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Research Article

Global Analysis of a Liénard System with Quadratic Damping

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In this paper, the global analysis of a Liénard equation with quadratic damping is studied. There are 22 different global phase portraits in the Poincaré disc. Every global phase portrait is given as well as the complete global bifurcation diagram. Firstly, the equilibria at finite and infinite of the Liénard system are discussed. The properties of the equilibria are studied. Then, the sufficient and necessary conditions of the system with closed orbits are obtained. The degenerate Bogdanov-Takens bifurcation is studied and the bifurcation diagrams of the system are given.

1. Introduction and Main Results

Liénard equations have a very wide application in many areas, such as mechanics, electronic technology, and modern biology; see [1–4]. People are strongly interested in the solution existence, vibration, and periodic solutions of Liénard equations, which promote the research of Liénard equations more and more deeply, as shown in [5–9]. All kinds of problems about Liénard equations are always the focus of the theory of differential equations. In 2016, Llibre [10] studied the centers of the analytic differential systems and analyzed the focus-center problem. H. Chen and X. Chen [11–13] investigated the dynamical behaviour of a cubic Liénard system with global parameters, analyzing the qualitative properties of all the equilibria and judging the existence of limit cycles and homoclinic loops for the whole parameter plane. They gave positive answers to Wang Kooij's [14] two conjectures and further properties of those bifurcation curves such as monotonicity and smoothness.

In 1977, Lins, de Melo, and Pugh studied the Liénard equations

$$\begin{aligned}\frac{dx}{dt} &= y - F(x),\\ \frac{dy}{dt} &= -x,\end{aligned} \tag{1}$$

where F is a polynomial of degree $n+1$, or equivalently,

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{2}$$

with $f(x) = F'(x)$. They proposed the following result.

Conjecture 1. *If $f(x)$ has degree n, then (1) has at most $[n/2]$ limit cycles ($[n/2]$ is the integer part of $n/2$, $n \geq 2$).*

$n = 2$ is proved by [15]; $n = 3$ is proved by [16]. The problem for $n > 3$ is still open. In 1988, Lloyd and Lynch [17] considered the similar problem for generalized Liénard equations

$$\begin{aligned}\frac{dx}{dt} &= y - F(x),\\ \frac{dy}{dt} &= -g(x),\end{aligned} \tag{3}$$

where F is a polynomial of degree $n+1$ and $g(x)$ is a polynomial of degree m. In most cases, they gave an upper bound for the number of small amplitude limit cycles that can bifurcate out of a single nondegenerate singularity. If we denote by $N(m,n)$ the uniform upper bound for the number of limit cycles (admitting a priori that $N(m,n)$ could be infinite), then the results in [17] give a lower bound for $N(m,n)$. In 1988 Coppel [18] proved that $N(2,1) = 1$. In [19–22], it was proved that $N(3,1) = 1$. Up to now, as far as we know, only these three cases have been completely investigated.

Consider the Liénard equations

$$\frac{dx}{dt} = y - F(x),$$
$$\frac{dy}{dt} = -x^{2m+1}, \tag{4}$$

where $F(x) = ax^3 + bx^2 + cx$, $a \neq 0$ and $m \in \mathbb{N}$. We only discuss $a > 0$, because the case $a < 0$ can be derived from the case $a > 0$ by using the transformation $x \longrightarrow -x$, $y \longrightarrow -y$, and $a \longrightarrow -a$. From the above two motivations, we shall give a complete classification for all the global phase portraits of the Liénard system (4).

We give the following theorem.

Theorem 2. *All phase portraits of system (4) can be given, as shown in Figures 1 and 2.*

The classifications of global phase portraits are explained in Section 2 and the infinite and finite critical points are discussed in Sections 3 and 4.

The paper is organized as follows. Section 2 explains the classification for all kinds of Liénard system (4). The infinite and finite critical points are discussed in Sections 3 and 4, respectively. Section 5 provides the sufficient and necessary condition for Liénard system (4) to have closed orbits.

2. Explanation of Global Dynamics

The bifurcation diagram and global phase portraits of system (4) for parameters a, b, c, m in all cases are shown in Figure 1.

For example, as shown in Figure 1 (k, l), if $b > 0$, the elliptic sector lies in the negative y-axis; if $b < 0$, the elliptic sector lies in the positive y-axis.

(A) Global phase portraits of $m = 0$: there exist infinite critical points A and B.

(1) Suppose $c < 0$ and $a > 0$. A unique stable limit cycle appears around the equilibrium O of system (4). If $c \leq -2$, O is an unstable node, and the global phase portrait is shown in Figure 1(a); if $-2 < c < 0$, O is an unstable focus, and the global phase portrait is shown in Figure 1(b).

(2) Suppose $c > 0$ and $a > 0$. There are no closed orbits in system (4). If $0 \leq c < 2$ and $a > 0$, O is a stable focus, and the global phase portrait is shown in Figure 1(c); if $c \geq 2$ and $a > 0$, O is a stable node, and the global phase portrait is shown in Figure 1(d).

(3) Suppose $c < 0$ and $a < 0$. There are no closed orbits in system (4). If $c \leq -2$, O is an unstable node, and the global phase portrait is shown in Figure 2(a); if $-2 < c < 0$, O is an unstable focus, and the global phase portrait is shown in Figure 2(b).

