}(t-0.5)$$

$$+ Dy(t-0.5) = U(t), \quad t \in [0,T],$$

$$E\widetilde{x}(t) + Ny(t) + F\widetilde{x}(t-0.5) + Gy(t-0.5) = V(t),$$

$$t \in [0,T], \quad (34)$$

$$\widetilde{x}(t) = (0,0)^T,$$

$$y(t) = 0,$$

$$t \in [-0.5,0],$$

and we have

$$M = \begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.12 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.001 & 0 \end{bmatrix},$$

$$N = -0.5,$$

$$F = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$G = 0.8,$$

$$U(t) = \begin{bmatrix} \sin(t) & \sin(t) \end{bmatrix}^T,$$

$$V(t) = 0. \quad (35)$$

TABLE 1: The error of numerical solutions.

T	$T = 1$	$T = 2$	$T = 3$	$T = 4$
$\tilde{x}_1(t)$	$3.0981e - 003$	$1.1308e - 003$	$2.3020e - 003$	$3.8279e - 003$
$\tilde{x}_2(t)$	$4.1365e - 003$	$1.1548e - 003$	$3.5419e - 003$	$5.4337e - 003$
$y_1(t)$	$1.9550e - 005$	$3.4345e - 004$	$1.6464e - 003$	$5.455e - 003$

Splitting the coefficient matrix, we can construct the *Jacobi* waveform relaxation method and deduce

$$M_1 = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.12 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.001 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$F_1 = F_2 = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$N_1 = -\frac{0.5}{1.1},$$

$$N_2 = -\frac{0.05}{1.1},$$

TABLE 2: Between the iteration allowable error and the number of iterations.

T	1.0×10^{-4}	1.0×10^{-6}	1.0×10^{-8}
$T = 1$	5	7	10
$T = 2$	4	7	10
$T = 3$	4	7	10
$T = 4$	5	8	10
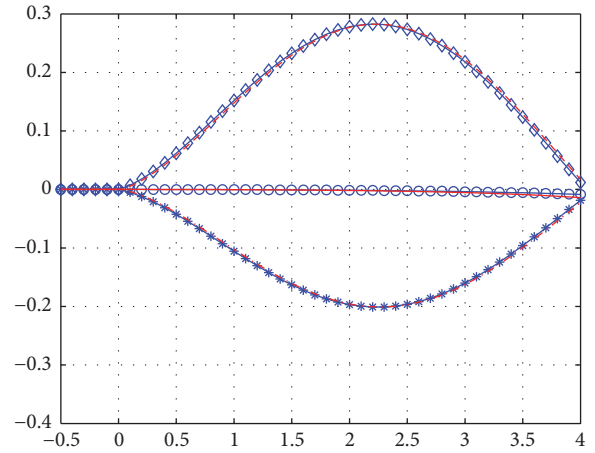
FIGURE 1: Numerical solutions by using the two methods for solving (33).

$$G_1 = 0.8,$$

$$G_2 = 0.$$

(36)

According to the iterative format (4), we take time step $h = 0.1$ and use the *Jacobi* waveform relaxation method to solve the initial value problem (33). Then, we take time step $h = 0.005$ and take the numerical solution of (33) obtained by the implicit Grünwald-Letnikov method as the approximate true solution. Then, the error of the numerical solution obtained by WR method is shown in Table 1.

The numerical solutions are plotted in Figure 1, where ∗, ◇, and ○, respectively, are numerical solutions of $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, and $y_1(t)$ by using the waveform relaxation.

Table 1 and Figure 1 show that the waveform relaxation method for obtaining the solution is accurate.

Table 2 shows that the iterative allowable error is smaller, and the number of corresponding iterations is higher. If the

iteration tolerance is the same, then the condition of iterations number is relatively stable.

Example 2. Consider the initial value problem of the linear fractional delay differential-algebraic equations

$$
\begin{bmatrix} 5 & 1 & 0 & 0 \\ 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} {}_a^C D_0^\alpha x(t) + \begin{bmatrix} 4 & 2 & 1 & 0.5 \\ 1 & 5 & 0.8 & 1.2 \\ 0.1 & 0.3 & 6 & 2 \\ 0.2 & 0.2 & 2 & 7 \end{bmatrix} x(t)
$$

$$
+ \begin{bmatrix} 3 & 1.2 & 4 & 1 \\ 1.4 & 5 & 2 & 6 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 4 \end{bmatrix} x(t-0.5) = \begin{bmatrix} t \\ \sin(t) \\ 1-\cos(t) \end{bmatrix},
\tag{37}
$$

$$
t \in [0, T],
$$

$$
x(t) = (0,0,0,0)^T, \quad t \in [-0.5, 0],
$$

where $\alpha = 0.5$, $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$; introduce $\tilde{x}(t) = (x_1(t), x_2(t))^T$, $y(t) = (x_3(t), x_4(t))^T$; system (37) can be written as

$$
M {}_a^C D_0^\alpha \tilde{x}(t) + A\tilde{x}(t) + By(t) + C\tilde{x}(t-0.5)
$$

