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Research Article

Multiperiod Telser's Safety-First Portfolio Selection with Regime Switching

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This paper investigates a multiperiod Telser's safety-first portfolio selection model with regime switching where the returns of the assets are assumed to depend on the market states modulated by a discrete-time Markov chain. The investor aims to maximize the expected terminal wealth and does not want the probability of the terminal wealth to fall short of a disaster level to exceed a predetermined number called the risk control level. Referring to Tchebycheff inequality, we modify Telser's safety-first model to the case that aims to maximize the expected terminal wealth subject to a constraint where the upper bound of the disaster probability is less than the risk control level. By the Lagrange multiplier technique and the embedding method, we study in detail the existence of the optimal strategy and derive the closed-form optimal strategy. Finally, by mathematical and numerical analysis, we analyze the effects of the disaster level, the risk control level, the transition matrix of the Markov chain, the expected excess return, and the variance of the risky return.

1. Introduction

Nowadays, the portfolio selection theory has been one of the main areas of research in the financial field. The earliest approach to portfolio optimization is the mean-variance approach pioneered by Markowitz [1]. In the past decades, Markowitz's mean-variance approach, where the variance of return is used as a risk measure, has received a lot of attention. In addition to the mean-variance criterion, there is another important school of thought called the safety-first criterion, which can be traced back to the work by Roy [2] based on the recognition that avoiding loss of a significant magnitude is a matter of great concern to most investors. According to Roy's safety-first rule, the investor aims to minimize the disaster probability of the final return falling below a prespecified critical return. In a follow-up paper, Kataoka [3] prespecifies the probability that the final return is less than a critical return and selects the strategy that maximizes the critical return. The third form of the safety-first criterion proposed by Telser [4] presents another form of the safety-first (TSF for short) criterion, which tries to maximize the expected final return subject to the constraint that the probability of the final return no greater than a disaster level is less than a predetermined acceptable number. The safety-first criterion can actually be regarded as a significant complement to the prevailing mean-variance criterion for portfolio optimization. First, the mean-variance approach views the risk as return variability, but in the real world, investors might perceive risk in different ways. For example, as Hagigi and Kluger [5] note, when the time horizon is long, the investor might not care much about the short-term fluctuation of the return. He might instead aim to maximize the expected return while ensuring that the probability of disaster is less than a given number. Second, the results obtained under the safety-first framework are different from those under a mean-variance criterion. The findings of Shefrin and Statman [6] indicate that, in general, optimal safety-first portfolios are not mean-variance efficient. Moreover, according to the empirical findings in Lopes [7], De Bondt [8], and Neugebauer [9], there actually exists a comparative advantage of the safety-first approach over deviation risk measures, such as the variance, because it seems to better fit with the way investors perceive risk.

Nowadays, people are more concerned about risk and try their best to minimize inevitable losses in the face of the intensified economic turmoil and political unrest over the past years. For example, Haque and Varela [10] apply safety-first portfolio principles to optimize the portfolios of risk-averse US investors considering the harmful influence of the 911 terrorist attacks on US financial markets. Therefore, the safety-first approach now receives as much attention as Markowitz's theory. In terms of Roy's safety-first criterion, Norkin and Boyko [11] consider a static portfolio selection model by improving Roy's safety-first approach to the case with a better estimation of the negative return probabilities. Li et al. [12] extend Roy's safety-first approach to a multiperiod setting. In view of Tchebycheff inequality, they adopt an approximation approach, which is to replace Roy's disaster probability by its upper bound, to obtain an analytical solution. Their paper represents the first pioneering work in dynamic safety-first. In some follow-up papers that adopt the solution scheme of Li et al. [12], Chiu and Li [13] study asset-liability management; Yan [14] deals with a continuous-time portfolio selection under the assumption that the evolution of the stock price is a jump-diffusion process. However, the approximation approach actually deviates from the original conceptual framework set by Roy. Therefore, Chiu et al. [15] study the dynamic Roy's original safety-first formulation and its application in asset and liability management. In addition, Li and Yao [16] investigate a continuous-time Roy's portfolio selection problem in a Black-Scholes setting and obtain closed-form solutions of the best constant-rebalanced portfolios. Li et al. [17] compare the optimal constant-rebalanced portfolio, dynamic-rebalanced portfolio, and buy-and-hold strategies under Roy's safety-first principle. In terms of Kataoka's safety-first (KSF for short) principle, Ding and Zhang [18] study a static KSF investment choice model. They obtain conditions under which the KSF model has a finite optimal strategy without normality assumption and derive the optimal portfolios in two cases where the short-sell is allowed or it is not allowed. Ding and Zhang [19] give a further study on KSF model with regular distribution by providing geometrical properties of the KSF model and establishing a model for risky asset's pricing. Nico [20] investigates a static Telser's safety portfolio model with two kinds of targets, the fixed target and the stochastic target, and tries to determine which target choice results in a better investment performance. Arzac and Bawa [21] analyze the existence of the optimal solution for the TSF model and derive the conclusion that when the asset returns are normally or stably Pareto distributed, the CAPM can be derived from the TSF model. Engels [22] gives an intuitive and analytical solution for the TSF model under the assumption that the portfolio returns are, respectively, normally and elliptically distributed. For more details about this topic, interested readers are referred to Pyle and Turnovsky [23], Levy and Sarnat [24], Bigman [25], Milevsky [26], Stutzer [27], and Haley and Whiteman [28].

From the above-mentioned papers, the common points of the existing portfolio optimization under these three safety-first criterions can be summarized as follows. They only consider the risk from the asset prices but do not take into account the risk resulting from the change of the financial market states. To fill the gap, this present paper investigates a multiperiod portfolio selection problem under the TSF criterion with regime switching, in which the asset returns depend on the market state modulated by a Markov chain. To the best of our knowledge, no work in the existing literature has considered this topic. In reality, financial markets usually have a finite number of states, and these states would switch among each other. The empirical analysis indicates that the returns of the assets are actually sensitive to the change of the market states. For example, the findings of Hardy [29] show that the regime-switching log-normal model is better than any other asset pricing model. For this reason, many papers have studied portfolio selection with regime switching. Among others, Zhou and Yin [30], Yin and Zhou [31], Çakmak and Özekici [32], Çelikyurt and Özekici [33], Chen et al. [34], Costa and Araujo [35], Wu and Li [36], Wu et al. [37], and Wu and Chen [38] consider the investment model with regime switching under a mean-variance criterion. Cheung and Yang [39], Zeng et al. [40], and Wu [41] study this topic for investors with a power utility. In contrast, this paper takes the step of investigating portfolio optimization under the TSF framework. Actually, the three basic safety-first models mentioned above have the same constraint condition but different optimization objectives. There are two reasons that the authors choose Telser's safety-first criterion. First, an overwhelming majority of portfolio selection models under safety-first criterion adopt Roy's safety-first criterion, while Telser's safety-first portfolio selection models deserve greater attention. In addition, the authors prefer Telser's criterion because it can take into account both utility maximization and downside risk control.

The rest of the paper is organized as follows. In Section 2, we introduce the model and separate its solving process into three steps. Prime notations and assumptions are also described in this section. Sections 3 and 4 are devoted to the existence and the explicit expressions of the optimal strategies for the auxiliary problem, the Lagrangian optimal control problem, and the original problem, respectively. Mathematical and numerical analysis of some results is given in Section 5. This paper is concluded in Section 6. Proofs of the lemmas and theorems are given in the appendixes.

2. Problem Formulation and Notations

This paper assumes that an investor accesses the market at time 1 with initial wealth $x_1 > 0$ and plans to invest her wealth in the financial market for T consecutive periods. Moreover, we assume that the financial market has multiple states $\{1, 2, \ldots, L\}$, and its dynamics are described by a time-homogeneous Markov chain $\{S_n,\ n = 1, 2, \ldots\}$ where S_n represents the market state at time n. There are one risk-free asset and one risky asset available in the financial market whose returns depend on the states of the financial market. Denote by $R_n(i)$ and $r_n^f(i)$, respectively, the random return of the risky asset and the risk-free return over period n (time interval $[n, n+1)$, $n = 1, 2, \ldots, T$) given $S_n = i$. In this paper, $R_n(i)$ is assumed to be independent of $R_m(j)$ for any given

$i, j \in \{1, 2, \ldots, L\}$ as long as $n \neq m$. The Markov chain and the returns are mutually independent in the following sense:

$$\begin{aligned} &P_n\left(S_{n+1} = j, R_n\left(S_n\right) \in B\right) \\ &\quad = P_n\left(S_{n+1} = j\right) P_n\left(R_n\left(S_n\right) \in B\right) \end{aligned} \tag{1}$$

for all $B \in \mathbf{B}(\mathbb{R}^{m+1})$, $j \in S$, and $n = 1, 2, \ldots, T$, where P_n is the probability based on the information up to time n and $\mathbf{B}(\mathbb{R}^{m+1})$ is the Borel σ-algebra on $\mathbb{R}^{m+1}$. Furthermore, we use the following notations in this paper.

(N1) The transition matrix of the Markov chain $\{S_n, n = 1, 2, \ldots\}$ is denoted by Q. The matrix Q^k is the kth power of Q. In particular, we define Q^0 as an identity matrix.

(N2) For any matrix $A_{L\times L}$ and any vector $a_{L\times 1}$, denote by $A(i)$ the ith row of A and $a(i)$ the ith component of a. Furthermore, let $A_a = \{A_a(i, j)\}_{L\times L}$ where $A_a(i, j) = A(i, j)a(j)$ and $\overline{A}$ be a column vector whose ith component is $\overline{A}(i) = \sum_{j=1}^{L} A(i, j)$.

(N3) If vectors a, b, c have the same dimension, then $a\cdot c/b$ denotes a vector whose ith entry is $a(i)c(i)/b(i)$ and a^2 a vector with $(a^2)(i) = [a(i)]^2$.