(4) Suppose $c > 0$ and $a < 0$. A unique unstable limit cycle appears around the equilibrium O of system (4). If $0 \leq c < 2$ and $a > 0$, O is a stable focus, and the global phase portrait is shown in Figure 2(c); if $c \geq 2$ and $a > 0$, O is a stable node, and the global phase portrait is shown in Figure 2(d).

(B) Global phase portraits of $m = 1$: there exist infinite critical points A_1 and B.

(1) Suppose $c > 0$ or $c = b = 0$, and $a > 0$. There are no closed orbits in system (4). O is a stable degenerate node, and the global phase portrait is shown in Figure 1(e).

(2) Suppose $c > 0$ and $a < 0$. A unique unstable limit cycle appears around the stable degenerate node O of system (4), and the global phase portrait is shown in Figure 2(e).

(3) Suppose $c < 0$ or $c = b = 0$, and $a < 0$. There are no closed orbits in system (4). O is an unstable degenerate node, and the global phase portrait is shown in Figure 2(f).

(4) Suppose $c < 0$ and $a > 0$. A unique stable limit cycle appears around the unstable degenerate node O of system (4), and the global phase portrait is shown in Figure 1(f).

(5) Suppose $c = 0$ and $b \neq 0$. There are no closed orbits in system (4). If $b > 0$, the elliptic sector lies in the positive y-axis, and the global phase portraits are shown in Figures 1(g) and 2(g); if $b < 0$, the elliptic sector lies in the negative y-axis, and the global phase portraits are shown in Figures 1(h) and 2(h).

(C) Global phase portraits of $m \geq 2$: there exists a unique infinite critical point B.

(1) Suppose $c > 0$ or $c = b = 0$, and $a > 0$. There are no closed orbits in system (4). O is a stable degenerate node, and the global phase portrait is shown in Figure 1(i).

(2) Suppose $c > 0$ and $a < 0$. A unique unstable limit cycle appears around the stable degenerate node O of system (4), and the global phase portrait is shown in Figure 2(i).

(3) Suppose $c < 0$ and $a > 0$. A unique stable limit cycle appears around the unstable degenerate node O of system (4), and the global phase portrait is shown in Figure 1(j).

(4) Suppose $c < 0$ or $c = b = 0$, and $a < 0$. There are no closed orbits in system (4). O is an unstable degenerate node, and the global phase portrait is shown in Figure 2(j).

(5) Suppose $c = 0$ and $b \neq 0$. There are no closed orbits in system (4). If $b > 0$, the elliptic sector lies in the positive y-axis, and the global phase portraits are shown in Figure 1(k) and the picture (k) in Figure 2; if $b < 0$, the elliptic sector lies in the negative y-axis, and the global phase portraits are shown in Figures 1(l) and 2(l).

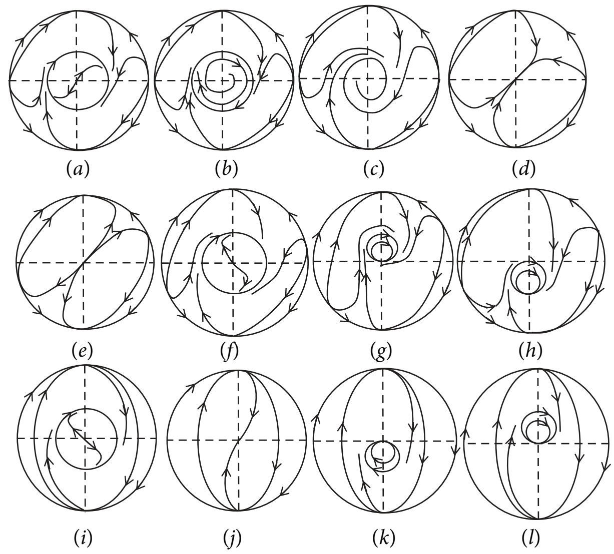


Figure 1: The global phase portraits of system (4) as the parameter $a > 0$.

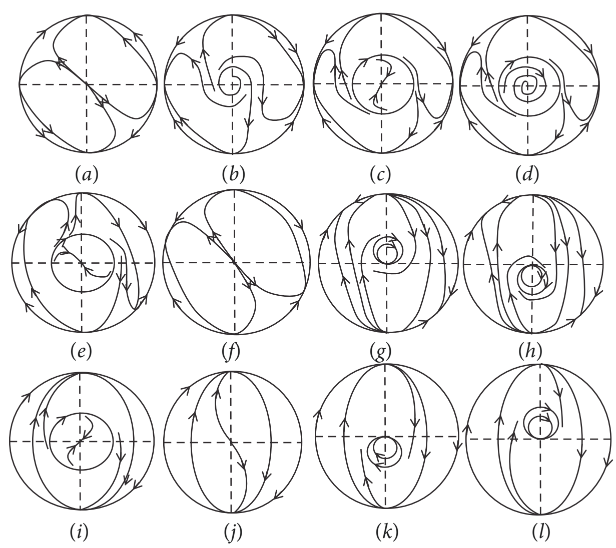


Figure 2: The global phase portraits of system (4) as the parameter $a < 0$.

