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Research Article

On the Optimal Dynamic Control Strategy of Disruptive Computer Virus

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Disruptive computer viruses have inflicted huge economic losses. This paper addresses the development of a cost-effective dynamic control strategy of disruptive viruses. First, the development problem is modeled as an optimal control problem. Second, a criterion for the existence of an optimal control is given. Third, the optimality system is derived. Next, some examples of the optimal dynamic control strategy are presented. Finally, the performance of actual dynamic control strategies is evaluated.

1. Introduction

The proliferation of computer networks has brought huge benefits to human society. Meanwhile, it offers a shortcut to spread computer viruses, inflicting large economic losses [1]. Consequently, containing the prevalence of digital viruses has been one of the major concerns in the field of cybersecurity. The spreading dynamics of computer virus has been widely adopted as the standard method for assessing the viral prevalence [2]. Since the seminal work by Kephart and White [3, 4], a multitude of computer virus-spreading models, ranging from the population-level models [5–12] and the network-level models [13–17] to the node-level models [18–22], have been proposed.

One of the central tasks in cybersecurity is to develop control strategies of computer virus so that, subject to limited budgets, the losses caused by computer infections are minimized [23]. In recent years, the optimal design problem of virus control strategies has been modeled as static optimization problems [24–28]. The optimal static control strategies, however, only apply to the small-timescale situations where the network state keeps unchanged. In the realistic situations where the network state is varying over time, the optimal design problem of virus control strategies can be modeled as dynamic optimal control problems [29–33]. The optimal dynamic control strategies outperform their static counterparts, because the former not only are more cost-effective but apply to different timescales.

A disruptive computer virus is defined as a computer virus whose life period consists of two consecutive phases: the latent phase and the disruptive phase. In the latent phase, a disruptive virus staying in a host does not perform any disruptive operations. Rather, the virus tries to infect as many hosts as possible by sending its copies to them. In the disruptive phase, a disruptive virus staying in a host performs a variety of operations that disrupt the host, such as distorting data, deleting data or files, and destroying the operating system. To assess the prevalence of disruptive viruses, a number of virus-spreading models, which are referred to as the Susceptible-Latent-Bursting-Susceptible (SLBS) models, have been suggested [34–38]. The main distinction between the SLBS models and the traditional SEIS models lies in that the latent hosts in the former possess strong infecting capability, whereas the exposed individuals in the latter possess no infecting capability at all. Recently, the basic SLBS models have been extended towards different directions [39–43]. At the population-level, Chen et al. [44] developed an optimal dynamic control strategy of disruptive viruses.

All of the above-mentioned SLBS models are population-level; that is, they are based on the assumption that every infected host in the population is equally likely to infect any other susceptible host. These models have two striking defects: (a) the personalized features of different hosts cannot be taken into consideration and (b) the impact of the structure of the virus-propagating network on the viral prevalence cannot be revealed by studying the models. To overcome these defects, Yang et al. [45] presented a node-level SLBS model. In our opinion, optimal dynamic control strategies of disruptive viruses should be developed at the node-level, so as to achieve the best cost-efficiency.

This paper is intended to develop at the node-level an optimal dynamic control strategy of disruptive computer viruses. First, the development problem is modeled as an optimal control problem. Second, a criterion for the existence of an optimal control for the optimal control problem is given. Third, the optimality system for the optimal control problem is presented. Next, some exemplar optimal dynamic control strategies are given. Finally, the difference between the cost-efficiency of an arbitrary control strategy and that of the optimal dynamic strategy is estimated.

The subsequent materials of this work are organized as follows. Section 2 presents the preliminary knowledge on optimal control theory. Sections 3 and 4 formulate and study the optimal control problem, respectively. Some numerical examples are given in Section 5. Section 6 estimates the aforementioned difference. Finally, Section 7 closes this work.

2. Fundamental Knowledge

For fundamental knowledge on optimal control theory, see [46].

Consider the following optimal control problem.

$$\begin{aligned} \underset{\mathbf{u}(\cdot)\in\mathscr{U}}{\text{Minimize}} \quad & J\left(\mathbf{u}(t)\right) = \int_0^T F\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt \\ \text{subject to} \quad & \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right), \\ & 0 \le t \le T. \end{aligned} \tag{P}$$

Lemma 1. *Problem* (P) *has an optimal control if the following five conditions hold simultaneously.*

(C_1) *$\mathscr{U}$ is closed and convex.*

(C_2) *There is $\mathbf{u}(\cdot) \in \mathscr{U}$ such that the adjunctive dynamical system is solvable.*

(C_3) *$\mathbf{f}(\mathbf{x}, \mathbf{u})$ is bounded by a linear function in $\mathbf{x}$.*

(C_4) *$F(\mathbf{x}, \mathbf{u})$ is convex on $\mathscr{U}$.*

(C_5) *$F(\mathbf{x}, \mathbf{u}) \ge c_1\|\mathbf{u}\|^\rho + c_2$ for some vector norm $\|\cdot\|$, $\rho > 1$, $c_1 > 0$, and c_2.*

3. Formulation of the Optimal Control Problem

Consider a population of N hosts (nodes) labelled $1, 2, \ldots, N$. As with the traditional SLBS models, assume that at any time every node in the population is in one of three possible states: *susceptible*, *latent*, and *disruptive*. Susceptible nodes are those that are not infected with any disruptive computer virus. Latent nodes are those that are infected with some disruptive viruses and all of them are in the latent phase. Disruptive nodes are those that are infected with some disruptive viruses and some of them are in the disruptive phase. Let $X_i(t) = 0$, 1, and 2 denote that at time t node i is susceptible, latent, and disruptive, respectively. Let

$$\begin{aligned} S_i(t) &= \Pr\left\{X_i(t) = 0\right\}, \\ L_i(t) &= \Pr\left\{X_i(t) = 1\right\}, \\ B_i(t) &= \Pr\left\{X_i(t) = 2\right\}. \end{aligned} \tag{1}$$

As $S_i(t) + L_i(t) + B_i(t) \equiv 1$ $(1 \le i \le N)$, the vector

$$\begin{aligned} &\mathbf{I}(t) \\ &\quad = \left(L_1(t), \ldots, L_N(t), B_1(t), \ldots, B_N(t)\right)^T \end{aligned} \tag{2}$$

probabilistically captures the state of the population at time t.

Suppose a dynamic control strategy will be carried out during the time frame $[0, T]$. Let us impose a set of statistical hypotheses as follows.

(H_1) A susceptible node i is infected by a latent node j at rate $\beta_{L,ij} \ge 0$. Let $\mathbf{A}_L = (\beta_{L,ij})_{N\times N}$.

(H_2) A susceptible node i is infected by a disruptive node j at rate $\beta_{B,ij} \ge 0$. Let $\mathbf{A}_B = (\beta_{B,ij})_{N\times N}$.

(H_3) Due to the outburst of latent viruses, a latent node i becomes disruptive at rate $\alpha_i > 0$. Let $\overline{\alpha} = \max_i \alpha_i$.

(H_4) Due to the action of new patches, at time t a latent node i becomes susceptible at a controllable rate $\gamma_{L,i}(t) \in L^2[0,T]$ and $\underline{\gamma_L} \le \gamma_{L,i}(t) \le \overline{\gamma_L}$. Hereafter, the symbol $L^2[0,T]$ stands for the set of all Lebesgue square integrable functions defined on the interval $[0,T]$. Moreover, the cost needed to achieve the rate at the infinitesimal time interval $[t, t+dt)$ is $p_i\gamma_{L,i}^\theta(t)dt$, $p_i > 0$, and $\theta > 0$. This accords with the intuition that the cost increases with $\gamma_{L,i}(t)$.