(N4) $R_n^e(i) = R_n(i) - r_n^f(i)$. $r_n^e(i) = \mathrm{E}[R_n^e(i)]$, which is assumed to be nonzero for $n = 1, 2, \ldots, T$. h_n, g_n, and q_n $(n = 1, 2, \ldots, T)$ are L-dimension column vectors whose ith components are, respectively,

$$\begin{aligned} h_n(i) &= \frac{\left(r_n^e(i)\right)^2}{\mathrm{E}\left[\left(R_n^e(i)\right)^2\right]}, \\ g_n(i) &= r_n^f(i)\frac{\operatorname{Var}\left(R_n^e(i)\right)}{\mathrm{E}\left[\left(R_n^e(i)\right)^2\right]}, \\ q_n(i) &= \left(r_n^f(i)\right)^2\frac{\operatorname{Var}\left(R_n^e(i)\right)}{\mathrm{E}\left[\left(R_n^e(i)\right)^2\right]}. \end{aligned} \tag{2}$$

$\widetilde{h}_n, \overline{h}_n$, and $\overline{\alpha}_n$ $(n = 2, 3, \ldots, T+1)$ are column vectors whose ith components are, respectively,

$$\begin{aligned} \widetilde{h}_n(i) &= \frac{\overline{\prod_{m=n}^{T} Q_{g_m}}(i)}{\prod_{m=n}^{T} Q_{q_m}(i)} h_{n-1}(i), \\ \overline{h}_n(i) &= \frac{\left(\overline{\prod_{m=n}^{T} Q_{g_m}}(i)\right)^2}{\left(\overline{\prod_{m=n}^{T} Q_{q_m}}(i)\right)^2} h_{n-1}(i), \\ \overline{\alpha}_n(i) &= \frac{\left(\overline{\prod_{m=n}^{T} Q_{g_m}}(i)\right)^2}{\overline{\prod_{m=n}^{T} Q_{q_m}}(i)} h_{n-1}(i). \end{aligned} \tag{3}$$

For the sake of convenience, we set

$$\begin{aligned} &\sum_{k=m}^{n} A_k = 0, \quad \text{if } n < m \text{ for any } \{A_k\}; \\ &\prod_{k=m}^{n} Q_{g_k} = I, \quad \text{if } n < m \text{ where } I \text{ is an identity matrix;} \\ &\prod_{k=m}^{n} Q_{g_k} = Q_{g_m} \times Q_{g_{m+1}} \times \cdots \times Q_{g_n}, \quad \text{if } n \geq m. \end{aligned} \tag{4}$$

If we define u_n as the amount invested in the risky asset at time n and W_n^u as the wealth under the strategy u at time n, then the wealth dynamics are

$$W_{n+1}^u = r_n^f\left(S_n\right) W_n^u + R_n^e\left(S_n\right) u_n, \quad n = 1, 2, \ldots, T. \tag{5}$$

In this paper, we consider the optimal investment choice with the TSF criterion where the investor does not want the probability of her final wealth falling below a disaster value γ to exceed the risk control level β. Hence the strategy u subject to $P\{W_{T+1}^u \leq \gamma \mid S_1 = i_1, W_1 = x_1\} \leq \beta$ is an admissible action, and then the investor tries to select an admissible action to maximize the expected terminal wealth. Given the initial market state $S_1 = i_1$ and the initial wealth x_1, we formulate the portfolio selection problem as follows:

$$\begin{aligned} \max_{u_1, u_2, \ldots, u_T} \quad & \mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) \\ \text{s.t.} \quad & W_{n+1}^u = r_n^f\left(S_n\right) W_n^u + R_n^e\left(S_n\right) u_n, \\ & \qquad n = 1, 2, \ldots, T, \\ & P\left\{W_{T+1}^u \leq \gamma \mid S_1 = i_1, W_1 = x_1\right\} \leq \beta, \end{aligned} \tag{6}$$

where γ stands for the disaster value and $0 < \beta < 1$ is a given real number representing the risk control level. Referring to Tchebycheff inequality, we have

$$\begin{aligned} &P\left\{W_{T+1}^u \leq \gamma \mid S_1 = i_1, W_1 = x_1\right\} \\ &\quad \leq \frac{\operatorname{Var}_{i_1, x_1}\left(W_{T+1}^u\right)}{\left[\mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) - \gamma\right]^2}, \quad \mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) > \gamma. \end{aligned} \tag{7}$$

This means that if the upper bound satisfies

$$\frac{\operatorname{Var}_{i_1, x_1}\left(W_{T+1}^u\right)}{\left[\mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) - \gamma\right]^2} \leq \beta, \tag{8}$$

then $P\{W_{T+1}^u \leq \gamma \mid S_1 = i_1, W_1 = x_1\} \leq \beta$. Therefore, we modify the above-mentioned problem as follows:

$$\begin{aligned} \max_{u_1, u_2, \ldots, u_T} \quad & \mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) \\ \text{s.t.} \quad & W_{n+1}^u = R_{n,0}\left(S_n\right) W_n^u + R_n^e\left(S_n\right) u_n, \\ & \qquad n = 1, 2, \ldots, T, \qquad (P(\gamma, \beta)) \\ & \operatorname{Var}_{i_1, x_1}\left(W_{T+1}^u\right) \\ & \quad \leq \beta\left[\mathrm{E}_{i_1, x_1}\left(W_{T+1}^u\right) - \gamma\right]^2. \end{aligned}$$

When we adopt the Tchebycheff inequality to replace the probability $P\{W_{T+1}^u \leq \gamma \mid S_1 = i_1, W_1 = x_1\}$ by its upper bound $\mathrm{Var}_{i_1,x_1}(W^u_{T+1})/[\mathrm{E}_{i_1,x_1}(W^u_{T+1})-\gamma]^2$, the resulting modified formulation degenerates to a mean-variance formulation, thus losing the spirit of safety-first, as indicated in Chiu et al. [15]. For this reason, we admit that this is a weakness of the current approach. Nonetheless, due to the absence of detailed knowledge or empirical estimates of the cumulative distribution function of the final wealth when the returns of the available assets are assumed to depend on the regimes, we have to fall back on the Tchebycheff inequality to calculate the maximum probability of the final wealth below γ and then use this approximation approach to derive the optimal strategy. In what follows, we aim to derive the optimal strategy and the optimal value function for the problem $(P(\gamma,\beta))$ and analyze the conditions that γ and β satisfy when the optimal strategy exists. In order to solve $(P(\gamma,\beta))$, we introduce a Lagrangian multiplier $\omega > 0$ and formulate the Lagrangian optimal control problem as follows:

$$\begin{aligned} \max_{u_1,u_2,\ldots,u_T} \quad & \left\{\mathrm{E}_{i_1,x_1}(W^u_{T+1}) - \omega \mathrm{Var}_{i_1,x_1}(W^u_{T+1}) + \omega\beta\left[\mathrm{E}_{i_1,x_1}(W^u_{T+1})-\gamma\right]^2\right\} \\ \text{s.t.} \quad & W^u_{n+1} = r^f_n(S_n)W^u_n + R^e_n(S_n)u_n, \quad n=1,2,\ldots,T. \end{aligned} \tag{$PL(\omega,\gamma,\beta)$}$$

The relationship of $(P(\gamma,\beta))$ and $(PL(\omega,\gamma,\beta))$ is summarized in Lemma 1.

Lemma 1. *Denote by $u_{PL}(\omega,\gamma,\beta)$ and $H(\omega,\gamma,\beta)$ the optimal strategy and the value function of the problem $(PL(\omega,\gamma,\beta))$, respectively. If*

$$\omega^* = \arg\min_{\omega>0} H(\omega,\gamma,\beta) \tag{9}$$

exists, then the optimal value function of $(P(\gamma,\beta))$ is $H(\omega^*,\gamma,\beta)$ and the optimal strategy is $u_{PL}(\omega^*,\gamma,\beta)$.

Now, we can obtain the optimal solution of $(P(\gamma,\beta))$ by solving $(PL(\omega,\gamma,\beta))$. However, the problem $(PL(\omega,\gamma,\beta))$ is not separable in the sense of dynamic programming due to the term $\mathrm{Var}_{i,x_1}(W^u_{T+1})$. In view of Zhu et al. [42], we construct an auxiliary problem $(A(\lambda,\omega))$ of $(PL(\omega,\gamma,\beta))$ as follows:

$$\begin{aligned} \max_{u_1,u_2,\ldots,u_T} \quad & \mathrm{E}_{i_1,x_1}\left[\lambda W^u_{T+1} - \omega(W^u_{T+1})^2\right] \\ \text{s.t.} \quad & W^u_{n+1} = r^f_n(S_n)W^u_n + R^e_n(S_n)u_n, \\ & n=1,2,\ldots,T. \end{aligned} \tag{$A(\lambda,\omega)$}$$

Define

$$\begin{aligned} &\Pi_A(\lambda,\omega) \\ &\quad = \{u \mid u \text{ is an optimal strategy of } A(\lambda,\omega)\}, \\ &\Pi_{PL}(\omega) \\ &\quad = \{u \mid u \text{ is an optimal strategy of } PL(\omega,\gamma,\beta)\}, \\ &d(u) = 1 - 2\omega\beta\gamma + 2\omega(1+\beta)\mathrm{E}_{i_1,x_1}(W^u_{T+1}), \end{aligned} \tag{10}$$

then the relationship between $\Pi_A(\lambda,\omega)$ and $\Pi_{PL}(\omega)$ is summarized in the following lemma.