3. Analysis of Equilibria

Clearly, system (4) has a unique equilibrium $O : (0, 0)$.

Lemma 3. *The type of equilibrium O in system (4) is shown as Table 1.*

Proof. Now we consider the case $m = 0$. The Jacobian matrix at O is

$$J = \begin{pmatrix} -c & 1 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

from which we obtain that $D = \det J = 1$, $T = \operatorname{trace} J = -c$. Further, O is a focus when $\Delta = T^2 - 4D = c^2 - 4 < 0$ and a node when $\Delta < 0$. Clearly, $\Delta = 0$ if and only if $c^2 = 4$. Therefore, O is a stable focus when $0 < c < 2$, an unstable focus when $-2 < c < 0$, a stable node when $c \geq 2$, and an unstable node when $c \leq -2$.

For the case that $c = 0$, we consider the case that the linear part of system (4) around O has eigenvalues $\alpha(c) \pm i\beta(c)$ for c near 0, in which $\alpha(\eta^2) = -c/2$. Obviously, $\alpha(0) = 0$ and $\beta(0) \neq 0$. Clearly $d\alpha(c)/dc = -1/2$.

Now, we need to compute the coefficients of Hopf bifurcation of order 1. According to the Hopf bifurcation theory [23], we obtain the following results for b outside the interval $(0, 1)$. By ([23] P.152), we can compute the coefficients of Hopf bifurcation of (4)

$$\operatorname{Re}C_1(a) = -\frac{a}{8}. \tag{6}$$

We can get $\operatorname{Re}C_1(a) > 0$ for $a < 0$; and we can get $\operatorname{Re}C_1(a) < 0$ for $a > 0$.

Table 1: Qualitative properties of equilibria O.

Possibilities a, b, c			Type and stability
$m = 0$	$c \le -2$		unstable node
	$-2 < c < 0$		unstable focus
	$c = 0$		unstable weak focus
	$0 < c < 2$		stable focus
	$c \ge 2$		stable node
$m \ge 1$	$c < 0$		unstable degenerate node
	$c > 0$		stable degenerate node
	$c = 0$	$b \ne 0$	cusp
		$b = 0$	stable degenerate node

We need to compute the sign of $c\mathrm{Re}C_1 d\alpha(c)/dc = ac/16$. When $ac \ge 0$, we can get $c\mathrm{Re}C_1 d\alpha(c)/dc \ge 0$; and when $ac > 0$, we can get $c\mathrm{Re}C_1 d\alpha(c)/dc > 0$. □

Therefore, we obtain the following lemma.

Lemma 4. *When $a > 0$ and $c < 0$, the equilibrium O of system (4) is an unstable weak focus with multiplicity 1, and there is a unique stable limit cycle bifurcating from O; when $a < 0$ and $c > 0$, the equilibrium O of system (4) is a stable weak focus with multiplicity 1, and there is a unique unstable limit cycle bifurcating from O; when $a > 0$ and $c > 0$, the equilibrium O of system (4) is an unstable weak focus with multiplicity 1, and there are no closed orbits near O; when $a < 0$ and $c \le 0$, the equilibrium O of system (4) is a stable weak focus with multiplicity 1, and there are no closed orbits near O.*

3.1. Degenerate Bogdanov-Takens Bifurcation. In another case $m \ge 1$ and $c \ne 0$, only one eigenvalue of linearization of system (4) at O equals zero. In fact, by a reversible transformation

$$\tilde{x} = y, \qquad \tilde{y} = x - \frac{y}{c}, \tag{7}$$

which changes the linearization of system (4) into Jordan canonical form near O, when $m = 1$, we get

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= -\left(\tilde{y} + \frac{\tilde{x}}{c}\right)^3, \\ \frac{d\tilde{y}}{dt} &= -c\tilde{y} - b\left(\tilde{y} + \frac{\tilde{x}}{c}\right)^2 - a\left(\tilde{y} + \frac{\tilde{x}}{c}\right)^3. \end{aligned} \tag{8}$$

Let the second equation of (8) equal zero, and we solve that $\tilde{y} = Y_1(\tilde{x}) := -b\tilde{x}^2/c^2 + o(|\tilde{x}|^2)$ by the Implicit Function Theorem. Substituting $\tilde{y}$ of the first equation of (8) by $Y_1(\tilde{x})$, we obtain that

$$\frac{d\tilde{x}}{dt} = -\frac{\tilde{x}^3}{c^3} + o\left(|\tilde{x}|^3\right). \tag{9}$$

When $c > 0$, O is a stable degenerate node; when $c < 0$, O is an unstable degenerate node.

In the remaining case that $m \ge 1$ and $c = 0$, the two eigenvalues of the linearization of system (4) at O are both zero but the linear part does not equal zero identically. System (4) is equivalent to this system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\left(3ax^2 + 2bx\right)y - x^{2m+1}. \end{aligned} \tag{10}$$

By Theorem 7.2 of [24, Chapter 2], when $b = 0$ and $a > 0$, O is a stable degenerate node; when $b = 0$ and $a < 0$, O is an unstable degenerate node.

When $b \ne 0$, we can get that an elliptic sector and a hyperbolic sector consist of the field of the O by Theorem 7.2 of [24, Chapter 2].

