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*Research Article*

# Certain Nonlinear Integral Inequalities and Their Applications

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Several new retarded integral inequalities of Gronwall-Bellman-Pachpatte type are presented which generalize the inequalities to the more general nonlinear case in the literature and provide explicit bounds on unknown functions. These results include many existing ones as special cases and can be used as tools in the qualitative analysis of certain classes of integrodifferential equations.

## 1. Introduction

It is well known that integral inequalities are very useful to investigate the existence, uniqueness, boundedness, oscillation, and other qualitative properties of solutions to differential equations and integrodifferential equations [1–19]. One of the fundamental inequalities is the Gronwall inequality, which was established in 1919 by Gronwall [1]. As a generalization of Gronwall inequality, Gronwall-Bellman inequality plays a key role in studying stability and asymptotic behavior of solutions to differential equations and integrodifferential equations. An important nonlinear generalization of Gronwall-Bellman inequality is Bihari's inequality [2]. In recent years, there has been much research activity concerning linear and nonlinear generalizations of Gronwall-Bellman-Bihari type inequalities. To mention a few, Pachpatte [3] presented a new integral inequality and studied the boundedness properties of some linear integrodifferential equations, one of which we give below for the convenience of the reader.

**Theorem 1** (see Pachpatte [3]). *Let $u, f, g, h \in C(R_+, R_+)$, $R_+ = [0, \infty)$, and $u_0$ be a nonnegative constant. If*

$$u(t) \le u_0 + \int_0^t \left(f(s)\,u(s) + h(s)\right) ds + \int_0^t f(s) \left(\int_0^s g(\tau)\,u(\tau)\,d\tau\right) ds, \tag{1}$$

*then*

$$u(t) \le u_0 + \int_0^t \Bigg[h(s) + f(s) \cdot \left(u_0 \exp\left(\int_0^s \left(f(\sigma) + g(\sigma)\right) d\sigma\right) + \int_0^s h(\sigma) \exp\left(\int_\sigma^s \left(f(\tau) + g(\tau)\right) d\tau\right) d\sigma\right)\Bigg] ds, \tag{2}$$

*for $t \in R_+$.*

Very recently, Abdeldaim and El-Deeb [4, 5] investigated the following integral inequalities which generalized the main results of [3].

**Theorem 2** (see [4, Theorem 2.1]). *Let $u, f, g, h \in C(R_+, R_+)$ and suppose that $\alpha \in C^1(R_+, R_+)$ are increasing functions with $\alpha(t) \le t$ and $\alpha(0) = 0$, for $t \in R_+$, and $u_0$ is a nonnegative constant. If*

$$u(t) \le u_0 + \int_0^{\alpha(t)} \left(f(s)\,u(s) + h(s)\right) ds + \int_0^{\alpha(t)} f(s) \left(\int_0^s g(\tau)\,u(\tau)\,d\tau\right) ds, \tag{3}$$

*then*

$$u(t) \le u_0 + \int_0^t \Bigg(\alpha'(s)\,h(\alpha(s)) + \alpha'(s)\,f(\alpha(s)) \cdot \exp\left(\int_0^{\alpha(s)} \left(f(\tau) + g(\tau)\right) d\tau\right) \Bigg(u_0$$

$$+\int_0^{\alpha(s)} h(\sigma)\exp\left(-\int_0^{\sigma}(f(\tau)+g(\tau))\,d\tau\right)d\sigma\Bigg)\Bigg)\,ds,\quad \forall t\in R_+. \tag{4}$$

**Theorem 3** (see [4, Theorem 2.2] and [5, Theorem 2.1]). *Let $u, f, g, h \in C(R_+, R_+)$ and assume that $\varphi, \varphi', \alpha \in C^1(R_+, R_+)$ are increasing functions with $\varphi' \le k$, $\varphi > 0$, $\alpha(t) \le t$, and $\alpha(0) = 0$, for $t \in R_+$, and $u_0$ and $k$ are positive constants. If*

$$\begin{aligned} u(t) &\le u_0 + \int_0^{\alpha(t)} \varphi(u(s))\left(f(s)\varphi(u(s)) + h(s)\right)ds \\ &\quad + \int_0^{\alpha(t)} \varphi(u(s)) f(s)\left(\int_0^s g(\tau)\varphi(u(\tau))\,d\tau\right)ds, \end{aligned} \tag{5}$$