(H_5) Due to the action of new patches, at time t a disruptive node i becomes susceptible at a controllable rate $\gamma_{B,i}(t) \in L^2[0,T]$ and $\underline{\gamma_B} \le \gamma_{B,i}(t) \le \overline{\gamma_B}$. Moreover, the cost needed to achieve the rate at the infinitesimal time interval $[t, t+dt)$ is $q_i\gamma_{B,i}^\theta(t)dt$, $q_i > 0$. This conforms to the intuition that the cost increases with $\gamma_{B,i}(t)$.

Figure 1 shows hypotheses (H_1)–(H_5) schematically.

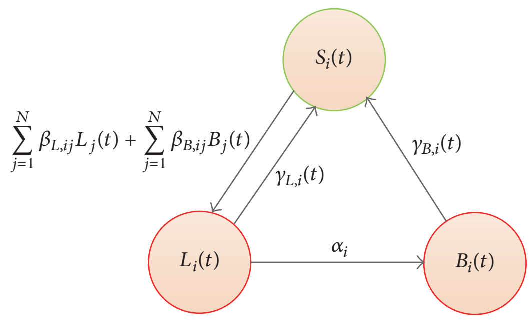

FIGURE 1: Diagram of assumptions (H_1)–(H_5).

Let $\Delta t > 0$ denote a very small time interval. Hypotheses (H_1)–(H_5) imply the following relations.

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=0\} \\
&\quad = \Delta t\sum_{j=1}^{N}\beta_{L,ij}L_j(t)+\Delta t\sum_{j=1}^{N}\beta_{B,ij}B_j(t)+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=0\}=o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=1\}=\alpha_i\Delta t+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=0 \mid X_i(t)=1\}=\gamma_{L,i}(t)\Delta t+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=0 \mid X_i(t)=2\}=\gamma_{B,i}(t)\Delta t+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=2\}=o(\Delta t).
\end{aligned} \tag{3}$$

As a result, we have

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=0 \mid X_i(t)=0\} \\
&\quad = 1-\Delta t\sum_{j=1}^{N}\beta_{L,ij}L_j(t) \\
&\qquad -\Delta t\sum_{j=1}^{N}\beta_{B,ij}B_j(t)+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=1\} \\
&\quad = 1-\alpha_i\Delta t-\gamma_{L,i}(t)\Delta t+o(\Delta t), \\
&\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=2\} \\
&\quad = 1-\gamma_{B,i}(t)\Delta t+o(\Delta t).
\end{aligned} \tag{4}$$

By the total probability formula, we get

$$\begin{aligned}
&L_i(t+\Delta t) \\
&\quad = S_i(t)\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=0\} \\
&\qquad + L_i(t)\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=1\} \\
&\qquad + B_i(t)\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=2\} \\
&\quad = [1-L_i(t)-B_i(t)]\,\Delta t\sum_{j=1}^{N}\left[\beta_{L,ij}L_j(t)+\beta_{B,ij}B_j(t)\right] \\
&\qquad + L_i(t)-\Delta t\left[\alpha_i+\gamma_{L,i}(t)\right]L_i(t)+o(\Delta t), \\
&B_i(t+\Delta t) \\
&\quad = S_i(t)\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=0\} \\
&\qquad + L_i(t)\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=1\} \\
&\qquad + B_i(t)\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=2\} \\
&\quad = \alpha_i\Delta t L_i(t)+B_i(t)-\gamma_{B,i}(t)\Delta t B_i(t)+o(\Delta t).
\end{aligned} \tag{5}$$

Transposing the terms $L_i(t)$ and $B_i(t)$ from the right to the left and dividing both sides by Δt, we get

$$\begin{aligned}
&\frac{L_i(t+\Delta t)-L_i(t)}{\Delta t} \\
&\quad = [1-L_i(t)-B_i(t)]\sum_{j=1}^{N}\left[\beta_{L,ij}L_j(t)+\beta_{B,ij}B_j(t)\right] \\
&\qquad -\left[\alpha_i+\gamma_{L,i}(t)\right]L_i(t)+\frac{o(\Delta t)}{\Delta t}, \\
&\frac{B_i(t+\Delta t)-B_i(t)}{\Delta t}=\alpha_iL_i(t)-\gamma_{B,i}(t)B_i(t)+\frac{o(\Delta t)}{\Delta t}.
\end{aligned} \tag{6}$$

Letting $\Delta t \to 0$, we get the following dynamical model.

$$\begin{aligned}
&\frac{dL_i(t)}{dt} \\
&\quad = [1-L_i(t)-B_i(t)]\sum_{j=1}^{N}\left[\beta_{L,ij}L_j(t)+\beta_{B,ij}B_j(t)\right] \\
&\qquad -\left[\alpha_i+\gamma_{L,i}(t)\right]L_i(t), \\
&\frac{dB_i(t)}{dt}=\alpha_iL_i(t)-\gamma_{B,i}(t)B_i(t),
\end{aligned} \tag{7}$$

where $t \geq 0$, $1 \leq i \leq N$. We refer to the model as the *controlled SLBS model*, where the control,

$$\gamma(t) = \left(\gamma_{L,1}(t), \ldots, \gamma_{L,N}(t), \gamma_{B,1}(t), \ldots, \gamma_{B,N}(t)\right)^T, \tag{8}$$

stands for a dynamic control strategy of disruptive computer viruses. The admissible set of controls is

$$\Gamma = \left\{\gamma(t) \in \left(L^2[0,T]\right)^{2N} \mid \underline{\gamma_L} \leq \gamma_{L,i}(t) \leq \overline{\gamma_L}, \underline{\gamma_B} \leq \gamma_{B,i}(t) \leq \overline{\gamma_B}, \ 0 \leq t \leq T, \ 1 \leq i \leq N\right\}. \tag{9}$$

Model (7) can be written in matrix notation as

$$\frac{d\mathbf{I}(t)}{dt} = \mathbf{f}\left(\mathbf{I}(t), \gamma(t)\right), \quad 0 \leq t \leq T. \tag{10}$$

Given a dynamic control strategy $\gamma(\cdot)$. The total loss can be measured by $\int_0^T \sum_{i=1}^N [L_i(t) + B_i(t)]dt$, and the total cost can be gauged by $\int_0^T \sum_{i=1}^N [p_i\gamma_{L,i}^\theta(t) + q_i\gamma_{B,i}^\theta(t)]dt$. As a result, the performance of a dynamic control strategy $\gamma(\cdot)$ can be measured by

$$J(\gamma(\cdot)) = \int_0^T \sum_{i=1}^N \left[L_i(t) + B_i(t) + p_i\gamma_{L,i}^\theta(t) + q_i\gamma_{B,i}^\theta(t)\right] dt. \tag{11}$$

Hence, developing an optimal dynamic control strategy of disruptive viruses can be modeled as solving the following optimal control problem.

$$\begin{aligned} &\underset{\gamma(\cdot)\in\Gamma}{\text{Minimize}} \quad J(\gamma(\cdot)) = \int_0^T \sum_{i=1}^N \left[L_i(t) + B_i(t) + p_i\gamma_{L,i}^\theta(t) + q_i\gamma_{B,i}^\theta(t)\right] dt \\ &\text{subject to} \quad \frac{d\mathbf{I}(t)}{dt} = \mathbf{f}\left(\mathbf{I}(t), \gamma(t)\right), \quad 0 \leq t \leq T, \\ &\qquad\qquad \mathbf{I}(0) = \mathbf{I}_0. \end{aligned} \tag{P*}$$

A solution to the optimal control problem (P*) stands for an optimal dynamic control strategy of disruptive viruses. For convenience, let

$$F\left(\mathbf{I}(t), \gamma(t)\right) = \sum_{i=1}^N \left[L_i(t) + B_i(t) + p_i\gamma_{L,i}^\theta(t) + q_i\gamma_{B,i}^\theta(t)\right]. \tag{12}$$

4. A Theoretical Study of the Optimal Control Problem

In this section, we shall study the optimal control problem (P*) presented in the previous section.