Lemma 5. *Suppose $m = 1, b = 0$, and $c = 0$, then there is a neighborhood V of the point $(0,0)$ in $\mathbb{R}$ such that system (16) displays a degenerate Bogdanov-Takens bifurcation near $O(0,0)$ when (ϵ, c) varies in V. More concretely, there exist six curves*

(a) $R^+ = \{(\epsilon, c) \mid \epsilon = 0, c < 0\}$,
(b) $R^- = \{(\epsilon, c) \mid \epsilon = 0, c < 0\}$,
(c) $H_1 = \{(\epsilon, c) \mid c = 0, \epsilon < 0\}$,
(d) $H_2 = \{(\epsilon, c) \mid c = \epsilon + O(\varepsilon^2), \epsilon > 0\}$,
(e) $HL = \{(\epsilon, c) \mid c = -(5/4)\epsilon + O(\varepsilon^{2/3}), \epsilon > 0\}$,
(f) $B = \{(\epsilon, c) \mid c = -c_0\epsilon + O(\varepsilon^{2/3}), \epsilon > 0, c_0 \approx 0.752\}$.

When $a < 0$, system (16) displays a bifurcation of equilibria, a Hopf bifurcation, a homoclinic bifurcation, and a double limit cycle bifurcation near O when (ϵ, c) pass through the curves $R^+ \bigcup R^-, H_1 \bigcup H_2, HL$, and B. c will be replaced with $-c$ when $a > 0$.

Proof. When $a < 0$, being the standard form of degenerate Bogdanov-Takens system as shown in [1], the equilibrium O of system (16) is a stable degenerate node. Thus, equilibrium O of system (4) is a stable degenerate node and a degenerate Bogdanov-Takens bifurcation of codimension-2 will occur near the stable degenerate node when parameter c crosses $c = 0$, respectively, with $b = 0$ and $m = 1$. By [16], we know the following two-parameter family provides a universal unfolding of (16).

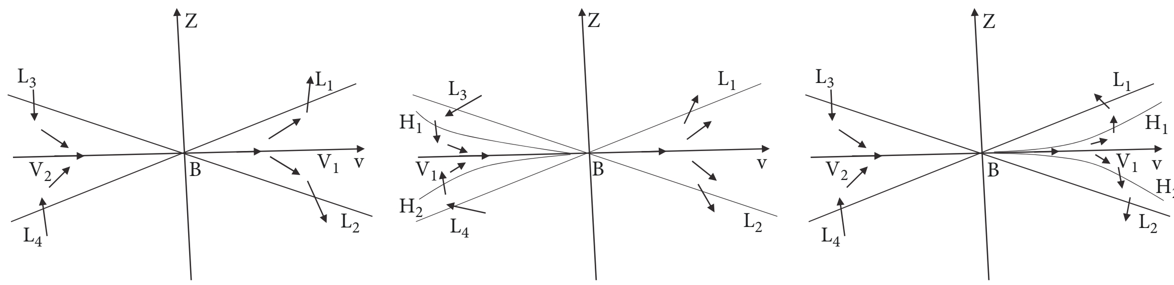
$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \epsilon x - cy - 3ax^2y - x^3. \end{aligned} \tag{11}$$

The bifurcation diagrams and phase portraits of (17) are shown in Figure 3.

When $a > 0$, with the transformation $y \longrightarrow -y$ and $dt = -dt$, we can know the following two-parameter family provides a universal unfolding of (16)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \epsilon x + cy + 3ax^2y - x^3. \end{aligned} \tag{12}$$

Therefore c will be replaced with $-c$ when $a > 0$. □



(a) $m = 2, a^2 \leq 4$ (b) $m = 2$ and $a < -2$, or $m \geq 3$ and $a < 0$ (c) $m = 2$ and $a > 2$, or $m \geq 3$ and $a > 0$

FIGURE 3: The qualitative properties of (0,0) of the system (4).

Lemma 6. *Suppose that $m = 1, b \neq 0$, and $c = 0$, then there is a neighborhood V_2 of the point $(0,0,0)$ in $\mathbb{R}$ such that system (16) displays a codimension-3 Bogdanov-Takens bifurcation near $O(0,0)$ when (μ_1, μ_2, c) varies in V_2.*

4. Equilibria at Infinity

In this section, we discuss the qualitative properties of the equilibria at infinity, which reflect the tendencies of x, y as going up by a large amount. With a Poincaré transformation $x = 1/z$, $y = u/z$, system (4) can be rewritten as

$$\frac{du}{dt} = -z^2 + u\left(a + bz + cz^2\right) - u^2z^2,$$
$$\frac{dz}{dt} = az + bz^2 + cz^3 - uz^3, \tag{13}$$

where $d\tau = dt/z^2$ and $m = 0$.