*then*

$$u(t) \le \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} (h(s) + f(s)\beta(s))\,ds\right), \quad \forall t \in R_+, \tag{6}$$

*where*

$$\begin{aligned} \beta(t) &\le \frac{\exp\left(\int_0^{\alpha(t)} (kh(s) + g(s))\,ds\right)}{\left(\varphi^{-1}(u_0) - \int_0^{\alpha(t)} kf(s)\exp\left(\int_0^s (kh(\sigma) + g(\sigma))\,d\sigma\right)ds\right)}, \\ \Phi(r) &= \int_1^r \frac{dt}{\varphi(t)}, \quad r > 1, \end{aligned} \tag{7}$$

*and $\Phi^{-1}$ and $\varphi^{-1}$ stand for the inverses of the functions $\Phi$ and $\varphi$, respectively.*

Note that the function $\varphi$ of inequalities established in [4, 5] satisfies $\varphi' \le k$; that is, $\varphi$ is global Lipchitz. However, in the real system there may exist the non-Lipschitz $\varphi$, such as $\varphi(t) = t^2$ or $\varphi(t) = e^t$, and then the obtained results can not be applied to this kind of systems. The natural question now is as follows: is it possible to relax this condition? The aim of this paper is to give an affirmative answer to this question.

In this paper, we study certain classes of integral inequalities which generalize the inequalities established in [3–5] to the more general nonlinear case. The obtained results can be used as tools in the study of qualitative theory of certain classes of integrodifferential equations with more general nonlinearities. At the end, two examples are provided to illustrate the main results.

## 2. Main Results

In what follows, $R$ denotes the set of real numbers, $C(M, S)$ and $C^1(M, S)$ denote the sets of all continuous functions and all continuously differentiable functions defined on set $M$ with range in the set $S$, respectively, and $f^{-1}$ stands for the inverse to $f$.

The following lemmas are very useful in the proof of our main results.

**Lemma 4** (see [6, Lemma 2.1]). *Assume that $a \ge 0$ and $m \ge n > 0$. Then*

$$a^{n/m} \le \frac{n}{m}K^{(n-m)/m}a + \frac{m-n}{m}K^{n/m}, \tag{8}$$

*for any $K > 0$.*

**Lemma 5** (see [7, Corollary 1]). *Assume that $u, k_i \in C(R_+, R_+)$, $i = 1, 2, 3$, and $a_0$ is a positive constant. If*

$$\begin{aligned} u(t) &\le a_0 + \int_0^t k_1(s)u(s)\,ds + \int_0^t k_2(s)u^2(s)\,ds \\ &\quad + \int_0^t k_3(s)u^3(s)\,ds, \end{aligned} \tag{9}$$

*then*

$$u(t) \le \frac{a_0\exp\left(\int_0^t k_1(s)\,ds\right)}{\sqrt{\left(1 - a_0\exp\left(\int_0^t k_1(s)\,ds\right)\int_0^t k_2(s)\,ds\right)^2 - 2a_0^2\exp\left(\int_0^t 2k_1(s)\,ds\right)\int_0^t k_3(s)\,ds}}, \quad t \in [0, T], \tag{10}$$

*where $T$ is the largest number such that*

$$\begin{aligned} &\exp\left(\int_0^T k_1(s)\,ds\right)\int_0^T k_2(s)\,ds < \frac{1}{a_0}, \\ &\frac{\exp\left(\int_0^T 2k_1(s)\,ds\right)\int_0^T k_3(s)\,ds}{\left(1 - a_0\exp\left(\int_0^T k_1(s)\,ds\right)\int_0^T k_2(s)\,ds\right)^2} < \frac{1}{2a_0^2}. \end{aligned} \tag{11}$$

**Theorem 6.** *Let $u_0$, $p$, $q$, and $r$ be nonnegative constants satisfying $p \ge q > 0$, $p \ge r > 0$, $u, f, g, h \in C(R_+, R_+)$, $\alpha \in C^1(R_+, R_+)$, $\alpha' \ge 0$, $\alpha(t) \le t$, and $\alpha(0) = 0$. If*

$$\begin{aligned} u^p(t) &\le u_0 + \int_0^{\alpha(t)} \left(f(s)u^q(s) + h(s)\right)ds \\ &\quad + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(\tau)u^r(\tau)\,d\tau\right)ds, \end{aligned} \tag{12}$$