Lemma 2. *For any $u^{PL} \in \Pi_{PL}(\omega)$, $u^{PL} \in \Pi_A(d(u^{PL}),\omega)$; conversely, if $\tilde{u} \in \Pi_A(\lambda,\omega)$, then a necessary condition for $\tilde{u} \in \Pi_{PL}(\omega)$ is*

$$\lambda = 1 - 2\omega\beta\gamma + 2\omega(1+\beta)\mathrm{E}_{i_1,x_1}\left(W^{\tilde{u}}_{T+1}\right). \tag{11}$$

The proof of Lemma 2 is similar to that of Zhu et al. [42]; thus, it is omitted here. Lemma 2 implies that $\Pi_{PL}(\omega) \subseteq \bigcup_\lambda \Pi_A(\lambda,\omega)$. We can obtain the optimal strategy of the problem $(PL(\omega,\gamma,\beta))$ by first solving the auxiliary problem $(A(\lambda,\omega))$ and then finding a suitable λ^* that can make $u \in \Pi_A(\lambda^*,\omega)$ become the optimal strategy of problem $(PL(\omega,\gamma,\beta))$. The second part of Lemma 2 gives the necessary condition that λ^* should satisfy. In the next section, we shall solve the auxiliary problem $(A(\lambda,\omega))$, which is separable in the sense of dynamic programming.

3. Solution to Problems $(A(\lambda,\omega))$ and $(PL(\omega,\gamma,\beta))$

We first introduce Lemma 3 to solve $(A(\lambda,\omega))$, and the proof can be found in Wu et al. [37].

Lemma 3. *Given any $n-k$ $(k\ge 1)$ vectors $a_{k+1} = (a_{k+1}(j), j \in S), \ldots, a_n = (a_n(j), j\in S)$, one has*

$$\mathrm{E}\left[\prod_{l=k+1}^{n} a_l(S_l) \mid S_k = i\right] = \overline{Q_{a_{k+1}}Q_{a_{k+2}}\cdots Q_{a_n}}(i), \quad n = k+1, k+2, \ldots. \tag{12}$$

Now we define the value functions

$$f_n(i,x_n) = \max_{u_n,\ldots,u_T} \mathrm{E}\left[\lambda W^u_{T+1} - \omega(W^u_{T+1})^2 \mid S_n = i, W_n = x_n\right]. \tag{13}$$

Then, according to Bellman's principle of optimality, we have

$$\begin{aligned} &f_n(i,x_n) \\ &\quad = \max_{u_n} \mathrm{E}\left[f_{n+1}(S_{n+1}, W^{u_n}_{n+1}) \mid S_n = i, W_n = x_n\right] \\ &\quad = \max_{u_n} \sum_{j\in S} Q(i,j)\,\mathrm{E}\left[f_{n+1}\left(j, r^f_n(i)x_n + R^e_n(i)u_n\right)\right] \end{aligned} \tag{14}$$

for $n=1,\ldots,T$, with the boundary condition

$$f_{T+1}(i,x_{T+1}) = \lambda x_{T+1} - \omega(x_{T+1})^2. \tag{15}$$

According to the recursive formulas (14)-(15), we have Theorem 4.

Theorem 4. *The value function of problem* $(A(\lambda,\omega))$ *is given by*

$$f_n(i,x_n) = -\omega q_n(i)\overline{\prod_{m=n+1}^{T} Q_{q_m}}(i)(x_n)^2 + \lambda g_n(i)\overline{\prod_{m=n+1}^{T} Q_{g_m}}(i)x_n + \frac{\lambda^2}{4\omega}\overline{\alpha}_{n+1}(i) + \frac{\lambda^2}{4\omega}\sum_{k=n+2}^{T+1}\left(Q^{k-(n+2)}\overline{Q_{\overline{\alpha}_k}}\right)(i), \tag{16}$$

and the corresponding optimal strategy is given by

$$\widetilde{u}_n(i,x_n) = \frac{r_n^e(i)}{\mathrm{E}\left[(R_n^e(i))^2\right]}\left[\frac{\lambda}{2\omega}\frac{\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)} - r_n^f(i)x_n\right] \tag{17}$$

for $n = 1,2,\ldots,T$ *and* $i = 1,2,\ldots,L$.

Proof. See Appendix A. □

In order to derive the solution of problem $(PL(\omega,\gamma,\beta))$, we give the explicit expressions for $\mathrm{E}_{i_1,x_1}(W_{T+1}^{\widetilde{u}})$ and $\mathrm{E}_{i_1,x_1}[(W_{T+1}^{\widetilde{u}})^2]$ in the following theorem.

Theorem 5. *Under the optimal strategy (17) of the auxiliary problem,*

$$\begin{aligned}\mathrm{E}_{i_1,x_1}\left(W_n^{\widetilde{u}}\right) &= g_1(i_1)\overline{\prod_{k=2}^{n-1}Q_{g_k}}(i_1)x_1 + \frac{\lambda}{2\omega}\sum_{k=2}^{n}\left[Q^{k-2}\left(\widetilde{h}_k\cdot\overline{\prod_{l=k}^{n-1}Q_{g_l}}\right)\right](i_1),\\ \mathrm{E}_{i_1,x_1}\left[\left(W_n^{\widetilde{u}}\right)^2\right] &= q_1(i_1)\overline{\prod_{k=2}^{n-1}Q_{q_k}}(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2}\sum_{k=2}^{n}\left[Q^{k-2}\left(\overline{h}_k\cdot\overline{\prod_{l=k}^{n-1}Q_{q_l}}\right)\right](i_1),\\ &\qquad n = 2,3,\ldots,T+1.\end{aligned} \tag{18}$$

Proof. See Appendix B. □

In order to obtain the optimal strategy for the problem $(PL(\omega,\gamma,\beta))$, we need to summarize some properties of the coefficient of $\mathrm{E}_{i_1,x_1}(W_{T+1}^{\widetilde{u}})$ and $\mathrm{E}_{i_1,x_1}[(W_{T+1}^{\widetilde{u}})^2]$. In view of the notations of $\overline{h}_n(i)$ and $\widetilde{h}_n(i)$, we first obtain

$$\begin{aligned}\sum_{k=2}^{T+1}\left[Q^{k-2}\left(\overline{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{q_l}}\right)\right](i_1) &= \sum_{k=2}^{T+1}Q^{k-2}(i_1)\left(\overline{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{q_l}}\right)\\ &= \sum_{k=2}^{T+1}Q^{k-2}(i_1)\left[\frac{\left(\overline{\prod_{l=k}^{T}Q_{g_l}}\right)^2}{\overline{\prod_{l=k}^{T}Q_{q_l}}}\cdot h_{k-1}\right],\\ \sum_{k=2}^{T+1}\left[Q^{k-2}\left(\widetilde{h}_k\cdot\overline{\prod_{l=k}^{n-1}Q_{g_l}}\right)\right](i_1) &= \sum_{k=2}^{T+1}Q^{k-2}(i_1)\left(\widetilde{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{g_l}}\right)\\ &= \sum_{k=2}^{T+1}Q^{k-2}(i_1)\left[\frac{\left(\overline{\prod_{l=k}^{T}Q_{g_l}}\right)^2}{\overline{\prod_{l=k}^{T}Q_{q_l}}}\cdot h_{k-1}\right].\end{aligned} \tag{19}$$

Therefore,

$$\sum_{k=2}^{T+1}\left[Q^{k-2}\left(\overline{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{q_l}}\right)\right](i_1) = \sum_{k=2}^{T+1}\left[Q^{k-2}\left(\widetilde{h}_k\cdot\overline{\prod_{l=k}^{n-1}Q_{g_l}}\right)\right](i_1). \tag{20}$$

For convenience, denote

$$\begin{aligned}\phi(i_1) &= g_1(i_1)\overline{\prod_{k=2}^{T}Q_{g_k}}(i_1),\\ \varphi(i_1) &= q_1(i_1)\overline{\prod_{k=2}^{T}Q_{q_k}}(i_1),\\ \xi(i_1) &= \sum_{k=2}^{T+1}\left[Q^{k-2}\left(\overline{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{g_l}}\right)\right](i_1).\end{aligned} \tag{21}$$

Then, by Theorem 5, $\mathrm{E}_{i_1,x_1}(W_{T+1}^{\widetilde{u}})$ and $\mathrm{E}_{i_1,x_1}[(W_{T+1}^{\widetilde{u}})^2]$ can be written as

$$\begin{aligned}\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\widetilde{u}}\right) &= \phi(i_1)x_1 + \frac{\lambda}{2\omega}\xi(i_1),\\ \mathrm{E}_{i_1,x_1}\left[\left(W_{T+1}^{\widetilde{u}}\right)^2\right] &= \varphi(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2}\xi(i_1).\end{aligned} \tag{22}$$

Lemma 6. $0 < \xi(i_1) < 1$, $(1-\xi(i_1))\varphi(i_1) - (\phi(i_1))^2 \geq 0$.

Proof. See Appendix C. □

Now, we begin to seek the optimal strategy of the problem $(PL(\omega,\gamma,\beta))$. To this end, we define a function $U(\lambda)$ as follows:

$$\begin{aligned} U(\lambda) &= \mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right) - \omega \mathrm{Var}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right) \\ &\quad + \omega\beta\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right) - \gamma\right]^2 \\ &= (1-2\omega\beta\gamma)\,\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right) \\ &\quad + \omega(1+\beta)\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right)\right]^2 \\ &\quad - \omega\mathrm{E}_{i_1,x_1}\left[\left(W_{T+1}^{\bar{u}}\right)^2\right] + \omega\beta\gamma^2. \end{aligned} \tag{23}$$

By (22), we have

$$\begin{aligned} U(\lambda) &= (1-2\omega\beta\gamma)\left(\phi(i_1)x_1 + \frac{\lambda}{2\omega}\xi(i_1)\right) \\ &\quad + \omega(1+\beta)\left(\phi(i_1)x_1 + \frac{\lambda}{2\omega}\xi(i_1)\right)^2 \\ &\quad - \omega\left(\varphi(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2}\xi(i_1)\right) + \omega\beta\gamma^2. \end{aligned} \tag{24}$$

Differentiating (24) with respect to λ, we obtain

$$\begin{aligned} U'(\lambda) &= \frac{1-2\omega\beta\gamma}{2\omega}\xi(i_1) - \frac{\lambda}{2\omega}\xi(i_1) \\ &\quad + (1+\beta)\left(\phi(i_1)x_1 + \frac{\lambda}{2\omega}\xi(i_1)\right)\xi(i_1), \end{aligned}$$

$$U''(\lambda) = \frac{\xi(i_1)}{2\omega}\left[(1+\beta)\xi(i_1) - 1\right]. \tag{25}$$