$$\frac{du}{dt} = -1 + uz^{2m-2}\left(a + bz + cz^2\right) - u^2z^{2m},$$
$$\frac{dz}{dt} = z^{2m-1}\left(a + bz + cz^2\right) - uz^{2m+1}, \tag{14}$$

where $d\tau = dt/z^{2m}$ and $m \geq 1$. System (13) has an equilibrium $A : (0,0)$ on the u–axis, and system (14) has an equilibrium $A_1 : (1/a, 0)$ when $m = 1$ and no equilibria when $m > 1$ on the u–axis, which corresponds to an equilibrium I_A at infinity on the x–axis. With another Poincaré transformation $x = v/z$, $y = 1/z$, system (4) is changed into

$$\frac{dv}{dt} = z^2 - \left(av^3 + bv^2z + cvz^2\right) + v^2z^2,$$
$$\frac{dz}{dt} = vz^3, \tag{15}$$

where $d\tau = dt/z^2$ and $m = 0$.

$$\frac{dv}{dt} = z^{2m} - z^{2m-2}\left(av^3 + bv^2z + cvz^2\right) + v^{2m+2},$$
$$\frac{dz}{dt} = v^{2m+1}z, \tag{16}$$

where $d\tau = dt/z^{2m}$ and $m \geq 1$. We only need to study the equilibrium $B : (0,0)$ of systems (15) and (16), which corresponds to an equilibrium I_B of system (4) at infinity on the y–axis.

Lemma 7. *Equilibria A and A_1 are unstable nodes when $a > 0$ and stable nodes when $a < 0$.*

System (16) provides an interesting example for highly degenerate equilibria when m is greater than 1. As m is unspecified, the lowest degree of nonzero terms in (16) is $2m$. One could not use the blowing-up methods as done in [24] $2m$ times to decompose the equilibrium B into simple ones. So a natural idea is to study the system with normal sectors, as in [24]. We will see that the method of normal sectors does not work in some cases, while we show how to apply the method of generalized normal sectors [24] (GNS for short).

Lemma 8. *For system (16), when $m = 0, 1$ and $a > 0$, there are infinite orbits approaching $B : (0,0)$ in two directions $\theta = \pi$, there is a unique orbit approaching $B : (0,0)$ in two directions $\theta = 0$, and there are infinite orbits leaving $B : (0,0)$ in two directions $\theta = 0$; when $m = 0, 1$ and $a < 0$, there are infinite orbits leaving $B : (0,0)$ in two directions $\theta = 0$, there is a unique orbit leaving $B : (0,0)$ in two directions $\theta = \pi$, and there are infinite orbits approaching $B : (0,0)$ in two directions $\theta = \pi$; when $m \geq 2$, there are infinite orbits leaving $B : (0,0)$ in two directions $\theta = 0$, and there are infinite orbits approaching $B : (0,0)$ in two directions $\theta = \pi$.*

Proof. It is equivalent to consider the equilibrium A of system (13). By Theorem II.3.1 in [24], we only need to discuss the orbits in exceptional directions, as seen in Frommer [25]. With the substitution $u = r\cos\theta$, $v = r\sin\theta$, system (13) can be written as

$$\frac{dr}{rd\theta} = \frac{H_1(\theta) + o(1)}{G_1(\theta) + o(1)}, \quad \text{as } r \longrightarrow 0, \tag{17}$$

where $G_1(\theta) = -\sin^3\theta$, $H_1(\theta) = \cos\theta\sin^2\theta$ when $m = 0$, $G_1(\theta) = -\sin^{2m+1}\theta$, $H_1(\theta) = \cos\theta\sin^{2m}\theta$ when $m \geq 1$. A necessary condition for $\theta = \theta_0$ to be an exceptional direction is that $G_1(\theta) = 0$. Obviously, $G_1(\theta)$ has two roots 0 and π. As in [24], except in these exceptional directions, no orbits connect B.

When $m = 0$, using the Briot-Bouquet transformation [24] $v = v$, $z = z_1v$, which desingularizes the degenerate equilibrium $D:(0,0)$ of system (15) in the directions of z_1-axis, we reduce (15) to the following form (18):

$$\begin{aligned}\frac{dv}{d\sigma}&=z_1{}^2v+v^3z_1{}^2-\left(a+bz_1+cz_1{}^2\right)v^2,\\ \frac{dz_1}{d\sigma}&=\left(a+bz_1+cz_1{}^2\right)vz_1-z_1{}^3,\end{aligned}\tag{18}$$

where $d\sigma = vd\tau$. We need to investigate the origin of (18) which is a degenerate equilibrium of system (18). In polar coordinates $v=r\cos\theta$ and $z_1=r\sin\theta$, we have

$$\begin{aligned}G(\theta)&=2a\sin\theta\cos^2\theta,\\ H(\theta)&=a\cos\theta\left(\sin^2\theta-\cos^2\theta\right),\end{aligned}\tag{19}$$

for system (18). The equation $G(\theta)=0$ has exactly four real roots 0, $\pi/2$, π, and $3\pi/2$, and we can check that

$$G'(0)H(0)=G'(\pi)H(\pi)=-2a^2<0.\tag{20}$$

By Theorem 3.7 of [24, Chapter 2], system (18) has a unique orbit approaching the origin in the direction $\theta=0$, a unique orbit leaving the origin in $\theta=\pi$ as $\tau\longrightarrow+\infty$, which are exactly the positive v-axis and the negative v-axis, respectively. And for $\theta=\pi/2$ and $\theta=3\pi/2$, we can check that $H(\pi/2)=H(3\pi/2)=0$.