*then, for any* $K > 0$,

$$u(t) \le \left[u_0 + \int_0^t \left(H(s) + \frac{q}{p}K^{(q-p)/p}\alpha'(s) f(\alpha(s)) \cdot \exp\left(\int_0^{\alpha(s)} J(\tau)\,d\tau\right) \cdot \left(u_0 + \int_0^s H(\sigma)\exp\left(-\int_0^{\alpha(\sigma)} J(\tau)\,d\tau\right)d\sigma\right)\right)ds\right]^{1/p}, \quad t \in R_+, \tag{13}$$

*where*

$$H(t) = \alpha'(t)h(\alpha(t)) + \alpha'(t) f(\alpha(t)) \cdot \left(\frac{p-q}{p}K^{q/p} + \frac{p-r}{p}K^{r/p}\int_0^{\alpha(t)} g(\tau)\,d\tau\right), \tag{14}$$

$$J(t) = \frac{q}{p}K^{(q-p)/p} f(t) + \frac{r}{q}K^{(r-q)/p} g(t).$$

*Proof.* Define a function $y$ by

$$y(t) = u_0 + \int_0^{\alpha(t)} \left(f(s)u^q(s) + h(s)\right)ds + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(\tau)u^r(\tau)\,d\tau\right)ds. \tag{15}$$

Then, $y(0) = u_0$, $y$ is nondecreasing, and

$$u(t) \le y^{1/p}(t), \quad u(\alpha(t)) \le y^{1/p}(\alpha(t)) \le y^{1/p}(t), \quad t \in R_+. \tag{16}$$

Differentiating $y$ and using (16), we conclude that

$$y'(t) = \alpha'(t)\left[f(\alpha(t))u^q(\alpha(t)) + h(\alpha(t))\right] + \alpha'(t) \cdot f(\alpha(t))\int_0^{\alpha(t)} g(\tau)u^r(\tau)\,d\tau \le \alpha'(t)h(\alpha(t)) + \alpha'(t) f(\alpha(t)) \cdot \left(y^{q/p}(t) + \int_0^{\alpha(t)} g(\tau)y^{r/p}(\tau)\,d\tau\right). \tag{17}$$

By virtue of Lemma 4, for any $K > 0$,

$$y^{q/p}(t) \le \frac{q}{p}K^{(q-p)/p}y(t) + \frac{p-q}{p}K^{q/p}, \quad y^{r/p}(t) \le \frac{r}{p}K^{(r-p)/p}y(t) + \frac{p-r}{p}K^{r/p}. \tag{18}$$

Combining (17) and (18), we deduce that

$$y'(t) \le H(t) + \alpha'(t) f(\alpha(t))\left(\frac{q}{p}K^{(q-p)/p}y(t) + \frac{r}{p}K^{(r-p)/p}\int_0^{\alpha(t)} g(\tau)y(\tau)\,d\tau\right). \tag{19}$$

Define another function $v$ by

$$v(t) = \frac{q}{p}K^{(q-p)/p}y(t) + \frac{r}{p}K^{(r-p)/p}\int_0^{\alpha(t)} g(\tau)y(\tau)\,d\tau. \tag{20}$$

Then $v$ is nondecreasing, $v(0) = qu_0K^{(q-p)/p}/p$, and

$$y(t) \le \frac{p}{q}K^{(p-q)/p}v(t), \quad y(\alpha(t)) \le \frac{p}{q}K^{(p-q)/p}v(\alpha(t)) \le \frac{p}{q}K^{(p-q)/p}v(t), \quad t \in R_+. \tag{21}$$

It follows from (19) that

$$y'(t) \le H(t) + \alpha'(t) f(\alpha(t))v(t). \tag{22}$$

Differentiating $v$ and using (22), we obtain

$$v'(t) = \frac{q}{p}K^{(q-p)/p}y'(t) + \frac{r}{p}K^{(r-p)/p}\alpha'(t)g(\alpha(t)) \cdot y(\alpha(t)) \le \frac{q}{p}K^{(q-p)/p}\left[H(t) + \alpha'(t) f(\alpha(t))v(t)\right] + \frac{r}{q}K^{(r-q)/p}\alpha'(t)g(\alpha(t)) \cdot v(t) = \frac{q}{p}K^{(q-p)/p}H(t) + \left(\frac{q}{p}K^{(q-p)/p}\alpha'(t) f(\alpha(t)) + \frac{r}{q}K^{(r-q)/p}\alpha'(t)g(\alpha(t))\right)v(t). \tag{23}$$