4.1. Existence of an Optimal Control. As a solution to the optimal control problem (P*) stands for an optimal dynamic control strategy of disruptive viruses, it is critical to show that there is such an optimal control. For that purpose, let us show that the five conditions in Lemma 1 hold true simultaneously.

Lemma 2. *The admissible set Γ is closed.*

Proof. Let $\gamma(t) = (\gamma_{L,1}(t), \ldots, \gamma_{L,N}(t), \gamma_{B,1}(t), \ldots, \gamma_{B,N}(t))^T$ be a limit point of Γ,

$$\gamma^{(n)}(t) = \left(\gamma_{L,1}^{(n)}(t), \ldots, \gamma_{L,N}^{(n)}(t), \gamma_{B,1}^{(n)}(t), \ldots, \gamma_{B,N}^{(n)}(t)\right)^T, \quad n = 1, 2, \ldots, \tag{13}$$

a sequence of points in Γ such that

$$\left\|\gamma^{(n)}(t) - \gamma(t)\right\|_2 = \left[\int_0^T \left|\gamma^{(n)}(t) - \gamma(t)\right|^2 dt\right]^{1/2} < \frac{1}{n}. \tag{14}$$

The completeness of $(L^2(0,T))^{2N}$ implies $\gamma(t) \in L^2(0,T)^{2N}$. Hence, the claim follows from the observation that

$$\begin{aligned} \underline{\gamma_L} &\leq \gamma_{L,i}(t) = \lim_{n\to\infty} \gamma_{L,i}^{(n)}(t) \leq \overline{\gamma_L}, \\ \underline{\gamma_B} &\leq \gamma_{B,i}(t) = \lim_{n\to\infty} \gamma_{B,i}^{(n)}(t) \leq \overline{\gamma_B}, \\ &\qquad 1 \leq i \leq N. \end{aligned} \tag{15}$$

□

Lemma 3. *The admissible set Γ is convex.*

Proof. Let

$$\begin{aligned}\gamma^{(1)}(t) &= \left(\gamma_{L,1}^{(1)}(t), \ldots, \gamma_{L,N}^{(1)}(t), \gamma_{B,1}^{(1)}(t), \ldots, \gamma_{B,N}^{(1)}(t)\right)^T \\ &\in \Gamma, \\ \gamma^{(2)}(t) &= \left(\gamma_{L,1}^{(2)}(t), \ldots, \gamma_{L,N}^{(2)}(t), \gamma_{B,1}^{(2)}(t), \ldots, \gamma_{B,N}^{(2)}(t)\right)^T \\ &\in \Gamma,\end{aligned} \tag{16}$$

and $0 < \kappa < 1$. As $(L^2[0,T])^{2N}$ is a real vector space, we get

$$\begin{aligned}&(1-\kappa)\gamma^{(1)}(t) + \kappa\gamma^{(2)}(t) \\ &\in \left(L^2[0,T]\right)^{2N}.\end{aligned} \tag{17}$$

So, the claim follows from the observation that

$$\begin{aligned}&\underline{\gamma_L} \le (1-\kappa)\gamma_{L,i}^{(1)}(t) + \kappa\gamma_{L,i}^{(2)}(t) \le \overline{\gamma_L}, \\ &\underline{\gamma_B} \le (1-\kappa)\gamma_{B,i}^{(1)}(t) + \kappa\gamma_{B,i}^{(2)}(t) \le \overline{\gamma_B}, \\ &\qquad 1 \le i \le N.\end{aligned} \tag{18}$$

□

Lemma 4. *There is $\gamma \in \Gamma$ such that model (7) is solvable.*

Proof. Substituting $\gamma(t) \equiv \overline{\gamma} = (\overline{\gamma_L}, \ldots, \overline{\gamma_L}, \overline{\gamma_B}, \ldots, \overline{\gamma_B})^T$ into model (7), we get

$$\frac{d\mathbf{I}(t)}{dt} = \mathbf{f}\left(\mathbf{I}(t), \overline{\gamma}\right), \quad 0 \le t \le T. \tag{19}$$

As $\mathbf{f}(\mathbf{I}, \overline{\gamma})$ is continuously differentiable, the claim follows from the Continuation Theorem for Differential Systems [47]. □

Lemma 5. *$\mathbf{f}(\mathbf{I}, \gamma)$ is bounded by a linear function in $\mathbf{I}$.*

Proof. The claim follows from the observation that, for $i = 1, 2, \ldots, N$,

$$\begin{aligned}&(1 - L_i - B_i)\sum_{j=1}^{N}\left(\beta_{L,ij}L_j + \beta_{B,ij}B_j\right) \\ &\quad - \left(\alpha_i + \gamma_{L,i}\right)L_i \le \sum_{j=1}^{N}\beta_{L,ij}L_j \\ &\quad + \sum_{j=1}^{N}\beta_{B,ij}B_j - \left(\alpha_i + \underline{\gamma_L}\right)L_i, \\ &\alpha_i L_i - \gamma_{B,i}B_i \le \alpha_i L_i - \underline{\gamma_B}B_i.\end{aligned} \tag{20}$$

□

Lemma 6. *$F(\mathbf{I}, \gamma)$ is convex on Γ if $\theta \ge 1$.*

Proof. The Hessian of F with respect to γ,

$$\begin{aligned}&\begin{bmatrix} \dfrac{\partial^2 F}{\partial\gamma_{L,1}^2} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{L,1}\partial\gamma_{L,N}} & \dfrac{\partial^2 F}{\partial\gamma_{L,1}\partial\gamma_{B,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{L,1}\partial\gamma_{B,N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 F}{\partial\gamma_{L,N}\partial\gamma_{L,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{L,N}^2} & \dfrac{\partial^2 F}{\partial\gamma_{L,N}\partial\gamma_{B,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{L,N}\partial\gamma_{B,N}} \\ \dfrac{\partial^2 F}{\partial\gamma_{B,1}\partial\gamma_{L,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{B,1}\partial\gamma_{L,N}} & \dfrac{\partial^2 F}{\partial\gamma_{B,1}^2} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{B,1}\partial\gamma_{B,N}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 F}{\partial\gamma_{B,N}\partial\gamma_{L,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{B,N}\partial\gamma_{L,N}} & \dfrac{\partial^2 F}{\partial\gamma_{B,N}\partial\gamma_{B,1}} & \cdots & \dfrac{\partial^2 F}{\partial\gamma_{B,N}^2} \end{bmatrix} \\ &= \theta\left(\theta - 1\right)\begin{bmatrix} p_1\gamma_{L,1}^{\theta-2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & p_N\gamma_{L,N}^{\theta-2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & q_1\gamma_{B,1}^{\theta-2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & q_N\gamma_{B,N}^{\theta-2} \end{bmatrix},\end{aligned} \tag{21}$$

is always positive semidefinite. This implies the convexity of F. □

Lemma 7. *$F(\mathbf{I},\gamma) \geq \min_i\{c_i, d_i\}\|\mathbf{I}\|_\theta^\theta$, where $\|\cdot\|_\theta$ stands for the θ-norm of vectors.*

Proof. We have

$$\begin{aligned} F(\mathbf{I},\gamma) &= \sum_{i=1}^{N}\left(L_i + B_i + p_i\gamma_{L,i}^{\theta} + q_i\gamma_{B,i}^{\theta}\right) \\ &\geq \min_{1\leq i\leq N}\{p_i, q_i\}\sum_{i=1}^{N}\left(\gamma_{L,i}^{\theta} + \gamma_{B,i}^{\theta}\right) \\ &= \min_{1\leq i\leq N}\{p_i, q_i\}\left\|\mathbf{I}\right\|_\theta^\theta . \end{aligned} \tag{22}$$

□

We are ready to present the main result of this subsection.

Theorem 8. *Problem* (P*) *has an optimal control if* $\theta > 1$.