Applying the Briot-Bouquet transformation $v=v_2z_1$, $z_1=z_1$, we can change system (18) into the following form:

$$\begin{aligned}\frac{dv_2}{ds}&=2v_2z_1+v_2{}^3z_1{}^3-2\left(a+bz_1+cz_1{}^2\right)v_2{}^2,\\ \frac{dz_1}{ds}&=\left(a+bz_1+cz_1{}^2\right)v_2z_1-z_1{}^2,\end{aligned}\tag{21}$$

where $ds=z_1d\sigma$. We need to investigate the origin of system (21) which is degenerate. In polar coordinates $v_2=r\cos\theta$ and $z_1=r\sin\theta$, we have

$$\begin{aligned}G(\theta)&=\sin\theta\cos\theta\left(3a\cos\theta-3\sin\theta\right),\\ H(\theta)&=a\cos\theta\sin^2\theta-\sin^3\theta-2a\cos^3\theta\\ &\quad+2\cos^2\theta\sin\theta,\end{aligned}\tag{22}$$

for system (21). The equation $G(\theta)=0$ has exactly six real roots 0, $\arctan a$, $\pi/2$, π, $\pi+\arctan a$, and $3\pi/2$ when $a>0$, $0,\pi-\arctan(-a)$, $\pi/2$, π, $2\pi-\arctan(-a)$, and $3\pi/2$ when $a<0$, and we can check that

$$\begin{aligned}&G'(0)H(0)=G'(\pi)H(\pi)=-6a^2<0,\\ &G'\left(\frac{\pi}{2}\right)H\left(\frac{\pi}{2}\right)=G'\left(\frac{3\pi}{2}\right)H\left(\frac{3\pi}{2}\right)=-3<0.\end{aligned}\tag{23}$$

By Theorem 3.7 of [24, Chapter 2] system (21) has a unique orbit approaching the origin in the direction $\theta=0$, a unique orbit leaving the origin in $\theta=\pi$, a unique orbit approaching the origin in $\theta=\pi/2$, and a unique orbit leaving the origin in $\theta=3\pi/2$ as $\tau\longrightarrow+\infty$, which are exactly the positive v_2-axis, the negative v_2-axis, the positive z_1-axis, and the negative z_1-axis, respectively. And for $\theta=\arctan a$ and $\theta=\pi+\arctan a$ when $a>0$ or $\theta=\pi-\arctan(-a)$ and $\theta=2\pi-\arctan(-a)$ when $a>0$, we can check that $H(\theta)=0$.

Applying the Briot-Bouquet transformation $v_2=v_2$, $z_1=z_3v_2$, we can change system (21) into the following form:

$$\begin{aligned}\frac{dv_2}{ds_1}&=-2av_2+2v_2z_3-2bv_2{}^2z_3-2cv_2{}^3z_3{}^2+v_2{}^5z_3{}^3,\\ \frac{dz_3}{ds_1}&=3az_3-3z_3{}^2+3bv_2z_3{}^2+3cv_2{}^2z_3{}^3-v_2{}^4z_3{}^4,\end{aligned}\tag{24}$$

where $ds_1 = v_2ds$. One can check that system (24) has exactly two equilibria $(0,0)$ and $(0,a)$ on the z_3-axis, and we only need to investigate the qualitative properties of $(0,a)$ which corresponds to the directions $\theta=\arctan a$ and $\theta=\pi+\arctan a$ when $a>0$ or $\theta=\pi-\arctan(-a)$ and $\theta=2\pi-\arctan(-a)$ when $a>0$, of system (21). Applying the transformation $\overline{v}_2=v_2$, $\overline{z}_3=z_3-a$, which translates the equilibrium $(0,a)$ to the origin, for simplicity, we denote $\overline{v}_2$ and $\overline{z}_3$ by v_2 and z_3, respectively, and system (24) can be written into the form

$$\begin{aligned}&\frac{dv_2}{ds_1}\\ &\quad=2v_2\overline{z}_3-2\left[b\left(\overline{z}_3+a\right)+cv_2\left(\overline{z}_3+a\right)^2\right]v_2{}^2\\ &\qquad+v_2{}^5\left(\overline{z}_3+a\right)^3,\\ &\frac{dz_3}{ds_1}\\ &\quad=3\left[-\overline{z}_3+bv_2\left(\overline{z}_3+a\right)+cv_2{}^2\left(\overline{z}_3+a\right)^2\right]\left(\overline{z}_3+a\right)\\ &\qquad-v_2{}^5\left(\overline{z}_4+a\right)^4,\end{aligned}\tag{25}$$

and we only need to analyze the qualitative properties of the origin of system (25).

Applying the transformation $v_2'=v_2$, $z_3'=abv_2-z_3$, and $ds_2=-ds_1$, for simplicity, we denote v_2' and z_3' by v_2 and z_3, respectively, and system (25) can be written as

$$\begin{aligned}\frac{dv_2}{ds_2}&=2z_3v_2+2\left(ab^2+a^2c\right)v_2^3-2bz_3v_2^2+h.o.t,\\ \frac{dz_3}{ds_2}&=3az_3+3z_3^2-3\left(a^2b^2+a^3c\right)v_2^2+2abz_3v_2\\ &\quad+h.o.t,\end{aligned}\tag{26}$$

and we only need to analyze the qualitative properties of the origin of system (26).