The optimal strategy of the problem $(PL(\omega,\gamma,\beta))$ exists if and only if $U''(\lambda) < 0$, that is, $(1+\beta)\xi(i_1) < 1$. Otherwise, the optimal solution of the problem $(PL(\omega,\gamma,\beta))$ does not exist. In view of Lemma 6, when the probability β satisfies

$$0 < \beta < \frac{1}{\xi(i_1)} - 1, \tag{26}$$

the optimal solution for the problem $(PL(\omega,\gamma,\beta))$ exists. When (26) holds, let $U'(\lambda) = 0$, and then we derive the optimal solution of $\max_{\lambda\in R} U(\lambda)$ at

$$\begin{aligned} \lambda^* &= \frac{1}{1-(1+\beta)\xi(i_1)} + 2\omega\frac{(1+\beta)\phi(i_1)x_1 - \beta\gamma}{1-(1+\beta)\xi(i_1)} \\ &:= c_1(i_1) + 2\omega c_2(i_1). \end{aligned} \tag{27}$$

We have verified that solving equation

$$\lambda = 1 - 2\omega\beta\gamma + 2\omega(1+\beta)\,\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\bar{u}}\right) \tag{28}$$

in Lemma 2 yields the same expression of λ^* as (27). Substituting (27) back into (17) gives the optimal policy of problem $(PL(\omega,\gamma,\beta))$, which is summarized in the following theorem.

Theorem 7. *The optimal strategy of the problem $(PL(\omega,\gamma,\beta))$ is given by*

$$\begin{aligned} u_n^{PL}(i,x_n) &= \frac{r_n^e(i)}{\mathrm{E}\left[(R_n^e(i))^2\right]}\left(\frac{\lambda^*}{2\omega}\frac{\overline{\prod_{m=n+1}^T Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^T Q_{q_m}}(i)} - r_n^f(i)x_n\right) \\ &= \frac{r_n^e(i)}{\mathrm{E}\left[(R_n^e(i))^2\right]}\left(\frac{1}{2\omega}\frac{1}{1-(1+\beta)\xi(i_1)}\frac{\overline{\prod_{m=n+1}^T Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^T Q_{q_m}}(i)} + \frac{(1+\beta)\phi(i_1)x_1 - \beta\gamma}{1-(1+\beta)\xi(i_1)}\frac{\overline{\prod_{m=n+1}^T Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^T Q_{q_m}}(i)} - r_n^f(i)x_n\right) \end{aligned} \tag{29}$$

for $n = 1,2,\ldots,T$ and $i = 1,2,\ldots,L$.

In the next section, we shall seek the optimal strategy of the problem $(P(\gamma,\beta))$ referring to the relationship between $(PL(\omega,\gamma,\beta))$ and $(P(\gamma,\beta))$.

4. Optimal Solution of Problem $(P(\gamma,\beta))$

We define a function $\Gamma(\omega)$ by substituting (27) into (24) as follows:

$$\begin{aligned} \Gamma(\omega) &= (1-2\omega\beta\gamma)\left(\phi(i_1)x_1 + \frac{c_1(i_1)+2\omega c_2(i_1)}{2\omega}\xi(i_1)\right) \\ &\quad + \omega(1+\beta)\left(\phi(i_1)x_1 + \frac{c_1(i_1)+2\omega c_2(i_1)}{2\omega}\xi(i_1)\right)^2 \\ &\quad - \omega\left(\varphi(i_1)(x_1)^2 + \frac{(c_1(i_1)+2\omega c_2(i_1))^2}{4\omega^2}\xi(i_1)\right) \\ &\quad + \omega\beta\gamma^2. \end{aligned} \tag{30}$$

Substituting $c_1(i_1)$ and $c_2(i_1)$ into (30) results in Lemma 8.

Lemma 8.

$$F(\omega)=\frac{\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}+\frac{\xi(i_1)}{1-(1+\beta)\xi(i_1)}\frac{1}{4\omega}+\frac{y(\gamma)}{1-(1+\beta)\xi(i_1)}\omega, \tag{31}$$

where

$$y(\gamma)=\beta(1-\xi(i_1))\gamma^2-2\beta\phi(i_1)x_1\gamma+\left[(1+\beta)(\phi(i_1))^2-(1-(1+\beta)\xi(i_1))\varphi(i_1)\right]\cdot(x_1)^2. \tag{32}$$

Proof. See Appendix D. □

According to Lemma 6 and (26), the coefficient of $1/4\omega$ is strictly greater than 0. Then, the formula (31) implies that the finite minimum value of $\Gamma(\omega)$ exists in $\omega>0$ if and only if $y(\gamma)>0$. If $y(\gamma)\le 0$ for some γ, then $\Gamma(\omega)$ is a decreasing function with respect to $\omega>0$, and then the minimum value of $\Gamma(\omega)$ does not exist. This means that the prespecified critical return γ has to satisfy specific conditions so that the problem $(P(\gamma,\beta))$ has the optimal solution. Considering that $y(\gamma)$ is a quadratic curve with respect to γ, we define

$$\begin{aligned}\Delta&=4\beta^2(\phi(i_1))^2(x_1)^2-4\beta(1-\xi(i_1))\\&\quad\cdot\left[(1+\beta)(\phi(i_1))^2-(1-(1+\beta)\xi(i_1))\varphi(i_1)\right]\\&\quad\cdot(x_1)^2=4\beta[1-(1+\beta)\xi(i_1)]\\&\quad\cdot\left[\varphi(i_1)-(\phi(i_1))^2-\xi(i_1)\varphi(i_1)\right](x_1)^2,\end{aligned} \tag{33}$$

which is not less than zero by (26) and Lemma 6. Hence, the roots of $y(\gamma)=0$ exist and are given as

$$y_1=\frac{\sqrt{\beta}\phi(i_1)-\sqrt{(1-(1+\beta)\xi(i_1))\left(\varphi(i_1)-(\phi(i_1))^2-\xi(i_1)\varphi(i_1)\right)}}{\sqrt{\beta}(1-\xi(i_1))}x_1, \tag{34}$$

$$y_2=\frac{\sqrt{\beta}\phi(i_1)+\sqrt{(1-(1+\beta)\xi(i_1))\left(\varphi(i_1)-(\phi(i_1))^2-\xi(i)\varphi(i_1)\right)}}{\sqrt{\beta}(1-\xi(i_1))}x_1. \tag{35}$$

When

$$\gamma\in\{\gamma\mid\gamma<y_1\}\cup\{\gamma\mid\gamma>y_2\}, \tag{36}$$

by solving the equation $\Gamma'(\omega)=0$, $\omega^*=\arg\min_{\omega>0}\Gamma(\omega)$ exists and is given as

$$\omega^*=\frac{1}{2}\sqrt{\frac{\xi(i_1)}{y(\gamma)}}. \tag{37}$$

Then, the corresponding optimal value of $(P(\gamma,\beta))$ at ω^* is

$$\min_{\omega>0}\ \Gamma(\omega)=\frac{\sqrt{\xi(i_1)y(\gamma)}+\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}. \tag{38}$$

Referring to (27) and (37), we have

$$\begin{aligned}\frac{\lambda^*}{2\omega^*}&=\frac{c_1(i_1)}{2\omega^*}+c_2(i_1)\\&=c_1(i_1)\sqrt{\frac{y(\gamma)}{\xi(i_1)}}\\&\quad+c_1(i_1)((1+\beta)\phi(i_1)x_1-\beta\gamma).\end{aligned} \tag{39}$$

It first follows from Theorem 5 that

$$\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\tilde{u}}\right)\Big|_{\lambda^*}\right]\Big|_{\omega^*}=\phi(i_1)x_1+\frac{\lambda^*}{2\omega^*}\xi(i_1). \tag{40}$$

Consequently, according to (39), we obtain

$$\begin{aligned}\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{\tilde{u}}\right)\Big|_{\lambda^*}\right]\Big|_{\omega^*}&=\phi(i_1)x_1+\left(c_1(i_1)\sqrt{\frac{y(\gamma)}{\xi(i_1)}}+c_1(i_1)((1+\beta)\phi(i_1)x_1-\beta\gamma)\right)\xi(i_1)\\&=\frac{\phi(i_1)x_1-\phi(i_1)(1+\beta)\xi(i_1)x_1+((1+\beta)\phi(i_1)x_1-\beta\gamma)\xi(i_1)+\sqrt{y(\gamma)\xi(i_1)}}{1-(1+\beta)\xi(i_1)}\\&=\frac{\sqrt{y(\gamma)\xi(i_1)}+\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}.\end{aligned} \tag{41}$$

Now, we have verified that

$$\min_{\omega>0} \Gamma(\omega) = \left[\mathrm{E}_{i_1,x_1} \left(W_{T+1}^{\tilde{u}} \right) \Big|_{\lambda^*} \right] \Big|_{\omega^*} = \frac{\sqrt{y(\gamma)\xi(i_1)} + \phi(i_1) x_1 - \beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}. \tag{42}$$

As mentioned in Section 2, γ also satisfies the condition

$$\gamma < \left[\mathrm{E}_{i_1,x_1} \left(W_{T+1}^{\tilde{u}} \right) \Big|_{\lambda^*} \right] \Big|_{\omega^*}. \tag{43}$$

Together with (36), we have

$$\begin{aligned} \gamma &\in \{\gamma \mid \gamma < y_1\} \cup \{\gamma \mid \gamma > y_2\} \\ &\cap \left\{ \gamma \mid \gamma < \left[\mathrm{E}_{i_1,x_1} \left(W_{T+1}^{\tilde{u}} \right) \Big|_{\lambda^*} \right] \Big|_{\omega^*} \right\}, \end{aligned} \tag{44}$$

which we will make further efforts to simplify later.