Proof. Lemmas 2–7 show that the five conditions in Lemma 1 are all met. Hence, the existence of an optimal control follows from Lemma 1. □

4.2. The Optimality System. As the optimality system for the optimal control problem (P*) offers a method for numerically solving the problem, it is critical to determine the optimality system. For that purpose, consider the corresponding Hamiltonian

$$\begin{aligned} &H(\mathbf{I}(t),\gamma(t),\lambda(t)) \\ &\quad= \sum_{i=1}^{N}\left[L_i(t) + B_i(t) + p_i\gamma_{L,i}^{\theta}(t) + q_i\gamma_{2i}^{\theta}(t)\right] \\ &\quad+ \sum_{i=1}^{N}\lambda_{L,i}(t) \\ &\quad\cdot\left\{[1 - L_i(t) - B_i(t)]\sum_{j=1}^{N}\left[\beta_{L,ij}L_j(t) + \beta_{B,ij}B_j(t)\right]\right. \\ &\quad\left.- [\alpha_i + \gamma_{L,i}(t)]L_i(t)\right\} + \sum_{i=1}^{N}\lambda_{B,i}(t)\left[\alpha_i L_i(t)\right. \\ &\quad\left.- \gamma_{B,i}(t)B_i(t)\right], \end{aligned} \tag{23}$$

where $\lambda(\cdot) = (\lambda_{L,1}(\cdot),\ldots,\lambda_{L,N}(\cdot),\lambda_{B,1}(\cdot),\ldots,\lambda_{B,N}(\cdot))^T$ is the adjoint.

Theorem 9. *Suppose $\gamma^*(\cdot)$ is an optimal control for problem* (P*) *with $\theta > 1$; $\mathbf{I}^*(\cdot)$ is the solution to the controlled SLBS model with $\gamma(\cdot) = \gamma^*(\cdot)$. Then, there exists $\lambda^*(\cdot)$ with $\lambda^*(T) = \mathbf{0}$ such that*

$$\begin{aligned} \frac{d\lambda_{L,i}^*(t)}{dt} &= -1 + \lambda_{L,i}^*(t)\left\{\alpha_i + \gamma_{L,i}^*(t)\right. \\ &\quad\left.+ \sum_{j=1}^{N}\left[\beta_{L,ij}L_j^*(t) + \beta_{B,ij}B_j^*(t)\right]\right\} \\ &\quad- \sum_{j=1}^{N}\beta_{L,ji}\left[1 - L_j^*(t) - B_j^*(t)\right]\lambda_{L,j}^*(t) \\ &\quad- \alpha_i\lambda_{B,i}^*(t), \\ \frac{d\lambda_{B,i}^*(t)}{dt} &= -1 + \gamma_{B,i}^*(t)\lambda_{B,i}^*(t) + \lambda_{L,i}^*(t) \\ &\quad\cdot\sum_{j=1}^{N}\left[\beta_{L,ij}L_j^*(t) + \beta_{B,ij}B_j^*(t)\right] \\ &\quad- \sum_{j}\beta_{B,ji}\left[1 - L_j^*(t) - B_j^*(t)\right]\lambda_{L,j}^*(t), \\ \gamma_{L,i}^*(t) &= \max\left\{\min\left\{\left[\frac{\lambda_{L,i}^*(t)L_i^*(t)}{\theta p_i}\right]^{1/(\theta-1)},\right.\right. \\ &\quad\left.\left.\overline{\gamma_L}\right\},\underline{\gamma_L}\right\}, \\ \gamma_{B,i}^*(t) &= \max\left\{\min\left\{\left[\frac{\lambda_{B,i}^*(t)B_i^*(t)}{\theta q_i}\right]^{1/(\theta-1)},\right.\right. \\ &\quad\left.\left.\overline{\gamma_B}\right\},\underline{\gamma_B}\right\}, \end{aligned} \tag{24}$$

where $0 \leq t \leq T$ and $1 \leq i \leq N$.

Proof. According to the Pontryagin Minimum Principle [26], there exists $\lambda^*(t)$ such that

$$\frac{d\lambda_{L,i}^*(t)}{dt} = -\frac{\partial H(\mathbf{I}^*(t),\gamma^*(t),\lambda^*(t))}{\partial L_i}, \quad 0 \leq t \leq T,\ 1 \leq i \leq N,$$

$$\frac{d\lambda_{B,i}^{*}(t)}{dt} = -\frac{\partial H\left(\mathbf{I}^{*}(t), \gamma^{*}(t), \lambda^{*}(t)\right)}{\partial B_i}, \quad 0 \le t \le T, \ 1 \le i \le N. \tag{25}$$

Thus, the first $2N$ equations in the claim follow by direct calculations. As the terminal cost is unspecified and the final state is free, the transversality condition $\lambda^{*}(T) = \mathbf{0}$ holds. By using the optimality condition

$$\gamma^{*}(t) = \arg\min_{\gamma(t)\in\Gamma} H\left(\mathbf{I}^{*}(t), \gamma(t), \lambda^{*}(t)\right), \tag{26}$$

we get (a) either

$$\frac{\partial H\left(\mathbf{I}^{*}(t), \gamma^{*}(t), \lambda^{*}(t)\right)}{\partial \gamma_{L,i}} = \theta p_i \left(\gamma_{L,i}^{*}(t)\right)^{\theta-1} - \lambda_{L,i}^{*}(t) L_i^{*}(t) = 0 \tag{27}$$

or $\gamma_{L,i}^{*}(t) = \underline{\gamma_L}$ or $\gamma_{L,i}^{*}(t) = \overline{\gamma_L}$ and (b) either

$$\frac{\partial H\left(\mathbf{I}^{*}(t), \gamma^{*}(t), \lambda^{*}(t)\right)}{\partial \gamma_{B,i}} = \theta q_i \left(\gamma_{B,i}^{*}(t)\right)^{\theta-1} - \lambda_{B,i}^{*}(t) B_i^{*}(t) = 0 \tag{28}$$

or $\gamma_{B,i}^{*}(t) = \underline{\gamma_B}$ or $\gamma_{B,i}^{*}(t) = \overline{\gamma_B}$. So, the last $2N$ equations in the claim follow.

By combining the above discussions, we get the optimality system for problem (P*) with $\theta > 1$ as follows.

$$\begin{aligned}
\frac{dL_i(t)}{dt} &= \left[1 - L_i(t) - B_i(t)\right] \cdot \sum_{j=1}^{N} \left[\beta_{L,ij} L_j(t) + \beta_{B,ij} B_j(t)\right] - \left[\alpha_i + \gamma_{L,i}(t)\right] L_i(t), \\
\frac{dB_i(t)}{dt} &= \alpha_i L_i(t) - \gamma_{B,i}(t) B_i(t), \\
\frac{d\lambda_{L,i}(t)}{dt} &= -1 + \lambda_{L,i}(t) \left\{\alpha_i + \gamma_{L,i}(t) + \sum_{j=1}^{N} \left[\beta_{L,ij} L_j(t) + \beta_{B,ij} B_j(t)\right]\right\} \\
&\quad - \sum_{j=1}^{N} \beta_{L,ji} \left[1 - L_j(t) - B_j(t)\right] \lambda_{L,j}(t) - \alpha_i \lambda_{B,i}(t), \\
\frac{d\lambda_{B,i}(t)}{dt} &= -1 + \gamma_{B,i}(t) \lambda_{B,i}(t) + \lambda_{L,i}(t) \cdot \sum_{j=1}^{N} \left[\beta_{L,ij} L_j(t) + \beta_{B,ij} B_j(t)\right] \\
&\quad - \sum_{j=1}^{N} \beta_{B,ji} \left[1 - L_j(t) - B_j(t)\right] \lambda_{L,j}(t), \\
\gamma_{L,i}(t) &= \max\left\{\min\left\{\left[\frac{\lambda_{L,i}(t) L_i(t)}{\theta p_i}\right]^{1/(\theta-1)}, \overline{\gamma_L}\right\}, \underline{\gamma_L}\right\}, \\
\gamma_{B,i}(t) &= \max\left\{\min\left\{\left[\frac{\lambda_{B,i}(t) B_i(t)}{\theta q_i}\right]^{1/(\theta-1)}, \overline{\gamma_B}\right\}, \underline{\gamma_B}\right\},
\end{aligned} \tag{29}$$

where $\mathbf{I}(0) = \mathbf{I}_0$, $\lambda(T) = \mathbf{0}$, $0 \le t \le T$, $1 \le i \le N$.