When $ab^2+a^2c\neq0$, there exists a function

$$z_3=X_2\left(v_2\right)=-\left(ab^2+a^2c\right)v_2^2+h.o.t\tag{27}$$

which can be derived from the second equation of system (26). Substitute the function (27) into the first equation of system (26), and we obtain that

$$\frac{dv_2}{ds_2} = a^3\left(1+2c^2\right)v_2{}^5 + h.o.t. \tag{28}$$

By Theorem 7.1 in [24, Chapter 2], we obtain that when $a > 0$, the origin of system (26) is an unstable node; we obtain that when $a < 0$, the origin of system (26) is a stable node. So, according to the method of the Briot-Bouquet transformation, the theorem of $m = 0$ is proved. Based on the proof of $m = 0$, we can also use the same method to get the same result of $m = 1$.

When $m > 1$, some difficulties are caused when we discuss orbits in the directions $\theta = 0, \pi$, because $G_1'(0) = H_1(0) = 0$, which does not match any conditions of the theorems in references, e.g., [24]. However, in what follows, we construct GNSes or some related open quasi–sectors which allow curves and orbits to be their boundaries, to determine how many orbits connect A in $\theta = 0, \pi$.

From $dz/dt = 0$ in (16), two horizontal isoclines are determined near $\theta = 0, \pi$: one is $V_1 := \{v \in \mathbf{R}_+ : z = 0\}$ and the other is $V_2 := \{v \in \mathbf{R}_- : z = 0\}$. Furthermore, let

$$\begin{aligned}
\mathscr{L}_1 &= \left\{(v,z) \in \mathbf{R}_+{}^2 : z = \sigma_1 v,\ 0 < r < \ell\right\},\\
\mathscr{L}_2 &= \left\{(v,z) \in \mathbf{R}_+ \times \mathbf{R}_- : z = \sigma_1 v,\ 0 < r < \ell\right\},\\
\mathscr{L}_3 &= \left\{(v,z) \in \mathbf{R}_- \times \mathbf{R}_+ : z = \sigma_1 v,\ 0 < r < \ell\right\},\\
\mathscr{L}_4 &= \left\{(v,z) \in \mathbf{R}_-{}^2 : z = \sigma_1 v,\ 0 < r < \ell\right\},
\end{aligned} \tag{29}$$

where $\sigma_1 > 0$ and σ_1 is closed to zero.

Case 1. $a < 0$. Notice that there are no vertical isoclines near $\theta = 0$ in (16). We claim that the open sector $\Delta\widehat{\mathscr{L}_1 B \mathscr{L}_2}$ is a GNS in class I. In fact, we have $dz/dt > 0$ between $\mathscr{L}_1$ and V_1 and $dz/dt < 0$ between $\mathscr{L}_2$ and V_1. So $dr/dt > 0$ in the closure $\mathrm{cl}\Delta\widehat{\mathscr{L}_1 B \mathscr{L}_2}/\{B\}$. Therefore, what we claim is proved by the definition of GNS. Lemma 1 in [26] guarantees that system (16) has infinitely many orbits in connection with (actually leaving from) B in $\Delta\widehat{\mathscr{L}_1 B \mathscr{L}_2}$. If $m = 2$ and $-2 \le a < 0$, we notice that there are no vertical isoclines near $\theta = \pi$ in (16). Hence in $\Delta\widehat{\mathscr{L}_3 B V_2}$ and $\Delta\widehat{\mathscr{L}_4 B V_2}$, we have $dv/dz < 0$ and $dv/dz > 0$, respectively, implying that infinitely many orbits connect B in the two sectors by Lemma 1 in [26]. If $m = 2$ and $a < -2$, or $m > 2$, from $dv/d\tau = 0$ in (16), we obtain vertical isoclines $\mathscr{H}_1 := \{(v,z) \in \mathbf{R}_-^2 : z^{2m} - av^3 z^{2m-2} + v^{2m+2}, 0 < r < \ell\}$ and $\mathscr{H}_2 := \{(v,z) \in \mathbf{R}_- \times \mathbf{R}_+ : z^{2m} - av^3 z^{2m-2} + v^{2m+2}, 0 < r < \ell\}$, where $\ell > 0$ is a sufficiently small constant. Obviously, $\mathscr{H}_i (i = 1, 2)$ is tangent to v–axis at B; hence in $\Delta\widehat{\mathscr{H}_1 B V_2}$ and $\Delta\widehat{\mathscr{H}_2 B V_2}$, we have $dv/dz > 0$ and $dv/dz < 0$, respectively, implying that infinitely many orbits connect B in the two sectors by Lemma 1 in [26].