Lemma 9. *Condition (44) can be reduced to $\gamma \in \{\gamma \mid \gamma < y_1\}$.*

Proof. See Appendix E. □

Under this circumstance, the optimal strategy and optimal value of problem $(P(\gamma,\beta))$ are summarized in the following theorem.

Theorem 10. *When $\gamma \in \{\gamma \mid \gamma < y_1\}$ and $0 < \beta < \min\{1, 1/\xi(i_1) - 1\}$, the optimal strategy of $(P(\gamma,\beta))$ is*

$$\begin{aligned} u_n^*(i, x_n) &= \frac{r_n^e(i)}{\mathrm{E}\left[(R_n^e(i))^2 \right]} \left(\frac{\lambda^*}{2\omega^*} \frac{\overline{\prod_{m=n+1}^{T} Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T} Q_{q_m}}(i)} - r_n^f(i) x_n \right) \\ &= \frac{r_n^e(i)}{\mathrm{E}\left[(R_n^e(i))^2 \right]} \left(\sqrt{\frac{y(\gamma)}{\xi(i_1)}} \frac{1}{1-(1+\beta)\xi(i_1)} \frac{\overline{\prod_{m=n+1}^{T} Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T} Q_{q_m}}(i)} + \frac{(1+\beta)\phi(i_1)x_1 - \beta\gamma}{1-(1+\beta)\xi(i_1)} \frac{\overline{\prod_{m=n+1}^{T} Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T} Q_{q_m}}(i)} - r_n^f(i) x_n \right), \end{aligned} \tag{45}$$

where λ^* and ω^* satisfy (27) and (37), respectively, and the corresponding optimal value satisfies (38).

Now, we tend to derive the variance of the terminal wealth under u^* in Theorem 10. According to (39) and (C.2), the variance of the terminal wealth under u^* is given as

$$\begin{aligned} \mathrm{Var}_{i_1,x_1}\left(W_{T+1}^{u^*} \right) &= \xi(i_1)(1-\xi(i_1)) \left(\frac{\lambda^*}{2\omega^*} - \frac{\phi(i_1)x_1}{1-\xi(i_1)} \right)^2 \\ &+ \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)} (x_1)^2 = \xi(i_1)(1 \\ &- \xi(i_1)) \left[c_1(i_1) \sqrt{\frac{y(\gamma)}{\xi(i_1)}} + c_1(i_1)((1+\beta)\phi(i_1)x_1 - \beta\gamma) - \frac{\phi(i_1)x_1}{1-\xi(i_1)} \right]^2 \\ &+ \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)} (x_1)^2 = \xi(i_1)(1 \\ &- \xi(i_1)) \left[c_1(i_1) \sqrt{\frac{y(\gamma)}{\xi(i_1)}} + c_1(i_1)\beta \left(\frac{\phi(i_1)x_1}{1-\xi(i_1)} - \gamma \right) \right]^2 \\ &+ \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)} (x_1)^2. \end{aligned} \tag{46}$$

Moreover, according to (40), we have $\lambda^*/2\omega^* = (\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*}) - \phi(i_1)x_1)/\xi(i_1)$. Thus, the relationship between the expected terminal wealth and the terminal risk is given as

$$\begin{aligned} \mathrm{Var}_{i_1,x_1}\left(W_{T+1}^{u^*} \right) &= \xi(i_1)(1-\xi(i_1)) \cdot \left(\frac{\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{u^*} \right) - \phi(i_1)x_1}{\xi(i_1)} - \frac{\phi(i_1)x_1}{1-\xi(i_1)} \right)^2 \\ &+ \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)} (x_1)^2 \\ &= \frac{1-\xi(i_1)}{\xi(i_1)} \left(\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{u^*} \right) - \frac{\phi(i_1)x_1}{1-\xi(i_1)} \right)^2 \\ &+ \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)} (x_1)^2. \end{aligned} \tag{47}$$

5. Analysis of the Obtained Results

5.1. Effects of γ and β. In view of Theorem 10, the risk control level β should be less than a given value; otherwise, the maximum value of the problem $(P(\gamma,\beta))$ can be positive infinite. In addition, to guarantee the existence of the optimal strategy of $(P(\gamma,\beta))$, the disaster level γ also needs to be less than y_1. An intuitive understanding of these results is that if β is large enough, then the constraint to the strategy might disappear. In other words, the condition $\mathrm{Var}_{i_1,x_1}(W_{T+1}^u) \le \beta[\mathrm{E}_{i_1,x_1}(W_{T+1}^u) - \gamma]^2$ might hold naturally when β is large enough. Without this constraint, the investor just aims to maximize the expected terminal wealth without any risk control, so the maximum value of $(P(\gamma,\beta))$ might be positive infinite. When $0<\beta<\min\{1, 1/\xi(i_1)-1\}$ holds, the disaster level γ should satisfy $\gamma < y_1$. Otherwise, $(P(\gamma,\beta))$ does not have the optimal solution. This conclusion also makes sense. On the one hand, in reality, in order to reflect the awareness of the risk control, the disaster level γ should not be a very large number. On the other hand, if the value of γ is very large, the probability that the wealth is less than γ is very high so that there might be no strategy satisfying the condition $P\{W_{T+1}^u \le \gamma \mid S_1 = i_1, W_1 = x_1\} \le \beta$. Because

$$P\left\{W_{T+1}^u \le \gamma \mid S_1 = i_1, W_1 = x_1\right\} \le \frac{\mathrm{Var}_{i_1,x_1}\left(W_{T+1}^u\right)}{\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^u\right) - \gamma\right]^2}, \tag{48}$$

there is also no strategy u satisfying the condition

$$\mathrm{Var}_{i_1,x_1}\left(W_{T+1}^u\right) \le \beta\left[\mathrm{E}_{i_1,x_1}\left(W_{T+1}^u\right) - \gamma\right]^2. \tag{49}$$

In what follows, we shall analyze the effects of the disaster level γ and the risk control level β on the optimal strategy u_n^* in Theorem 10, the expected terminal wealth $\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*})$ and the variance of the terminal wealth $\mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$. The obtained results are summarized in Theorems 11-12.

Theorem 11. *When $\gamma < y_1$, the optimal strategy u_n^* of the problem $(P(\gamma,\beta))$, the expected terminal wealth $\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*})$, and the variance of the terminal wealth $\mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$ are decreasing along with the disaster level γ.*

Proof. See Appendix F. □

We will explain the results in Theorem 11. When γ is a very small number and especially when γ is negative and small enough, the constraint $\mathrm{Var}_{i_1,x_1}(W_{T+1}^u) \le \beta[\mathrm{E}_{i_1,x_1}(W_{T+1}^u) - \gamma]^2$ might disappear, causing the investor to invest more wealth in the risky asset in order to obtain more expected terminal wealth. At the same time, a higher expected return is often accompanied by higher risk; thus, we have a larger $\mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$ when γ is smaller.

Theorem 12. *When $\gamma < y_1$ and $0<\beta<\min\{1, 1/\xi(i_1)-1\}$, the optimal strategy u_n^* of the problem $(P(\gamma,\beta))$, the expected terminal wealth $\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*})$, and the variance of the terminal wealth $\mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$ are increasing along with the risk control level β.*

Proof. See Appendix G. □

When the risk control level β is larger, the investor has more tolerance for wealth that is less than the disaster level. In other words, he will tend to invest more wealth in the risky asset. Consequently, the expected terminal wealth is larger, along with a higher terminal risk. Theorems 11 and 12 indicate that β and γ have the opposite influence on the obtained results. People with different attitudes toward the disaster level and the risk control level will have different investment behavior.

5.2. Effects of the Regime Switching. In this subsection, we numerically analyze the effects of the mechanism of regime switching on some obtained results. To this end, we investigate how the transition matrix, $r^e(\cdot)$, and $\mathrm{Var}(R^e(\cdot))$ at each state affect the investment strategy, the disaster level γ, and the risk control level β. Suppose that there are three market states; the initial wealth x_1 is 10; the investor adjusts the strategy every three months and there are 12 time periods; that is, $T = 12$. For convenience, we assume that the risk-free return is assumed to be a constant 1.0135 over time and the return of the risky asset depends on the market states only. Therefore, for convenience, denote by $r^e(i)$ the expected excess return at state i and $\mathrm{Var}(R^e(i))$ the variance of the excess return at state i.

5.2.1. Effects of $r^e(\cdot)$ and $\mathrm{Var}(R^e(\cdot))$. In this part, we first study the strategy at the initial time as an example to show the effects of $r^e(\cdot)$ and $\mathrm{Var}(R^e(\cdot))$. Then, their influence on β and γ is also investigated. To this end, let $r^e(1) = 0.30$, $r^e(2) = 0.2$, $r^e(3) = 0.1$, $\mathrm{Var}(R^e(1)) = 0.4$, $\mathrm{Var}(R^e(2)) = 0.4$, $\mathrm{Var}(R^e(3)) = 0.3$, and the transition matrix be $Q = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.4 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$. Because state 1 has the highest Sharpe ratio and state 3 has the lowest, we call state 1 the best state, state 2 a normal state, and state 3 the worst state.

We first demonstrate the impact of $r^e(\cdot)$ and $\mathrm{Var}(R^e(\cdot))$. To this end, we assume that $\beta = 0.2$, $\gamma = 11$, and increase $r^e(1)$ from 0.3 to 0.33 with the step size 0.01 while other parameters are kept the same value given above. In a similar way, when we increase $\mathrm{Var}(R^e(1))$ from 0.4 to 0.43, we do not change the values of other parameters. Thus, the influence of the market states on the strategy is demonstrated in Table 1, which indicates that $u_1^*(\cdot, x_1)$ is increasing along with $r^e(1)$ and is decreasing with respect to $\mathrm{Var}(R^e(1))$. Moreover, Table 1 also shows that (a) $u_1^*(\cdot, x_1)$ is very sensitive to the change in $r^e(1)$ and $\mathrm{Var}(R^e(1))$; (b) $u_1^*(1, x_1)$, the optimal investment strategy at state 1, is the most sensitive to the change in $r^e(1)$ and $\mathrm{Var}(R_n^e(1))$. For example, when $r^e(1)$ is increased from 0.3 to 0.31, that is, the growth rate is $(0.31-0.3)/0.3 = 3.33\%$, the growth rates of the investment strategy at each state are $(2.1485-1.8123)/1.8123 = 18.55\%$, $(1.1707-1.0666)/1.0666 = 9.76\%$, $(0.6977-0.6456)/0.6456 = 8.07\%$, respectively. When the growth rate of $\mathrm{Var}(R^e(1))$ is $(0.41-0.4)/0.4 = 2.5\%$, the decrement rates of the investment strategy at each state are 6.76%, 3.16%, and 2.66%,

Table 1: Influence of $r^e(1)$ and $\mathrm{Var}(R^e(1))$ on the strategy.