By applying the forward-backward Euler scheme to the optimality system, we can obtain the numerical solution to the optimal control problem (P*), that is, an optimal dynamic control strategy of disruptive viruses. □

5. Numerical Examples

This section gives some examples of the optimal dynamic control strategy of disruptive computer viruses. Given a dynamic control strategy $\gamma(t)$. Define the average control (AC) function, the average cumulative loss (ACL) function, the average cumulative cost (ACC) function, and the average cumulative performance (ACP) function as follows.

$$\mathrm{AC}(t) = \frac{1}{N} \sum_{i=1}^{N} \left[\gamma_{L,i}(t) + \gamma_{B,i}(t)\right], \quad 0 \le t \le T,$$

$$\mathrm{ACL}(t) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \left[L_i(s) + B_i(s)\right] ds, \quad 0 \le t \le T,$$

$$\mathrm{ACC}(t) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \left[p_i\gamma_{L,i}^{\theta}(s) + q_i\gamma_{B,i}^{\theta}(s)\right] ds, \quad 0 \le t \le T,$$

$$\mathrm{ACP}(t) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \left[L_i(s) + B_i(s) + p_i\gamma_{L,i}^{\theta}(s) + q_i\gamma_{B,i}^{\theta}(s)\right] ds, \quad 0 \le t \le T. \tag{30}$$

These functions form an evaluation criterion of dynamic control strategies of disruptive viruses.

5.1. Scale-Free Network. Scale-free networks are a large class of networks having widespread applications. For our purpose, generate a scale-free network G with $N = 100$ nodes using the Barabasi-Albert method [48].

Example 10. Consider an optimal control problem (P^*) on the virus-spreading network G, where the parameters and the initial conditions are set as follows.

(a) $T = 200$, $\theta = 2$, $\underline{\gamma_L} = 0$, $\overline{\gamma_L} = 0.2$, $\underline{\gamma_B} = 0.1$, and $\overline{\gamma_B} = 0.3$.

(b) $\beta_{L,ij} = 0.005$ and $\beta_{B,ij} = 0.001$, $(i, j) \in E(G)$.

(c) $\alpha_i = 0.1$ and $p_i = q_i = 1$, $i \in V(G)$.

(d) $L_i(0) = 0.1$ and $B_i(0) = 0$, $1 \le i \le N$.

For the optimal dynamic control strategy to the optimal control problem and some static control strategies, the AC functions, the ACL functions, the ACC functions, and the ACP function are shown in Figure 2.

5.2. Small-World Network. Small-world networks are another large class of networks having widespread applications. For our purpose, generate a small-world network G with $N = 100$ nodes using the Watts-Strogatz method [49].

Example 11. Consider an optimal control problem (P^*) on the virus-spreading network G, where the parameters and the initial conditions are set as follows.

(a) $T = 200$, $\theta = 2$, $\underline{\gamma_L} = 0$, $\overline{\gamma_L} = 0.2$, $\underline{\gamma_B} = 0.1$, and $\overline{\gamma_B} = 0.3$.

(b) $\beta_{L,ij} = 0.005$ and $\beta_{B,ij} = 0.001$, $(i, j) \in E(G)$.

(c) $\alpha_i = 0.1$ and $p_i = q_i = 1$, $i \in V(G)$.

(d) $L_i(0) = 0.1$ and $B_i(0) = 0$, $1 \le i \le N$.

For the optimal dynamic control strategy to the optimal control problem and some static control strategies, the AC functions, the ACL functions, the ACC functions, and the ACP function are shown in Figure 3.

5.3. Realistic Network. Consider a network G with $N = 300$ nodes cut out from the database of Stanford University [50].

Example 12. Consider an optimal control problem (P^*) on the virus-spreading network G, where the parameters and the initial conditions are set as follows.

(a) $T = 200$, $\theta = 2$, $\underline{\gamma_L} = 0$, $\overline{\gamma_L} = 0.2$, $\underline{\gamma_B} = 0.1$, and $\overline{\gamma_B} = 0.3$.

(b) $\beta_{L,ij} = 0.005$ and $\beta_{B,ij} = 0.001$, $(i, j) \in E(G)$.

(c) $\alpha_i = 0.1$ and $p_i = q_i = 1$, $i \in V(G)$.

(d) $L_i(0) = 0.1$ and $B_i(0) = 0$, $1 \le i \le N$.

For the optimal dynamic control strategy to the optimal control problem and some static control strategies, the AC functions, the ACL functions, the ACC functions, and the ACP function are shown in Figure 4.

6. Performance Evaluation

The previous discussions manifest that if the parameters in the optimal control problem (P^*) are all available, then an optimal dynamic control strategy can be obtained by numerically solving the optimality system. In realistic scenarios, however, some of these parameters might be unavailable. In such situations, it is necessary to estimate the performance of an actual dynamical control strategy in comparison with that of the optimal dynamical control strategy. Now let us present such an estimation.

Theorem 13. *Consider the optimal control problem* (P^*). *Let* $\gamma^*(\cdot)$ *be the optimal dynamic control strategy,* $\gamma(\cdot)$ *an arbitrary dynamic control strategy. Then,*

$$\left|J(\gamma(\cdot)) - J(\gamma^*(\cdot))\right| \le \frac{2Nc_1}{c_2}\left(e^{c_2T} - 1 - c_2T - \frac{c_2^2T^2}{2}\right) + \sum_{i=1}^{N} p_i \int_0^T \left|\gamma_{L,i}^{\theta}(t) - \gamma_{L,i}^{*\theta}(t)\right| dt + \sum_{i=1}^{N} q_i \int_0^T \left|\gamma_{B,i}^{\theta}(t) - \gamma_{B,i}^{*\theta}(t)\right| dt, \tag{31}$$