Integrating the latter inequality from 0 to $t$, we conclude that

$$v(t) \le \frac{q}{p}K^{(q-p)/p}\exp\left(\int_0^{\alpha(t)} J(\tau)\,d\tau\right) \cdot \left(u_0 + \int_0^t H(\sigma)\exp\left(-\int_0^{\alpha(\sigma)} J(\tau)\,d\tau\right)d\sigma\right). \tag{24}$$

Therefore,

$$y'(t) \le H(t) + \frac{q}{p}K^{(q-p)/p}\alpha'(t) f(\alpha(t)) \cdot \exp\left(\int_0^{\alpha(t)} J(\tau)\,d\tau\right) \cdot \left(u_0 + \int_0^t H(\sigma)\exp\left(-\int_0^{\alpha(\sigma)} J(\tau)\,d\tau\right)d\sigma\right), \tag{25}$$

which yields

$$y(t) \le u_0 + \int_0^t \left[ H(s) + \frac{q}{p} K^{(q-p)/p} \alpha'(s) f(\alpha(s)) \cdot \exp\left( \int_0^{\alpha(s)} J(\tau)\, d\tau \right) \left( u_0 + \int_0^s H(\sigma) \exp\left( -\int_0^{\alpha(\sigma)} J(\tau)\, d\tau \right) d\sigma \right) \right] ds. \tag{26}$$

By virtue of $u(t) \le y^{1/p}(t)$, (13) holds. This completes the proof. □

*Remark 7.* If $p = q = r = 1$, then Theorem 6 reduces to Theorem 2.

*Remark 8.* If $p = q = r = 1$ and $\alpha(t) = t$, then Theorem 6 reduces to Theorem 1.

**Theorem 9.** *Let $u, f, g, h \in C(R_+, R_+)$. Assume further that $\psi, \psi' \in C^1(R_+, R_+)$, $\psi' > 0$, $\psi'' \ge 0$, $\varphi \in C^1(R_+, R_+)$, $\varphi > 0$, $0 \le \varphi' \le a\varphi + b$, $a, b \ge 0$, $\alpha \in C^1(R_+, R_+)$, $\alpha' \ge 0$, $\alpha(t) \le t$, $\alpha(0) = 0$, and $u_0$ is a positive constant. If, for $t \in R_+$,*

$$\psi(u(t)) \le u_0 + \int_0^{\alpha(t)} \varphi(u(s)) \left( f(s) \varphi(u(s)) + h(s) \right) ds + \int_0^{\alpha(t)} f(s) \varphi(u(s)) \left( \int_0^s g(\tau) \varphi(u(\tau))\, d\tau \right) ds, \tag{27}$$

*then*

$$u(t) \le \psi^{-1} \left[ G^{-1} \left( G(u_0) + \int_0^{\alpha(t)} \left( h(s) + f(s) \beta\left(\alpha^{-1}(s)\right) \right) ds \right) \right], \quad t \in [0, T], \tag{28}$$

*where*

$$\begin{aligned} \beta(t) &= \frac{b_0 \exp\left( \int_0^t k_1(s)\, ds \right)}{\sqrt{\left( 1 - b_0 \exp\left( \int_0^t k_1(s)\, ds \right) \int_0^t k_2(s)\, ds \right)^2 - 2b_0^2 \exp\left( \int_0^t 2k_1(s)\, ds \right) \int_0^t k_3(s)\, ds}}, \quad t \in [0, T], \\ k_1(t) &= \alpha'(t) g(\alpha(t)) + \frac{b}{\psi'(\psi^{-1}(u_0))} \alpha'(t) h(\alpha(t)), \\ k_2(t) &= \frac{1}{\psi'(\psi^{-1}(u_0))} \alpha'(t) \left( a h(\alpha(t)) + b f(\alpha(t)) \right), \\ k_3(t) &= \frac{a}{\psi'(\psi^{-1}(u_0))} \alpha'(t) f(\alpha(t)), \\ b_0 &= \varphi\left( \psi^{-1}(u_0) \right), \end{aligned} \tag{29}$$