	$u_1^*(1,x_1)$	$u_1^*(2,x_1)$	$u_1^*(3,x_1)$
$r^e(1)=0.30$	1.8123	1.0666	0.6456
$r^e(1)=0.31$	2.1485	1.1707	0.6977
$r^e(1)=0.32$	2.6113	1.2977	0.7593
$r^e(1)=0.33$	3.2883	1.4555	0.8330
$\mathrm{Var}(R^e(1))=0.40$	1.8123	1.0666	0.6456
$\mathrm{Var}(R^e(1))=0.41$	1.6897	1.0329	0.6284
$\mathrm{Var}(R^e(1))=0.42$	1.5824	1.0024	0.6127
$\mathrm{Var}(R^e(1))=0.43$	1.4878	0.9747	0.5984

Table 2: Influence of $r^e(1)$ and $\mathrm{Var}(R^e(1))$ on β and γ.

	$1/\xi(\cdot)-1$	$y_1(\cdot)$
$r^e(1)=0.30$	(0.3327, 0.3801, 0.4254)	(11.7459, 11.7459, 11.7459)
$r^e(1)=0.31$	(0.3101, 0.3589, 0.4016)	(11.7459, 11.7459, 11.7459)
$r^e(1)=0.32$	(0.2891, 0.3390, 0.3793)	(11.7459, 11.7459, 11.7459)
$r^e(1)=0.33$	(0.2696, 0.3202, 0.3583)	(11.7459, 11.7459, 11.7459)
$\mathrm{Var}(R^e(1))=0.40$	(0.3327, 0.3801, 0.4254)	(11.7459, 11.7459, 11.7459)
$\mathrm{Var}(R^e(1))=0.41$	(0.3414, 0.3882, 0.4345)	(11.7459, 11.7459, 11.7459)
$\mathrm{Var}(R^e(1))=0.42$	(0.3501, 0.3962, 0.4435)	(11.7459, 11.7459, 11.7459)
$\mathrm{Var}(R^e(1))=0.43$	(0.3586, 0.4041, 0.4523)	(11.7459, 11.7459, 11.7459)

respectively. As for the impact of $r^e(i)$ and $\mathrm{Var}(R^e(i))$, $i=2,3$, here we emphasize that two similar experiments have been conducted and the results also indicate that $u_1^*(i,x_1)$ is the most sensitive to the change in $r^e(i)$ and $\mathrm{Var}(R^e(i))$, $i=2,3$.

Second, we want to know the influence of $r^e(\cdot)$ and $\mathrm{Var}(R^e(\cdot))$ on γ and β. Because $0<\beta<1/\xi(i_1)-1$ and $\gamma<y_1(i_1)$, actually we just need to observe how $1/\xi(i_1)-1$ and $y_1(i_1)$ change along with $r^e(\cdot)$ and $\mathrm{Var}(R^e(\cdot))$. In Table 2, both the rows and the columns indicate that a worse market state with a smaller excess return $r^e(1)$ or a larger variance $\mathrm{Var}(R^e(1))$ results in a longer interval, which β belongs to. We explain this phenomenon as follows. In a worse market state, the probability that the running wealth is below the disaster level might be increasing, leading to a larger risk control level. Two similar experiments have also been conducted to study the influence of $r^e(i)$ and $\mathrm{Var}(R^e(i))$, $i=2,3$, and we obtain similar results. That is, a worse market environment leads to a larger interval $[0,1/\xi(\cdot)-1]$. As for $y_1(\cdot)$, Table 2 shows that y_1 has almost no change regardless of the excess return and the variance. When there is only one market state, referring to (34), y_1 can be simplified as $y_1=(\phi/(1-\xi))x_1=x_1\prod_{n=1}^{T}r_n^f$, which shows that the investor will regard the value less than the risk-free return of the initial wealth x_1 from the initial time to the terminal time as a disaster level when the risk of the market-state fluctuation is neglected. In this subsection, substituting $x_1=10$ and $r_n^f=1.0135$ into $x_1\prod_{n=1}^{T}r_n^f$ yields $y_1\approx 11.7459$. These findings suggest that the value of y_1 in Table 2 is roughly equal to $x_1\prod_{n=1}^{T}r_n^f$. In other words, the investor almost does not consider the financial risk when she sets the value range of the disaster level. She would like to choose the risk-free return of the initial wealth as the disaster level.

5.2.2. Influence of the Transition Matrix. In this part, the influence of the transition matrix Q is studied. To do this, we change $Q(i)$, the ith ($i=1,2,3$) row of Q, while keeping the values of other parameters, and then we obtain Tables 3 and 4. We find from Table 3 that when the transition probability staying at the best state (state 1) is increased from 0.3 to 0.7, the optimal strategy $u_1^*(\cdot,x_1)$ is increasing accordingly. In particular, strategy $u_1^*(1,x_1)$ at state 1 is increased the most rapidly. However, when the probability of staying at state 2 or state 3 is increased, $u^*(\cdot,x_1)$ is decreased accordingly. Table 4 suggests that a high probability of staying at the best market state yields a shorter interval $[0,1/\xi(\cdot)-1]$, while the maximum value of the disaster level will also not be affected by the transition matrix.

6. Conclusion

This paper investigates a multiperiod Telser's safety-first portfolio selection problem with regime switching. There are one risk-free asset and one risky asset available in the financial market whose returns depend on the market states. The investor aims to maximize the expected terminal wealth subject to a constraint that the probability of the terminal wealth no greater than a disaster value is less than a

Table 3: Influence of the transition matrix Q on the strategy.

	$u_1^*(1,x_1)$	$u_1^*(2,x_1)$	$u_1^*(3,x_1)$
$Q(1) = [0.3, 0.4, 0.3]$	1.8123	1.0666	0.6456
$Q(1) = [0.5, 0.4, 0.1]$	3.2986	1.4842	0.8450
$Q(1) = [0.7, 0.2, 0.1]$	12.1759	2.2759	1.1666
$Q(2) = [0.4, 0.3, 0.3]$	1.8123	1.0666	0.6456
$Q(2) = [0.3, 0.5, 0.2]$	1.7873	1.0521	0.6395
$Q(2) = [0.2, 0.7, 0.1]$	1.7496	1.0301	0.6302
$Q(3) = [0.3, 0.3, 0.4]$	1.8123	1.0666	0.6456
$Q(3) = [0.3, 0.2, 0.5]$	1.6685	0.9969	0.5980
$Q(3) = [0.3, 0.1, 0.6]$	1.5168	0.9214	0.5471

Table 4: Influence of the transition matrix Q on β and γ.

	$1/\xi(\cdot) - 1$	$y_1(\cdot)$
$Q(1) = [0.3, 0.4, 0.3]$	(0.3327, 0.3801, 0.4254)	(11.7459, 11.7459, 11.7459)
$Q(1) = [0.5, 0.4, 0.1]$	(0.2650, 0.3173, 0.3554)	(11.7459, 11.7459, 11.7459)
$Q(1) = [0.7, 0.2, 0.1]$	(0.2161, 0.2707, 0.3036)	(11.7459, 11.7459, 11.7459)
$Q(2) = [0.4, 0.3, 0.3]$	(0.3327, 0.3801, 0.4254)	(11.7459, 11.7459, 11.7459)
$Q(2) = [0.3, 0.5, 0.2]$	(0.3351, 0.3835, 0.4285)	(11.7459, 11.7459, 11.7459)
$Q(2) = [0.2, 0.7, 0.1]$	(0.3388, 0.3889, 0.4335)	(11.7459, 11.7459, 11.7459)
$Q(3) = [0.3, 0.3, 0.4]$	(0.3327, 0.3801, 0.4254)	(11.7459, 11.7459, 11.7459)
$Q(3) = [0.3, 0.2, 0.5]$	(0.3475, 0.3977, 0.4525)	(11.7459, 11.7459, 11.7459)
$Q(3) = [0.3, 0.1, 0.6]$	(0.3673, 0.4212, 0.4901)	(11.7459, 11.7459, 11.7459)

predetermined acceptable number. Referring to Tchebycheff inequality, we modify Telser's safety-first model to the problem $(P(\gamma,\beta))$ that aims to maximize the expected terminal wealth subject to a constraint that the upper bound of the disaster probability is less than the risk control level. We find that when the risk control level and the disaster value satisfy $0 < \beta < 1/\xi(i_1) - 1$ and $\gamma < y_1$, the optimal strategy $u_n^*(i,x_n)$ of the problem $(P(\gamma,\beta))$ exists and is obtained by the Lagrange multiplier technique and the embedding method. We investigate the effects of the disaster level γ, the risk control level β, the expected excess return $r^e(i)$, the variance $\mathrm{Var}(R^e(i))$, and the transition matrix Q. Mathematics analysis indicates that the optimal strategy $u_n^*(i,x_n)$, the expected terminal wealth $\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*})$, and the variance of the terminal wealth $\mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$ are decreasing along with the disaster level γ, while they are increasing with respect to the risk control level β. By numerical analysis, we find the following: (a) the excess returns $r^e(i)$, $i = 1,2,3$ have a positive effect, while the variances $\mathrm{Var}(R^e(i)$, $i = 1,2,3$ have a negative impact on the investment strategy; (b) a smaller expected excess return or a larger variance of the risky return leads to a longer interval that the risk control level β lies in. However, the disaster value γ is almost not affected by the expected excess return or the variance of the risky return and is roughly equal to the risk-free return of the initial wealth from the initial time to the terminal time; (c) when the probability of staying at the best state is increased, the investment amount is increased accordingly. Meanwhile, a higher probability staying at the best market state yields a shorter interval that β belongs to, while the disaster value γ is also insensitive to the change in the transition matrix.