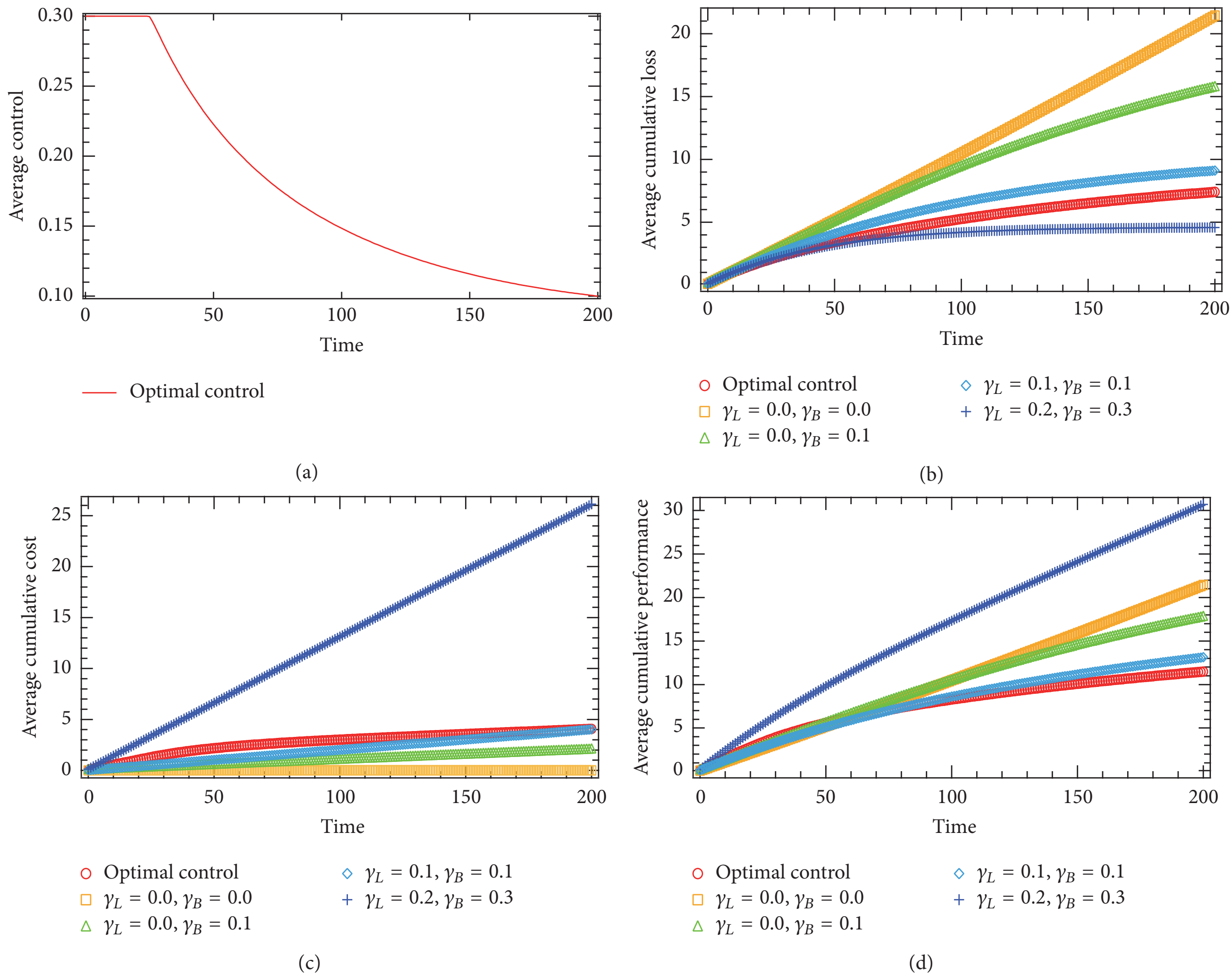

Figure 2: (a) The AC functions, (b) the ACL functions, (c) the ACC functions, and (d) the ACP functions for the optimal dynamic control strategies in Example 10.

where

$$c_1 = \max\left\{2\max_{1\le i\le N}\left\{\sum_{j=1}^{N}\left|\beta_{L,ij}\right|\right\} + 2\max_{1\le i\le N}\left\{\sum_{j=1}^{N}\left|\beta_{B,ij}\right|\right\} + \overline{\gamma_L} - \underline{\gamma_L}, \overline{\gamma_B} - \underline{\gamma_B}\right\}, \quad (32)$$

$$c_2 = 2\max_{1\le i\le N}\left\{\sum_{j=1}^{N}\left|\beta_{L,ij}\right|\right\} + \overline{\alpha} + \overline{\gamma_L} + \max\left\{2\max_{1\le i\le N}\left\{\sum_{j=1}^{N}\left|\beta_{B,ij}\right|\right\}, \overline{\gamma_B}\right\}.$$

Proof. Let $\|\cdot\|$ denote the ∞-norm. Let $\mathbf{I}^*(\cdot) = (\mathbf{L}^*(\cdot)^T, \mathbf{B}^*(\cdot)^T)^T$ denote the solution to the SLBS model with control $\gamma^*(t)$ and $\mathbf{I}(\cdot) = (\mathbf{L}(\cdot)^T, \mathbf{B}(\cdot)^T)^T$ the solution to the SLBS model with control $\gamma(\cdot)$. As

$$\begin{aligned}
\mathbf{L}(t) = \mathbf{L}_0^* &+ \int_0^t \operatorname{diag}\left(1 - B_i(s) - L_i(s)\right) \cdot \mathbf{A}_L \mathbf{L}(s)\, ds \\
&+ \int_0^t \operatorname{diag}\left(1 - B_i(s) - L_i(s)\right) \cdot \mathbf{A}_B \mathbf{B}(s)\, ds \\
&- \int_0^t \operatorname{diag}\left(\alpha_i + \gamma_{L,i}(s)\right) \mathbf{L}(s)\, ds, \\
\mathbf{L}^*(t) = \mathbf{L}_0^* &+ \int_0^t \operatorname{diag}\left(1 - B_i^*(s) - L_i^*(s)\right) \cdot \mathbf{A}_L \mathbf{L}^*(s)\, ds
\end{aligned}$$

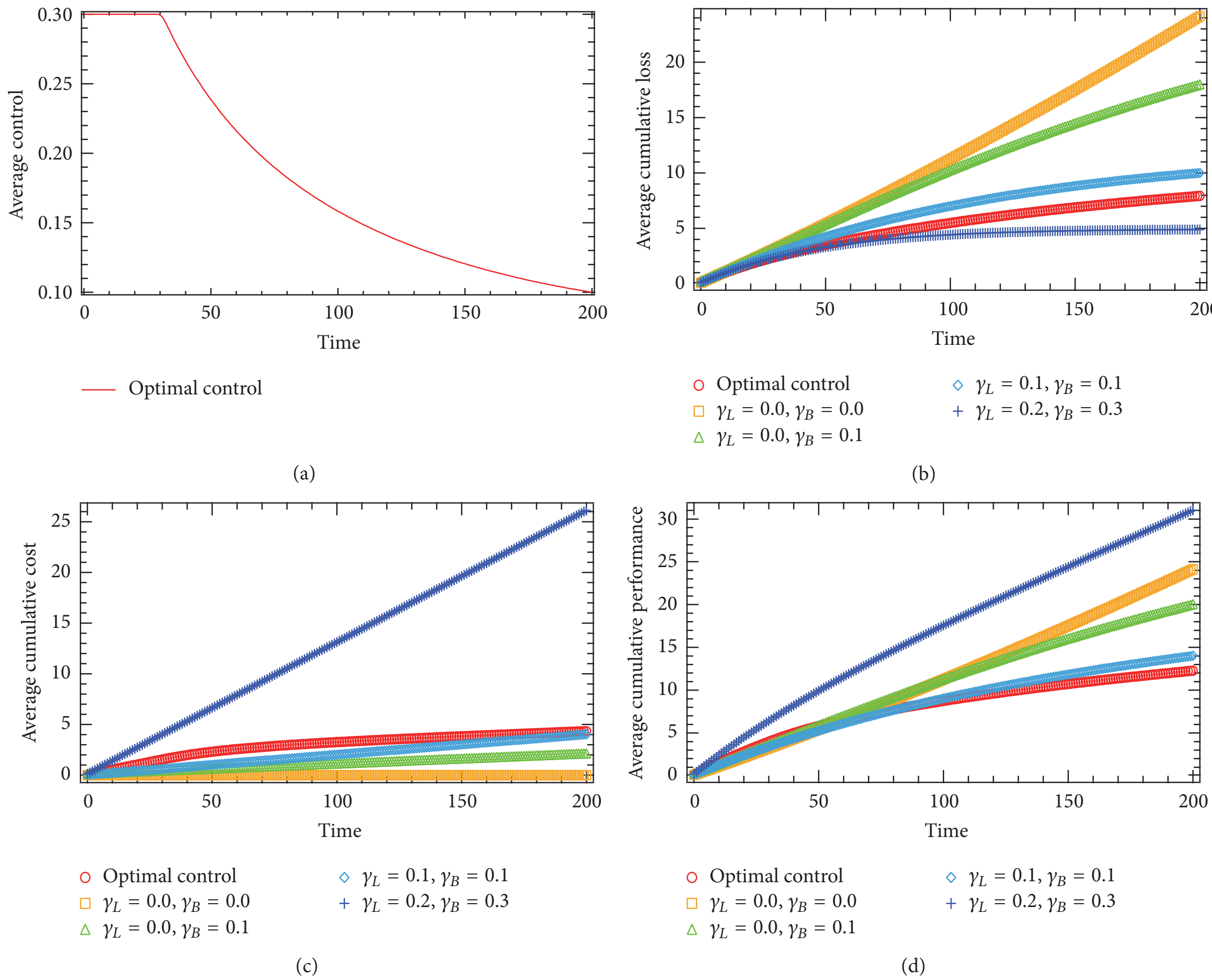

FIGURE 3: (a) The AC functions, (b) the ACL functions, (c) the ACC functions, and (d) the ACP functions for the optimal dynamic control strategies in Example 11.