*$T$ is the largest number such that*

$$\begin{aligned} &\exp\left( \int_0^T k_1(s)\, ds \right) \int_0^T k_2(s)\, ds < \frac{1}{b_0}, \\ &\frac{\exp\left( \int_0^T 2k_1(s)\, ds \right) \int_0^T k_3(s)\, ds}{\left( 1 - b_0 \exp\left( \int_0^T k_1(s)\, ds \right) \int_0^T k_2(s)\, ds \right)^2} < \frac{1}{2b_0^2}, \\ &G(r) = \int_{r_0}^r \frac{1}{\varphi(\psi^{-1}(t))} dt, \quad r > r_0 > 0. \end{aligned} \tag{30}$$

*Proof.* Define a function $z$ by

$$z(t) = u_0 + \int_0^{\alpha(t)} \varphi(u(s)) \left( f(s) \varphi(u(s)) + h(s) \right) ds + \int_0^{\alpha(t)} f(s) \varphi(u(s)) \left( \int_0^s g(\tau) \varphi(u(\tau))\, d\tau \right) ds. \tag{31}$$

Then, $z(0) = u_0$, $z$ is nondecreasing, and

$$\begin{aligned} u(t) &\le \psi^{-1}(z(t)), \\ u(\alpha(t)) &\le \psi^{-1}(z(\alpha(t))) \le \psi^{-1}(z(t)), \quad t \in R_+. \end{aligned} \tag{32}$$

It is not difficult to obtain

$$\begin{aligned} z'(t) &= \alpha'(t) \varphi(u(\alpha(t))) \left( f(\alpha(t)) \varphi(u(\alpha(t))) + h(\alpha(t)) \right) + \alpha'(t) f(\alpha(t)) \varphi(u(\alpha(t))) \cdot \int_0^{\alpha(t)} g(\tau) \varphi(u(\tau))\, d\tau \\ &\le \alpha'(t) h(\alpha(t)) \cdot \varphi\left( \psi^{-1}(z(t)) \right) + \alpha'(t) f(\alpha(t)) \varphi\left( \psi^{-1}(z(t)) \right) \end{aligned}$$

$$\times\left(\varphi\left(\psi^{-1}(z(t))\right)+\int_0^{\alpha(t)} g(\tau)\varphi\left(\psi^{-1}(z(\tau))\right)d\tau\right)=\alpha'(t)h(\alpha(t))\cdot\varphi\left(\psi^{-1}(z(t))\right)+\alpha'(t)f(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)\cdot w(t), \tag{33}$$

where

$$w(t)=\varphi\left(\psi^{-1}(z(t))\right)+\int_0^{\alpha(t)} g(\tau)\varphi\left(\psi^{-1}(z(\tau))\right)d\tau, \tag{34}$$

$w$ is nondecreasing, $w(0)=b_0$, and $\varphi(\psi^{-1}(z(t)))\le w(t)$. Differentiating $w$, we have

$$\begin{aligned} w'(t)&=\varphi'\left(\psi^{-1}(z(t))\right)\left(\psi^{-1}(z(t))\right)'z'(t)+\alpha'(t)g(\alpha(t))\varphi\left(\psi^{-1}(z(\alpha(t)))\right)\\ &=\varphi'\left(\psi^{-1}(z(t))\right)\frac{1}{\psi'\left(\psi^{-1}(z(t))\right)}z'(t)+\alpha'(t)g(\alpha(t))\varphi\left(\psi^{-1}(z(\alpha(t)))\right). \end{aligned} \tag{35}$$

Noting that $\psi$ is increasing and $\psi'$ is nondecreasing, we get $\psi'(\psi^{-1}(z(t)))\ge\psi'(\psi^{-1}(u_0))$, and so

$$\begin{aligned} w'(t)&\le\frac{1}{\psi'\left(\psi^{-1}(u_0)\right)}\left(a\varphi\left(\psi^{-1}(z(t))\right)+b\right)\cdot\left[\alpha'(t)h(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)+\alpha'(t)f(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)w(t)\right]+\alpha'(t)\\ &\quad\cdot g(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)\le\frac{1}{\psi'\left(\psi^{-1}(u_0)\right)}(aw(t)+b)\left[\alpha'(t)h(\alpha(t))w(t)+\alpha'(t)f(\alpha(t))w^2(t)\right]\\ &\quad+\alpha'(t)g(\alpha(t))w(t)=k_1(t)w(t)+k_2(t)w^2(t)+k_3(t)w^3(t). \end{aligned} \tag{36}$$

Integrating the latter inequality from 0 to $t$, we have

$$w(t)\le b_0+\int_0^t k_1(s)w(s)\,ds+\int_0^t k_2(s)w^2(s)\,ds+\int_0^t k_3(s)w^3(s)\,ds. \tag{37}$$