$$
+ Dy(t-0.5) = U(t), \quad t \in [0, T],
$$

$$
E\tilde{x}(t) + Ny(t) + F\tilde{x}(t-0.5) + Gy(t-0.5) = V(t),
$$

$$
t \in [0, T], \tag{38}
$$

$$
\tilde{x}(t) = (0, 0)^T,
$$

$$
y(t) = (0, 0)^T,
$$

$$
t \in [-0.5, 0],
$$

and we have

$$
M = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix},
$$

$$
A = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 1 & 0.5 \\ 0.8 & 2 \end{bmatrix},
$$

$$
C = \begin{bmatrix} 3 & 1.2 \\ 1.4 & 5 \end{bmatrix},
$$

$$
D = \begin{bmatrix} 4 & 1 \\ 2 & 6 \end{bmatrix},
$$

$$
E = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.2 \end{bmatrix},
$$

$$
N = \begin{bmatrix} 6 & 2 \\ 2 & 7 \end{bmatrix},
$$

$$
F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
G = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix},
$$

$$
U(t) = \begin{bmatrix} t & \sin(t) \end{bmatrix}^T,
$$

$$
V(t) = 1 - \cos(t).
\tag{39}
$$

Splitting the coefficient matrix, we can construct the Gauss-Seidel waveform relaxation method and deduce

$$
M_1 = \begin{bmatrix} 5 & 0 \\ 2 & 6 \end{bmatrix},
$$

$$
M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},
$$

$$
B_1 = \begin{bmatrix} 1 & 0 \\ 0.8 & 1.2 \end{bmatrix},
$$

$$
B_2 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix},
$$

$$
C_1 = \begin{bmatrix} 3 & 0 \\ 1.4 & 5 \end{bmatrix},
$$

$$
C_2 = \begin{bmatrix} 0 & 1.2 \\ 0 & 0 \end{bmatrix},
$$

$$
D_1 = \begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix},
$$

$$
D_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

$$
E_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.2 \end{bmatrix},
$$

$$
E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
N_1 = \begin{bmatrix} 6 & 0 \\ 2 & 7 \end{bmatrix},
$$

TABLE 3: The error of numerical solutions.

T	$T = 1$	$T = 2$	$T = 3$	$T = 4$
$\tilde{x}_1(t)$	$4.1571e - 003$	$4.9982e - 003$	$4.7782e - 003$	$3.4877e - 003$
$\tilde{x}_2(t)$	$1.2799e - 003$	$1.2010e - 003$	$3.2493e - 003$	$3.0521e - 003$
$y_1(t)$	$4.1742e - 003$	$4.1408e - 003$	$1.8243e - 003$	$1.2053e - 003$
$y_2(t)$	$1.0734e - 003$	$1.2718e - 003$	$7.4072e - 004$	$1.1129e - 004$

TABLE 4: Between the iteration allowable error and the number of iterations.

T	1.0×10^{-4}	1.0×10^{-6}	1.0×10^{-8}
$T = 1$	4	5	7
$T = 2$	3	5	7
$T = 3$	4	6	7
$T = 4$	4	6	7

$$N_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

$$(40)$$

According to the iterative format (4), we take time step $h = 0.1$ and use the *Gauss-Seidel* waveform relaxation method to solve the initial value problem (37). Then, we take time step $h = 0.005$ and take the numerical solution of (37) obtained by the implicit Grünwald-Letnikov method as the approximate true solution. Then, the error of the numerical solution obtained by WR method is shown in Table 3.

The numerical solutions are plotted in Figure 2, where $*$, $\diamond$, and $\bigcirc$, respectively, are numerical solutions of $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, and $y_1(t)$ by using the waveform relaxation.

Table 3 and Figure 2 show that the waveform relaxation method for obtaining the solution is accurate.

Table 4 shows that the iterative allowable error is smaller, and the number of corresponding iterations is higher. If the iteration tolerance is the same, then the condition of iterations number is relatively stable.

4. Conclusion

In this paper, the discrete waveform relaxation method has been extended to solve linear fractional delay differential equations. Some illustrative examples show that this method
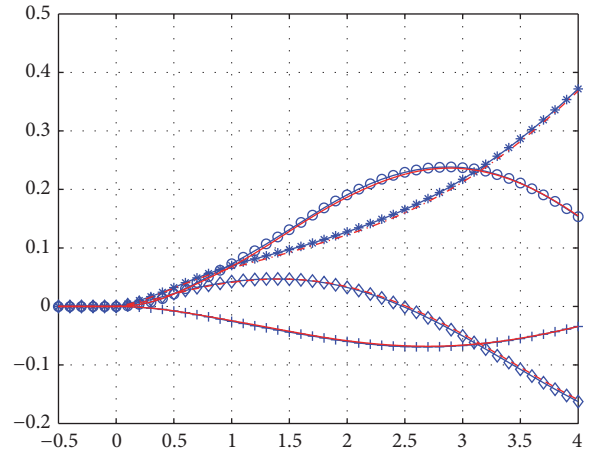

FIGURE 2: Numerical solutions by using the two methods for solving (34).

gives very good approximation to the exact solution. The discrete waveform relaxation method is a promising method to handle linear fractional delay differential-algebraic equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Acknowledgments

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