$$
\begin{aligned}
&+ \int_0^t \operatorname{diag}\left(1 - B_i^*(s) - L_i^*(s)\right) \\
&\cdot \mathbf{A}_B \mathbf{B}^*(s)\, ds \\
&- \int_0^t \operatorname{diag}\left(\alpha_i + \gamma_{L,i}^*(s)\right) \mathbf{L}^*(s)\, ds,
\end{aligned} \tag{33}
$$

we get

$$
\begin{aligned}
\mathbf{L}(t) - \mathbf{L}^*(t) &= \int_0^t \operatorname{diag}\left(1 - B_i(s) - L_i(s)\right) \\
&\cdot \mathbf{A}_L \left[\mathbf{L}(s) - \mathbf{L}^*(s)\right] ds \\
&+ \int_0^t \operatorname{diag}\left(1 - B_i(s) - L_i(s)\right) \\
&\cdot \mathbf{A}_B \left[\mathbf{B}(s) - \mathbf{B}^*(s)\right] ds \\
&- \int_0^t \operatorname{diag}\left(L_i(s) - L_i^*(s) + B_i(s) - B_i^*(s)\right) \\
&\cdot \mathbf{A}_L \mathbf{L}^*(s)\, ds \\
&- \int_0^t \operatorname{diag}\left(L_i(s) - L_i^*(s) + B_i(s) - B_i^*(s)\right) \\
&\cdot \mathbf{A}_B \mathbf{B}^*(s)\, ds - \int_0^t \operatorname{diag}\left(\alpha_i + \gamma_{L,i}(s)\right) \\
&\cdot \left[\mathbf{L}(s) - \mathbf{L}^*(s)\right] ds - \int_0^t \operatorname{diag}\left(\gamma_{L,i}(s) - \gamma_{L,i}^*(s)\right) \\
&\cdot \mathbf{L}^*(s)\, ds.
\end{aligned} \tag{34}
$$

So,

$$
\begin{aligned}
\left\|\mathbf{L}(t) - \mathbf{L}^*(t)\right\| &\le \left\|\mathbf{A}_L\right\| \int_0^t \left\|\operatorname{diag}\left(1 - B_i(s) - L_i(s)\right)\right\| \\
&\cdot \left\|\mathbf{L}(s) - \mathbf{L}^*(s)\right\| ds + \left\|\mathbf{A}_B\right\| \\
&\cdot \int_0^t \left\|\operatorname{diag}\left(1 - B_i(s) - L_i(s)\right)\right\| \\
&\cdot \left\|\mathbf{B}(s) - \mathbf{B}^*(s)\right\| ds + \left\|\mathbf{A}_L\right\|
\end{aligned}
$$

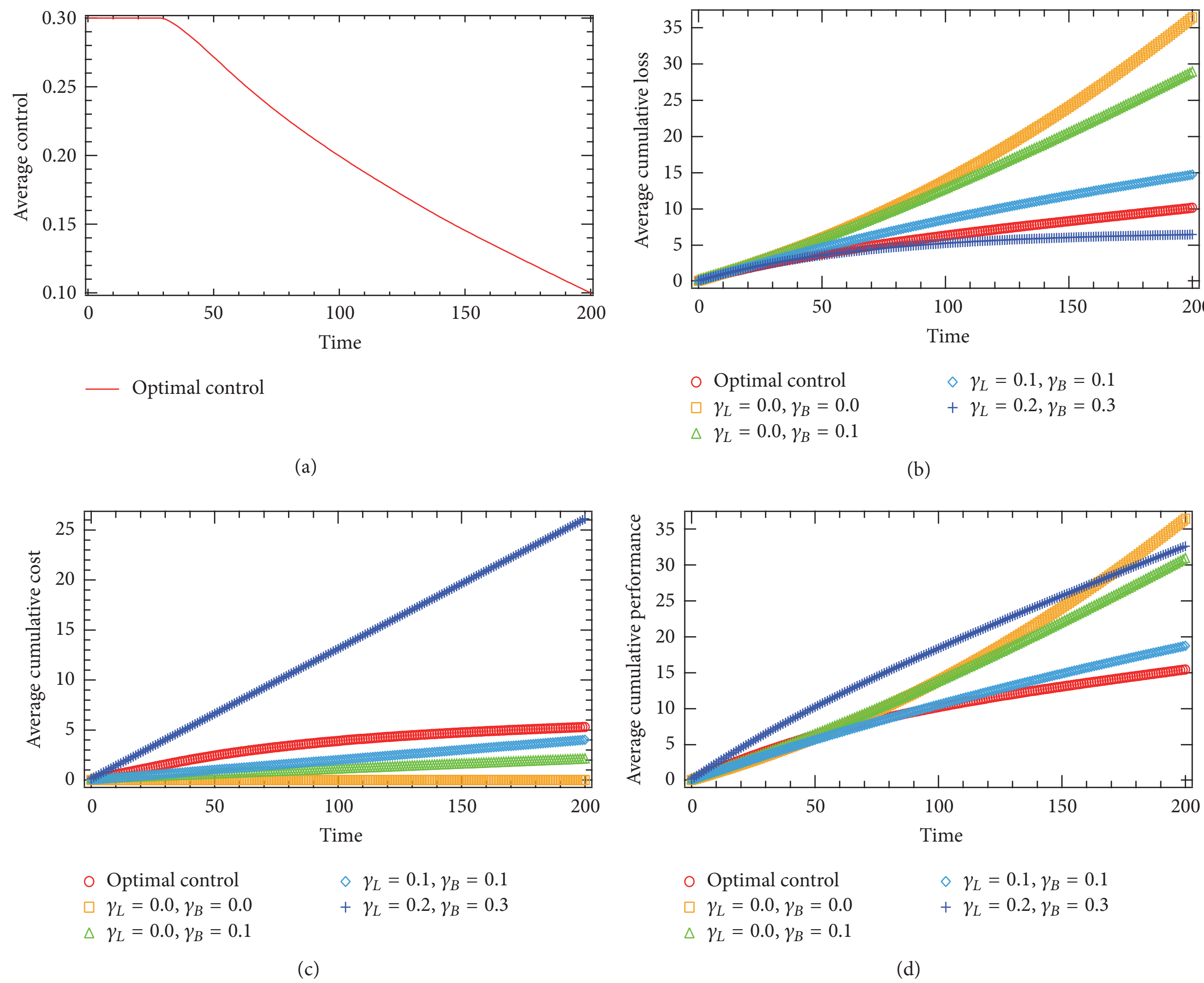

FIGURE 4: (a) The AC functions, (b) the ACL functions, (c) the ACC functions, and (d) the ACP functions for the optimal dynamic control strategies in Example 12.