By virtue of Lemma 5,

$$w(t)\le\beta(t),\quad t\in[0,T]. \tag{38}$$

This and (33) imply that

$$z'(t)\le\alpha'(t)h(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)+\alpha'(t)f(\alpha(t))\varphi\left(\psi^{-1}(z(t))\right)\beta(t), \tag{39}$$

and thus

$$\frac{z'(t)}{\varphi\left(\psi^{-1}(z(t))\right)}\le\alpha'(t)h(\alpha(t))+\alpha'(t)f(\alpha(t))\beta(t). \tag{40}$$

Integrating (40) from 0 to $t$,

$$z(t)\le G^{-1}\left(G(u_0)+\int_0^{\alpha(t)}\left(h(s)+f(s)\beta\left(\alpha^{-1}(s)\right)\right)ds\right), \tag{41}$$

which yields (28) due to (32). The proof is complete. □

*Remark 10.* The condition $\varphi'\le k$ in [4, 5] is relaxed to $\varphi'\le a\varphi+b$, where $a$ and $b$ are nonnegative constants. On the basis of this new assumption, the classes of $\varphi$ are enlarged. For example, $\varphi(t)=e^t$ satisfies $\varphi'\le a\varphi+b$ when choosing $a=1$ and $b=0$.

If we choose $\psi(u)=u^p$, $\varphi(u)=u^q$, and $p>q\ge1$ in Theorem 9, then $\varphi'(u)=qu^{q-1}$. Letting $\Phi(u)=u^{q-1}-u^q$, $u>0$, we have $u^{q-1}\le u^q+(1-1/q)^{q-1}/q$, and hence $\varphi'(u)=qu^{q-1}\le qu^q+(1-1/q)^{q-1}=q\varphi(u)+(1-1/q)^{q-1}$. Therefore, $\varphi$ satisfies $\varphi'(u)\le a\varphi(u)+b$ with $a=q$ and $b=(1-1/q)^{q-1}$, and then we can obtain the following result.

**Corollary 11.** *Assume that $u,f,g,h\in C(R_+,R_+)$, $\alpha\in C^1(R_+,R_+)$, $\alpha'\ge0$, $\alpha(t)\le t$, $\alpha(0)=0$, $u_0$ is a positive constant, and, for $t\in R_+$,*

$$u^p(t)\le u_0+\int_0^{\alpha(t)}u^q(s)\left(f(s)u^q(s)+h(s)\right)ds+\int_0^{\alpha(t)}f(s)u^q(s)\left(\int_0^s g(\tau)u^q(\tau)\,d\tau\right)ds. \tag{42}$$

*Then*

$$u(t)\le\left(u_0^{(p-q)/p}+\frac{p-q}{p}\int_0^{\alpha(t)}\left(h(s)+f(s)\beta\left(\alpha^{-1}(s)\right)\right)ds\right)^{1/(p-q)},\quad t\in[0,T], \tag{43}$$

*where $p>q\ge1$,*

$$\beta(t) = \frac{u_0^{q/p} \exp\left(\int_0^t k_1(s)\,ds\right)}{\sqrt{\left(1 - u_0^{q/p} \exp\left(\int_0^t k_1(s)\,ds\right) \int_0^t k_2(s)\,ds\right)^2 - 2u_0^{2q/p} \exp\left(\int_0^t 2k_1(s)\,ds\right) \int_0^t k_3(s)\,ds}},$$
$$k_1(t) = \alpha'(t)\, g(\alpha(t)) + \frac{1}{p}\left(1 - \frac{1}{q}\right)^{q-1} u_0^{1/p-1} \alpha'(t)\, h(\alpha(t)),$$
$$k_2(t) = \frac{1}{p} u_0^{1/p-1} \alpha'(t) \left(q h(\alpha(t)) + \left(1 - \frac{1}{q}\right)^{q-1} f(\alpha(t))\right),$$
$$k_3(t) = \frac{q}{p} u_0^{1/p-1} \alpha'(t)\, f(\alpha(t)), \tag{44}$$

*and $T$ is the largest number such that*

$$\exp\left(\int_0^T k_1(s)\,ds\right) \int_0^T k_2(s)\,ds < \frac{1}{u_0^{q/p}},$$
$$\frac{\exp\left(\int_0^T 2k_1(s)\,ds\right) \int_0^T k_3(s)\,ds}{\left(1 - u_0^{q/p} \exp\left(\int_0^T k_1(s)\,ds\right) \int_0^T k_2(s)\,ds\right)^2} < \frac{1}{2u_0^{2q/p}}. \tag{45}$$