Appendix

A. Proof of Theorem 4

Proof. Referring to (14) and (15), when $n = T$, we have

$$\begin{aligned} f_T(i,x_T) &= \max_{u_T}\sum_{j\in S}Q(i,j) \\ &\quad\cdot \mathrm{E}\left[f_{T+1}\left(j, r_T^f(i)x_T + R_T^e(i)u_T\right)\right] \\ &= \max_{u_T}\left[\lambda r_T^e(i)u_T - 2\omega x_T r_T^f(i)r_T^e(i)u_T\right. \\ &\quad \left.- \omega\mathrm{E}\left[\left(R_T^e(i)\right)^2\right]\left(u_T\right)^2\right] + \lambda r_T^f(i)x_T - \omega\left(r_T^f(i)\right)^2 \\ &\quad\cdot\left(x_T\right)^2. \end{aligned} \tag{A.1}$$

Obviously, the optimal solution of (A.1) exists and is

$$\tilde{u}_T(i,x_T) = \frac{r_T^e(i)}{\mathrm{E}\left[\left(R_T^e(i)\right)^2\right]}\left(\frac{\lambda}{2\omega} - r_T^f(i)x_T\right). \tag{A.2}$$

Substituting (A.2) into (A.1) yields

$$\begin{aligned} f_T(i,x_T) &= \lambda r_T^f(i)\frac{\operatorname{Var}(R_T^e(i))}{\operatorname{E}[(R_T^e(i))^2]}x_T \\ &\quad - \omega(r_T^f(i))^2\frac{\operatorname{Var}(R_T^e(i))}{\operatorname{E}[(R_T^e(i))^2]}(x_T)^2 \\ &\quad + \frac{\lambda^2}{4\omega}\frac{(r_T^e(i))^2}{\operatorname{E}[(R_T^e(i))^2]} \\ &= \lambda g_T(i)x_T - \omega q_T(i)(x_T)^2 \\ &\quad + \frac{\lambda^2}{4\omega}\overline{\alpha}_{T+1}(i). \end{aligned} \tag{A.3}$$

Hence, (16) and (17) hold true for $n = T$. Now, we assume that (16) and (17) are true for $n+1$; then, for n,

$$\begin{aligned} &f_n(i,x_n) \\ &= \max_{u_n}\sum_{j\in S}Q(i,j)\operatorname{E}[f_{n+1}(j, r_n^f(i)x_n + R_n^e(i)u_n)], \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} &\sum_{j\in S}Q(i,j)\operatorname{E}[f_{n+1}(j, r_n^f(i)x_n + R_n^e(i)u_n)] \\ &= -\omega\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)(r_n^f(i))^2(x_n)^2 + \lambda\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i) \\ &\quad \cdot r_n^f(i)x_n - \omega\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i) \\ &\quad \cdot [2x_n r_n^f(i)r_n^e(i)u_n + \operatorname{E}[(R_n^e(i))^2](u_n)^2] \\ &\quad + \lambda\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i)r_n^e(i)u_n + \frac{\lambda^2}{4\omega} \\ &\quad \cdot \sum_{k=n+2}^{T+1}(Q^{k-(n+2)}\overline{Q_{\overline{\alpha}_k}})(i). \end{aligned} \tag{A.5}$$

It is clear that $\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i) > 0$ according to the definition of $q_n(i)$. Hence, the optimal solution of (A.5) exists and is given as

$$\begin{aligned} &\widetilde{u}_n(i,x_n) \\ &= \frac{r_n^e(i)}{\operatorname{E}[(R_n^e(i))^2]}\left(\frac{\lambda}{2\omega}\frac{\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)} - r_n^f(i)x_n\right). \end{aligned} \tag{A.6}$$

Substituting (A.6) into (A.5), we obtain

$$\begin{aligned} f_n(i,x_n) &= \max_{u_n}\sum_{j\in S}Q(i,j) \\ &\quad \cdot \operatorname{E}[f_{n+1}(j, r_n^f(i)x_n + R_n^e(i)u_n)] \\ &= -\omega\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)(r_n^f(i))^2(x_n)^2 + \lambda\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i) \\ &\quad \cdot r_n^f(i)x_n + \frac{\lambda^2}{4\omega}\sum_{k=n+2}^{T+1}(Q^{k-(n+2)}\overline{Q_{\overline{\alpha}_k}})(i) \\ &\quad + \omega\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i) \\ &\quad \cdot \frac{(r_n^e(i))^2}{\operatorname{E}[(R_n^e(i))^2]}\left(\frac{\lambda}{2\omega}\frac{\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i)}{\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)}\right. \\ &\quad \left. - r_n^f(i)x_n\right)^2 = -\omega q_n(i)\overline{\prod_{m=n+1}^{T}Q_{q_m}}(i)(x_n)^2 \\ &\quad + \lambda g_n(i)\overline{\prod_{m=n+1}^{T}Q_{g_m}}(i)x_n + \frac{\lambda^2}{4\omega}\overline{\alpha}_{n+1}(i) + \frac{\lambda^2}{4\omega} \\ &\quad \cdot \sum_{k=n+2}^{T+1}(Q^{k-(n+2)}\overline{Q_{\overline{\alpha}_k}})(i). \end{aligned} \tag{A.7}$$

Equations (A.6) and (A.7) indicate that (16) and (17) hold for n. By induction, the conclusions of Theorem 4 are true. □

B. Proof of Theorem 5

Proof. Referring to (17), for $n = 1, 2, \ldots, T$, the wealth dynamics become

$$\begin{aligned} W_{n+1}^{\widetilde{u}} &= r_n^f(S_n)W_n^{\widetilde{u}} + R_n^e(S_n)\widetilde{u}_n \\ &= r_n^f(S_n)\left[1 - R_n^e(S_n)\frac{r_n^e(S_n)}{\operatorname{E}[(R_n^e(i))^2]}\right]W_n^{\widetilde{u}} \\ &\quad + \frac{\lambda}{2\omega}\frac{\overline{\prod_{m=n+1}^{T}Q_{g_m}}(S_n)}{\overline{\prod_{m=n+1}^{T}Q_{q_m}}(S_n)}R_n^e(S_n)\frac{r_n^e(S_n)}{\operatorname{E}[(R_n^e(i))^2]}, \end{aligned} \tag{B.1}$$

which leads to

$$\begin{aligned} &\operatorname{E}(W_{n+1}^{\widetilde{u}} \mid S_1, S_2, \ldots, S_n) \\ &= g_n(S_n)\operatorname{E}(W_n^{\widetilde{u}} \mid S_1, S_2, \ldots, S_n) + \frac{\lambda}{2\omega}\widetilde{h}_{n+1}(S_n). \end{aligned} \tag{B.2}$$

By induction and noting that $E(W_n^{\tilde{u}} \mid S_1, S_2, \ldots, S_n) = E(W_n^{\tilde{u}} \mid S_1, S_2, \ldots, S_{n-1})$, we derive

$$E\left(W_{n+1}^{\tilde{u}} \mid S_1, \ldots, S_n\right) = g_1(S_1)\left(\prod_{k=2}^{n} g_k(S_k)\right) W_1 + \frac{\lambda}{2\omega}\sum_{k=2}^{n+1}\widetilde{h}_k(S_{k-1})\prod_{l=k}^{n} g_l(S_l). \tag{B.3}$$

In view of the tower property of the expectation, we first have

$$\begin{aligned} E_{i_1,x_1}\left(W_n^{\tilde{u}}\right) &= E_{i_1,x_1}\left[E\left(W_n^{\tilde{u}} \mid S_1, S_2, \ldots, S_n\right)\right] \\ &= g_1(i_1) E_{i_1}\left(\prod_{k=2}^{n-1} g_k(S_k)\right) x_1 \\ &\quad + \frac{\lambda}{2\omega}\sum_{k=2}^{n} E_{i_1}\left[\widetilde{h}_k(S_{k-1})\prod_{l=k}^{n-1} g_l(S_l)\right]. \end{aligned} \tag{B.4}$$

Further, by Lemma 3, we derive

$$\begin{aligned} E_{i_1,x_1}\left(W_n^{\tilde{u}}\right) &= g_1(i_1)\overline{\prod_{k=2}^{n-1} Q_{g_k}}(i_1) x_1 + \frac{\lambda}{2\omega} \\ &\cdot \sum_{k=2}^{n}\sum_{j\in S} Q^{k-2}(i_1, j)\widetilde{h}_k(j) \\ &\cdot E\left(\prod_{l=k}^{n-1} g_l(S_l) \mid S_{k-1} = j\right) = g_1(i_1)\overline{\prod_{k=2}^{n-1} Q_{g_k}}(i_1) x_1 \\ &+ \frac{\lambda}{2\omega}\sum_{k=2}^{n}\sum_{j\in S} Q^{k-2}(i_1, j)\widetilde{h}_k(j)\overline{\prod_{l=k}^{n-1} Q_{g_l}}(j) = g_1(i_1) \\ &\cdot \overline{\prod_{k=2}^{n-1} Q_{g_k}}(i_1) x_1 + \frac{\lambda}{2\omega} \\ &\cdot \sum_{k=2}^{n}\left[Q^{k-2}\left(\widetilde{h}_k \cdot \overline{\prod_{l=k}^{n-1} Q_{g_l}}\right)\right](i_1). \end{aligned} \tag{B.5}$$