$$
\begin{aligned}
&\cdot \int_0^t \left\|\operatorname{diag}\left(L_i(s) - L_i^*(s) + B_i(s) - B_i^*(s)\right)\right\| \\
&\cdot \left\|\mathbf{L}^*(s)\right\| ds + \left\|\mathbf{A}_B\right\| \\
&\cdot \int_0^t \left\|\operatorname{diag}\left(L_i(s) - L_i^*(s) + B_i(s) - B_i^*(s)\right)\right\| \\
&\cdot \left\|\mathbf{B}^*(s)\right\| ds + \int_0^t \left\|\operatorname{diag}\left(\alpha_i + \gamma_{L,i}(s)\right)\right\| \\
&\cdot \left\|\mathbf{L}(s) - \mathbf{L}^*(s)\right\| ds \\
&+ \int_0^t \left\|\operatorname{diag}\left(\gamma_{L,i}(s) - \gamma_{L,i}^*(s)\right)\right\| \cdot \left\|\mathbf{L}^*(s)\right\| ds \\
&\le \left(2\left\|\mathbf{A}_L\right\| + 2\left\|\mathbf{A}_B\right\| + \overline{\gamma_L} - \underline{\gamma_L}\right) t + \left(\left\|\mathbf{A}_L\right\| + \overline{\alpha} + \overline{\gamma_L}\right) \\
&\cdot \int_0^t \left\|\mathbf{L}(s) - \mathbf{L}^*(s)\right\| ds + \left\|\mathbf{A}_B\right\| \\
&\cdot \int_0^t \left\|\mathbf{B}(s) - \mathbf{B}^*(s)\right\| ds.
\end{aligned}
\tag{35}
$$

As

$$
\begin{aligned}
\mathbf{B}(t) &= \mathbf{B}_0^* + \int_0^t \operatorname{diag}(\alpha_i)\, \mathbf{L}(s)\, ds \\
&\quad - \int_0^t \operatorname{diag}\left(\gamma_{B,i}(s)\right) \mathbf{B}(s)\, ds, \\
\mathbf{B}^*(t) &= \mathbf{B}_0^* + \int_0^t \operatorname{diag}(\alpha_i)\, \mathbf{L}^*(s)\, ds \\
&\quad - \int_0^t \operatorname{diag}\left(\gamma_{B,i}^*(s)\right) \cdot \mathbf{B}^*(s)\, ds,
\end{aligned}
\tag{36}
$$

we get

$$\begin{aligned}\mathbf{B}(t)-\mathbf{B}^*(t) &= \int_0^t \operatorname{diag}(\alpha_i)\left[\mathbf{L}(s)-\mathbf{L}^*(s)\right]ds \\ &\quad - \int_0^t \operatorname{diag}\left(\gamma_{B,i}(s)\right)\left[\mathbf{B}(s)-\mathbf{B}^*(s)\right]ds \\ &\quad - \int_0^t \operatorname{diag}\left(\gamma_{B,i}(s)-\gamma_{B,i}^*(s)\right)\mathbf{B}^*(s)\,ds.\end{aligned} \tag{37}$$

Thus,

$$\begin{aligned}&\|\mathbf{B}(t)-\mathbf{B}^*(t)\| \\ &\quad\le \int_0^t \|\operatorname{diag}(\alpha_i)\|\cdot\|\mathbf{L}(s)-\mathbf{L}^*(s)\|\,ds \\ &\qquad + \int_0^t \|\operatorname{diag}(\gamma_{B,i}(s))\|\cdot\|\mathbf{B}(s)-\mathbf{B}^*(s)\|\,ds \\ &\qquad + \int_0^t \left\|\operatorname{diag}\left(\gamma_{B,i}(s)-\gamma_{B,i}^*(s)\right)\right\|\cdot\|\mathbf{B}^*(s)\|\,ds \\ &\quad\le \left(\overline{\gamma_B}-\underline{\gamma_B}\right)t+\overline{\alpha}\int_0^t\|\mathbf{L}(s)-\mathbf{L}^*(s)\|\,ds \\ &\qquad + \overline{\gamma_B}\int_0^t\|\mathbf{B}(s)-\mathbf{B}^*(s)\|\,ds.\end{aligned} \tag{38}$$

As $\|\mathbf{I}(t)-\mathbf{I}^*(t)\| = \max\{\|\mathbf{L}(t)-\mathbf{L}^*(t)\|, \|\mathbf{B}(t)-\mathbf{B}^*(t)\|\}$, we get

$$\begin{aligned}\left\|\mathbf{I}(t)-\mathbf{I}^*(t)\right\| &\le \max\left\{2\left\|\mathbf{A}_L\right\|+2\left\|\mathbf{A}_B\right\|\right. \\ &\quad \left.+\overline{\gamma_L}-\underline{\gamma_L},\overline{\gamma_B}-\underline{\gamma_B}\right\}t+\left(\left\|\mathbf{A}_L\right\|+\overline{\alpha}\right. \\ &\quad \left.+\overline{\gamma_L}\right)\int_0^t\left\|\mathbf{L}(s)-\mathbf{L}^*(s)\right\|ds \\ &\quad +\max\left\{\left\|\mathbf{A}_B\right\|,\overline{\gamma_B}\right\} \\ &\quad \cdot\int_0^t\left\|\mathbf{B}(s)-\mathbf{B}^*(s)\right\|ds \le c_1 t \\ &\quad + c_2\int_0^t\left\|\mathbf{I}(s)-\mathbf{I}^*(s)\right\|ds.\end{aligned} \tag{39}$$

Applying the Gronwall inequality [47], we get

$$\begin{aligned}\left\|\mathbf{I}(t)-\mathbf{I}^*(t)\right\| &\le c_1 t + c_1 c_2\int_0^t s e^{c_2(t-s)}ds \\ &= \frac{c_1}{c_2}\left(e^{c_2 t}-1-c_2 t\right).\end{aligned} \tag{40}$$

Hence, we deduce that

$$\begin{aligned}&\left|J(\gamma(\cdot))-J(\gamma^*(\cdot))\right| \\ &\quad\le \sum_{i=1}^N\int_0^T\left|L_i(t)-L_i^*(t)\right|dt \\ &\qquad + \sum_i\int_0^T\left|B_i(t)-B_i^*(t)\right|dt \\ &\qquad + \sum_i p_i\int_0^T\left|\gamma_{L,i}^\theta(t)-\gamma_{L,i}^{*\theta}(t)\right|dt \\ &\qquad + \sum_{i=1}^N q_i\int_0^T\left|\gamma_{B,i}^\theta(t)-\gamma_{B,i}^{*\theta}(t)\right|dt \\ &\quad\le 2N\int_0^T\left\|\mathbf{I}(t)-\mathbf{I}^*(t)\right\|dt \\ &\qquad + \sum_{i=1}^N p_i\int_0^T\left|\gamma_{L,i}^\theta(t)-\gamma_{L,i}^{*\theta}(t)\right|dt \\ &\qquad + \sum_{i=1}^N q_i\int_0^T\left|\theta_i^k(t)-\theta_i^{*k}(t)\right|dt \\ &\quad\le \frac{2Nc_1}{c_2}\left(e^{c_2 T}-1-c_2 T-\frac{c_2^2T^2}{2}\right) \\ &\qquad + \sum_{i=1}^N p_i\int_0^T\left|\gamma_{L,i}^\theta(t)-\gamma_{L,i}^{*\theta}(t)\right|dt \\ &\qquad + \sum_{i=1}^N q_i\int_0^T\left|\gamma_{B,i}^\theta(t)-\gamma_{B,i}^{*\theta}(t)\right|dt.\end{aligned} \tag{41}$$

□

Although this estimation is rough, it takes the first step towards the accurate performance evaluation of actual dynamic control strategies of disruptive computer viruses.

7. Conclusions and Remarks

This paper has studied the problem of containing disruptive computer viruses in a cost-effective way. The problem has been modeled as an optimal control problem. A criterion for the existence of an optimal control has been given, and the optimality system has been derived. Some examples of the optimal dynamic control strategy have been presented. Finally, the performance of an actual control strategy of disruptive viruses has been estimated.

Towards this direction, there are a number of problems that are worth studying. First, the bandwidth resources consumed in the virus control process should be measured and incorporated in the cost. Second, the optimal dynamic control problem should be investigated under sophisticated epidemic models such as the impulsive epidemic models [51, 52], the stochastic epidemic models [53–55], and the epidemic models on time-varying networks [56–58]. Last, it is rewarding to apply the methodology developed in this paper to the optimal dynamic control of rumor spreading [59–61].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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