*Proof.* By virtue of $\varphi(u) = u^q$ and $\psi(u) = u^p$, we obtain $\varphi(\psi^{-1}(u)) = u^{q/p}$ and

$$G(r) = \int_{r_0}^r \frac{1}{\varphi(\psi^{-1}(t))} dt = \int_{r_0}^r t^{-q/p} dt$$
$$= \frac{p}{p-q}\left(r^{(p-q)/p} - r_0^{(p-q)/p}\right), \tag{46}$$
$$G^{-1}(r) = \left(\frac{p-q}{p} r + r_0^{(p-q)/p}\right)^{p/(p-q)}.$$

An application of Theorem 9 implies that

$$u(t) \le \psi^{-1}\left[G^{-1}\left(G(u_0) + \int_0^{\alpha(t)} \left(h(s) + f(s)\beta\left(\alpha^{-1}(s)\right)\right) ds\right)\right]$$
$$= \left(u_0^{(p-q)/p} + \frac{p-q}{p} \cdot \int_0^{\alpha(t)} \left(h(s) + f(s)\beta\left(\alpha^{-1}(s)\right)\right) ds\right)^{1/(p-q)}. \tag{47}$$

This completes the proof. □

## 3. Applications

In this section, we will use the inequalities established in Theorems 6 and 9 to study the boundedness of certain integrodifferential equations.

*Example 1.* Consider the following integral equation:

$$u^p(t) = a(t) + \int_0^t G\left(s, u(\alpha(s)), \int_0^s N(\tau, u(\alpha(\tau)))\,d\tau\right) ds, \quad t \in R_+, \tag{48}$$

where $a(t)$ is a continuous function defined on $R^+$, $|a(t)| \le A$, $A$ is a nonnegative constant, and $G \in C(R_+ \times R \times R, R)$ and $N \in C(R_+ \times R, R)$ satisfy

$$|G(t,x,y)| \le b(t)\left(|x|^q + |y|\right) + c(t),$$
$$|N(t,z)| \le d(t)|z|^r, \quad t \in R_+, \tag{49}$$

where $p$, $q$, and $r$ are nonnegative constants satisfying $p \ge q > 0$, $p \ge r > 0$, and $b, c, d \in C(R_+, R_+)$. Then every solution $u$ of (48) satisfies

$$|u(t)|^p \le A + \int_0^t \left[b(s)|u(\alpha(s))|^q + c(s)\right] ds + \int_0^t b(s) \cdot \left(\int_0^s d(\tau)|u(\alpha(\tau))|^r d\tau\right) ds = A$$
$$+ \int_0^{\alpha(t)} \frac{1}{\alpha'(\alpha^{-1}(s))}\left[b\left(\alpha^{-1}(s)\right)|u(s)|^q + c\left(\alpha^{-1}(s)\right)\right] ds$$
$$+ \int_0^{\alpha(t)} \frac{b\left(\alpha^{-1}(s)\right)}{\alpha'(\alpha^{-1}(s))}\left(\int_0^s \frac{d\left(\alpha^{-1}(\tau)\right)}{\alpha'(\alpha^{-1}(\tau))}|u(\tau)|^r d\tau\right) ds. \tag{50}$$

Now, using Theorem 6, we obtain that, for any $K > 0$,

$$|u(t)| \le \left[A + \int_0^t \left(H(s) + \frac{q}{p} K^{(q-p)/p} b(s) \cdot \exp\left(\int_0^s \tilde{J}(\tau)\,d\tau\right)\right.\right.$$

$$\cdot\left(A+\int_0^s H(\sigma)\exp\left(-\int_0^\sigma \tilde{J}(\tau)\,d\tau\right)d\sigma\right)\Bigg)ds\Bigg]^{1/p},\quad t\in R_+, \tag{51}$$

where

$$\begin{aligned} H(t) &= c(t) + b(t)\left(\frac{p-q}{p}K^{q/p}+\frac{p-r}{p}K^{r/p}\int_0^t d(\tau)\,d\tau\right), \\ \tilde{J}(t) &= \frac{q}{p}K^{(q-p)/p}b(t)+\frac{r}{q}K^{(r-q)/p}d(t), \end{aligned} \tag{52}$$

which illustrates that the solution of (48) is global boundedness.