Case 2. $a > 0$. Based on the proof of $a < 0$, we can also use the same method to get the same result of $a > 0$. We can give the three cases as shown in Figure 3. □

5. Nonexistence and Uniqueness of Closed Orbits

Let us consider the Liénard system

$$\begin{aligned}
\frac{dx}{dt} &= y - F(x),\\
\frac{dy}{dt} &= -g(x),
\end{aligned} \tag{30}$$

in which $F(x)$ and $g(x)$ are continuous functions on $\mathbb{R}$ satisfying locally Lipschitz condition. We assume that

$$\begin{aligned}
&F(0) = 0,\\
&xg(x) > 0 \quad \text{if } x \neq 0.
\end{aligned} \tag{31}$$

Then the origin is the only critical point. Let $M = \min\{\int_0^\infty g(x)dx, \int_0^{-\infty} g(x)dx\}$ (M may be ∞) and let

$$w = G(x) = \int_0^x |g(x)|\,dx. \tag{32}$$

Then by (31), $G(x)$ is strictly increasing. We denote the inverse function of $G(x)$ by $G^{-1}(w)$.

In article [27], Sugie and Hara gave the following condition on $F(x)$ and $g(x)$ under which system (30) has no periodic solutions except the origin.

Lemma 9 (see [27]). *Suppose that*

$$F\left(G^{-1}(w)\right) \neq F\left(G^{-1}(-w)\right) \quad \textit{for } 0 < w < M. \tag{33}$$

Then system (30) has no nonconstant periodic solutions.

Let $x_i(z)$ be the inverse function of $z = G(x)$ and $(-1)^{i+1}x \ge 0$, where $i = 1, 2$; (30) will be equations (E_1) and (E_2) in domains $x \ge 0$ and $x \le 0$, respectively.

$$\begin{aligned}
\frac{dz}{dy} &= F_i(z) - y,\\
&(E_i)_{i=1,2},
\end{aligned} \tag{34}$$

where $F_i(z) = F(x_i(z))$.

Lemma 10 (see [28]). *Assume $f(x)$ and $g(x)$ are continuous functions in $(-\infty, +\infty)$, $xg(x) > 0$ for $x \neq 0$, $G(\pm\infty) = +\infty$, and verify*

(1) $\exists a \ge 0$, $F_1(z) \le 0 \le F_2(z)$ *for* $0 < z < a$, $F_1(z) \not\equiv F_2(z)$, $F_1(z) > 0$ *for* $z > a$,
(2) $F_2{}'(z) \le 0$ *for* $F_2(z) < 0$,
(3) $F_1(z)F_1{}'(z)$ *is non-decrease for* $z > a$,
(4) *when* $F_1(z) = F_2(u)$ *for* $u \ge z > a$, *we have* $F_1{}'(z) \ge F_2{}'(u)$. *Then system (30) has at most one limit cycle in* $(-\infty, +\infty)$; *if it exists, it must be simple and stable.*

Lemma 11. *When $ac \ge 0$, system (4) has no closed orbits; when $ac < 0$, system (4) has a unique closed orbit.*

Proof. We can easily compute

$$G^{-1}(w) = \sqrt[2m+2]{w(2m+2)},$$
$$G^{-1}(-w) = -\sqrt[2m+2]{w(2m+2)} \quad \text{for } w > 0. \tag{35}$$

Obviously,

$$F\left(G^{-1}(w)\right) - F\left(G^{-1}(-w)\right) = 2\sqrt[2m+2]{w(2m+2)}\left[a\sqrt[m+1]{w(2m+2)} + c\right]. \tag{36}$$

When $ac \geq 0$, $F(G^{-1}(w)) \neq F(G^{-1}(-w))$ for $w > 0$. Therefore, system (4) has no closed orbits by Theorem 4.5 of [24, Chapter 2] 5 when $ac \geq 0$.

When $ac < 0$, we only discuss $a > 0$, since the proof of the case $a < 0$ is reduced to that of the case $a > 0$ by the transformations $y \longrightarrow -y$ and $t \longrightarrow -t$.

(1) $b > 0$. The equation $F(x) = 0$ has three roots $0, x_1, x_2$, where $x_i = (-b \pm \sqrt{b^2 - 4ac})/(2a)$, $i = 1, 2$. We can get $\sqrt{c/a} > x_1$. Therefore, $F(x) < F(-x)$ for $0 < x < x_1$ and $F(x) > 0$ for $x > x_1$. Because $F(-x) < 0$ for $x > -x_2$, we can easily compute $F'(-x) = -3ax^2 + 2bx - c = -2ax^2 + bx - ax^2 + bx - c < 0$. When $x > x_1$, we can get $F(x) > 0$ and $F(x)$ is an increase function, $F'(x) > 0$ and $F''(x) = 6ax + 2b$ is also an increase function. Therefore, $F(x)F'(x)$ is non-decrease for $x > x_1$. When $F(x_3) = F(-x_4)$ for $x_4 \geq x_3 > x_1$, we can get $ax_3(x_3 - x_4) + b(x_3 - x_4) + c = 0$. Therefore, $F'(x_3) - F'(x_4) = a({x_3}^2 + 2x_3x_4 + 3{x_4}^2) > 0$. So, system (30) has at most one limit cycle in $(-\infty, +\infty)$; if it exists, it must be simple and stable.

(2) $b < 0$. The proof of the case (2) is reduced to that of the case (1) by the transformations $y \longrightarrow -y$ and $x \longrightarrow -x$.

The existence of limit cycles can be proved by Theorem 1.3 in [24, Chapter 2]. Thus, system (30) has a unique stable limit cycle. □

Conflicts of Interest

The author declares that he has no conflicts of interest.

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