*Example 2.* Consider the integrodifferential equation

$$\left(u^p(t)\right)' = F\left(t, u(\alpha(t)), \int_0^t H(\tau, u(\alpha(\tau)))\,d\tau\right), \quad u^p(0)=C,\quad t\in R_+, \tag{53}$$

where $C\neq 0$ is a constant, $F\in C(R_+\times R\times R, R)$, and $H\in C(R_+\times R, R)$.

Assume that

$$\begin{aligned} |F(t,U,V)| &\le |U|^q\left[f(t)\left(|U|^q+|V|\right)+h(t)\right], \\ |H(t,W)| &\le g(t)|W|^q, \quad t\in R_+, \end{aligned} \tag{54}$$

where $f, g, h\in C(R_+, R_+)$. Then the solution $u$ of (53) satisfies

$$u^p(t) = C + \int_0^t F\left(s, u(\alpha(s)), \int_0^s H(\tau, u(\alpha(\tau)))\,d\tau\right)ds, \tag{55}$$

and hence

$$\begin{aligned} |u(t)|^p &\le |C| + \int_0^t |u(\alpha(s))|^q\left[f(s)|u(\alpha(s))|^q + h(s)\right]ds + \int_0^t f(s)|u(\alpha(s))|^q \cdot\left(\int_0^s g(\tau)|u(\alpha(\tau))|^q\,d\tau\right)ds = |C| \\ &+ \int_0^{\alpha(t)} \frac{|u(s)|^q}{\alpha'(\alpha^{-1}(s))}\left[f\left(\alpha^{-1}(s)\right)|u(s)|^q + h\left(\alpha^{-1}(s)\right)\right]ds + \int_0^{\alpha(t)} \frac{f\left(\alpha^{-1}(s)\right)}{\alpha'(\alpha^{-1}(s))}|u(s)|^q \cdot\left(\int_0^s \frac{g\left(\alpha^{-1}(\tau)\right)}{\alpha'(\alpha^{-1}(\tau))}|u(\tau)|^q\,d\tau\right)ds. \end{aligned} \tag{56}$$

A suitable application of Corollary 11 to (56) yields

$$\begin{aligned} |u(t)| &\le \left(|C|^{(p-q)/p} + \frac{p-q}{p}\cdot\int_0^{\alpha(t)}\frac{1}{\alpha'(\alpha^{-1}(s))}\left(h\left(\alpha^{-1}(s)\right) + f\left(\alpha^{-1}(s)\right)\beta\left(\alpha^{-1}(s)\right)\right)ds\right)^{1/(p-q)} \\ &= \left(|C|^{(p-q)/p} + \frac{p-q}{p}\int_0^t\left(h(s) + f(s)\beta(s)\right)ds\right)^{1/(p-q)}, \quad t\in[0,T], \end{aligned} \tag{57}$$

where

$$\begin{aligned} \beta(t) &= \frac{|C|^{q/p}\exp\left(\int_0^t k_1(s)\,ds\right)}{\sqrt{\left(1-|C|^{q/p}\exp\left(\int_0^t k_1(s)\,ds\right)\int_0^t k_2(s)\,ds\right)^2 - 2|C|^{2q/p}\exp\left(\int_0^t 2k_1(s)\,ds\right)\int_0^t k_3(s)\,ds}}, \\ k_1(t) &= g(t) + \frac{1}{p}\left(1-\frac{1}{q}\right)^{q-1}|C|^{1/p-1}h(t), \\ k_2(t) &= \frac{1}{p}|C|^{1/p-1}\left(qh(t) + \left(1-\frac{1}{q}\right)^{q-1}f(t)\right), \\ k_3(t) &= \frac{q}{p}|C|^{1/p-1}f(t) \end{aligned} \tag{58}$$

and $T$ is the largest number such that

$$\exp\left(\int_0^T k_1(s)\,ds\right)\int_0^T k_2(s)\,ds < \frac{1}{|C|^{q/p}},$$
$$\frac{\exp\left(\int_0^T 2k_1(s)\,ds\right)\int_0^T k_3(s)\,ds}{\left(1-|C|^{q/p}\exp\left(\int_0^T k_1(s)\,ds\right)\int_0^T k_2(s)\,ds\right)^2} < \frac{1}{2\,|C|^{2q/p}}. \tag{59}$$

## Competing Interests

The authors declare that they have no competing interests.

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