By (B.1), we obtain

$$\begin{aligned} \left(W_{n+1}^{\tilde{u}}\right)^2 &= \left(r_n^f(S_n)\right)^2\left[1 - R_n^e(S_n)\frac{r_n^e(S_n)}{E\left[\left(R_n^e(S_n)\right)^2\right]}\right]^2 \\ &\cdot \left(W_n^{\tilde{u}}\right)^2 + \frac{\lambda}{\omega}\frac{\overline{\prod_{m=n+1}^{T} Q_{g_m}}(S_n)}{\overline{\prod_{m=n+1}^{T} Q_{q_m}}(S_n)} R_n^e(S_n) \\ &\cdot \frac{r_n^e(S_n)}{E\left[\left(R_n^e(S_n)\right)^2\right]} \times r_n^f(S_n) \\ &\cdot \left[1 - R_n^e(S_n)\frac{r_n^e(S_n)}{E\left[\left(R_n^e(S_n)\right)^2\right]}\right] W_n^{\tilde{u}} \\ &+ \frac{\lambda^2}{4\omega^2}\frac{\left(\overline{\prod_{m=n+1}^{T} Q_{g_m}}(S_n)\right)^2}{\left(\overline{\prod_{m=n+1}^{T} Q_{q_m}}(S_n)\right)^2}\left(r_n^e(S_n)\right)^2 \\ &\cdot \frac{\left(R_n^e(S_n)\right)^2}{\left[E\left[\left(R_n^e(S_n)\right)^2\right]\right]^2}. \end{aligned} \tag{B.6}$$

Taking the expectation of both sides yields

$$\begin{aligned} E&\left(\left(W_{n+1}^{\tilde{u}}\right)^2 \mid S_1, \ldots, S_n\right) \\ &= q_n(S_n) E\left(\left(W_n^{\tilde{u}}\right)^2 \mid S_1, \ldots, S_{n-1}\right) \\ &\quad + \frac{\lambda^2}{4\omega^2}\overline{h}_{n+1}(S_n). \end{aligned} \tag{B.7}$$

Similarly by induction, we obtain

$$\begin{aligned} E&\left(\left(W_n^{\tilde{u}}\right)^2 \mid S_1, \ldots, S_{n-1}\right) \\ &= q_1(S_1)\prod_{k=2}^{n-1} q_k(S_k)(W_1)^2 \\ &\quad + \frac{\lambda^2}{4\omega^2}\sum_{k=2}^{n}\overline{h}_k(S_{k-1})\prod_{l=k}^{n-1} q_l(S_l). \end{aligned} \tag{B.8}$$

According to (B.8) and Lemma 3, we obtain

$$\begin{aligned} E_{i_1,x_1}&\left(\left(W_n^{\tilde{u}}\right)^2\right) = E_{i_1,x_1}\left[E\left(\left(W_n^{\tilde{u}}\right)^2 \mid S_1, S_2, \ldots, S_n\right)\right] \\ &= q_1(i_1) E_{i_1}\left[\prod_{k=2}^{n-1} q_k(S_k)\right](x_1)^2 + \frac{\lambda^2}{4\omega^2} \\ &\cdot E_{i_1}\left[\sum_{k=2}^{n}\overline{h}_k(S_{k-1})\prod_{l=k}^{n-1} q_l(S_k)\right] = q_1(i_1) \\ &\cdot \overline{\prod_{k=2}^{n-1} Q_{q_k}}(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2}\sum_{k=2}^{n}\sum_{j\in S} Q^{k-2}(i_1, j)\overline{h}_k(j) \\ &\cdot E\left[\prod_{l=k}^{n-1} q_l(S_k) \mid S_{k-1} = j\right] = q_1(i_1)\overline{\prod_{k=2}^{n-1} Q_{q_k}}(i_1) \\ &\cdot (x_1)^2 + \frac{\lambda^2}{4\omega^2}\sum_{k=2}^{n}\sum_{j\in S} Q^{k-2}(i_1, j)\overline{h}_k(j)\overline{\prod_{l=k}^{n-1} Q_{q_l}}(j) \\ &= q_1(i_1)\overline{\prod_{k=2}^{n-1} Q_{q_k}}(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2} \\ &\cdot \sum_{k=2}^{n}\left[Q^{k-2}\left(\overline{h}_k \cdot \overline{\prod_{l=k}^{n-1} Q_{q_l}}\right)\right](i_1). \end{aligned} \tag{B.9}$$

□

C. Proof of Lemma 6

Proof. First, we have

$$\begin{aligned}\xi(i_1) &= \sum_{k=2}^{T+1}\left[Q^{k-2}\left(\widetilde{h}_k\cdot\overline{\prod_{l=k}^{T}Q_{g_l}}\right)\right](i_1)\\ &= \sum_{k=2}^{T+1}Q^{k-2}(i_1)\left[\frac{\left(\overline{\prod_{l=k}^{T}Q_{g_l}}\right)^2}{\overline{\prod_{l=k}^{T}Q_{q_l}}}\cdot h_{k-1}\right].\end{aligned}\tag{C.1}$$

Because $r_n^e(i)$ is assumed to be nonzero, $h_n(i) = (r_n^e(i))^2/E[(R_n^e(i))^2]$ is positive. Together with $q_n(i) > 0$, we have $\xi(i_1) > 0$. Next, we will prove $\xi(i_1) < 1$. For convenience, it first follows from (22) that

$$\begin{aligned}&\operatorname{Var}_{i_1,x_1}\left(W_{T+1}^{\tilde{u}}\right)\\ &= \varphi(i_1)(x_1)^2 + \frac{\lambda^2}{4\omega^2}\xi(i_1)\\ &\quad -\left(\phi(i)x_1 + \frac{\lambda}{2\omega}\xi(i_1)\right)^2\\ &= \xi(i_1)(1-\xi(i_1))\frac{\lambda^2}{4\omega^2} - \phi(i_1)\xi(i_1)\frac{\lambda}{\omega}x_1\\ &\quad + \left(\varphi(i_1) - (\phi(i_1))^2\right)(x_1)^2\\ &= \xi(i_1)(1-\xi(i_1))\left(\frac{\lambda}{2\omega} - \frac{\phi(i_1)x_1}{1-\xi(i_1)}\right)^2\\ &\quad + \frac{\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1)}{1-\xi(i_1)}(x_1)^2.\end{aligned}\tag{C.2}$$

Because λ can take any value in $(-\infty, +\infty)$ and the variance should be greater than zero, the coefficient of $\lambda^2/4\omega^2$ should be greater than zero. That is, $\xi(i_1) < 1$. In addition, $\varphi(i_1) - (\phi(i_1))^2 - \xi(i_1)\varphi(i_1) \geq 0$ is immediately obtained when λ takes a special value $\lambda = 2\omega\phi(i_1)x_1/(1-\xi(i_1))$. □

D. Proof of Lemma 8

Proof. According to (30), we first have

$$\begin{aligned}\Gamma(\omega) &= \phi(i_1)x_1 - \xi(i_1)\beta c_1(i_1)\gamma + \xi(i_1)c_2(i_1) - 2\beta\gamma(\phi(i_1)x_1 + \xi(i_1)c_2(i_1))\omega + \frac{c_1(i_1)\xi(i_1)}{2\omega} + \omega(1+\beta)\\ &\quad\cdot\left[(\phi(i_1))^2(x_1)^2 + 2\phi(i_1)\xi(i_1)x_1\left(\frac{c_1(i_1)}{2\omega} + c_2(i_1)\right) + \left(\frac{(c_1(i_1))^2}{4\omega^2} + \frac{c_1(i_1)c_2(i_1)}{\omega} + (c_2(i_1))^2\right)(\xi(i_1))^2\right]\\ &\quad - \omega\left(\varphi(i_1)(x_1)^2 + \left(\frac{(c_1(i_1))^2}{4\omega^2} + \frac{c_1(i_1)c_2(i_1)}{\omega} + (c_2(i_1))^2\right)\xi(i_1)\right) + \omega\beta\gamma^2 = \phi(i_1)x_1 - \xi(i_1)\beta c_1(i_1)\gamma + \xi(i_1)\\ &\quad\cdot c_2(i_1) - 2\beta\gamma(\phi(i_1)x_1 + \xi(i_1)c_2(i_1))\omega + \frac{c_1(i_1)\xi(i_1)}{2\omega} + (1+\beta)(\phi(i_1))^2(x_1)^2\omega + (1+\beta)\phi(i_1)\xi(i_1)x_1c_1(i_1)\\ &\quad + 2(1+\beta)\phi(i_1)\xi(i_1)x_1c_2(i_1)\omega + (1+\beta)(\xi(i_1))^2\frac{(c_1(i_1))^2}{4\omega} + (1+\beta)(\xi(i_1))^2c_1(i_1)c_2(i_1) + (1+\beta)(\xi(i_1))^2\\ &\quad\cdot(c_2(i_1))^2\omega - \varphi(i_1)(x_1)^2\omega - \frac{(c_1(i_1))^2\xi(i_1)}{4\omega} - c_1(i_1)c_2(i_1)\xi(i_1) - (c_2(i_1))^2\xi(i_1)\omega + \beta\gamma^2\omega.\end{aligned}\tag{D.1}$$

Combining like terms yields

$$\begin{aligned}\Gamma(\omega) &= \phi(i_1)x_1 - \xi(i_1)\beta c_1(i_1)\gamma + \xi(i_1)c_2(i_1) + (1+\beta)\xi(i_1)c_1(i_1)\phi(i_1)x_1 + (1+\beta)(\xi(i_1))^2c_1(i_1)c_2(i_1) - c_1(i_1)c_2(i_1)\xi(i_1)\\ &\quad + \left[2\xi(i_1) + ((1+\beta)\xi(i_1) - 1)\xi(i_1)c_1(i_1)\right]\frac{c_1(i_1)}{4\omega}\\ &\quad + \left[(1+\beta)(\phi(i_1))^2(x_1)^2 - 2\beta\gamma(\phi(i_1)x_1 + \xi(i_1)c_2(i_1)) + 2(1+\beta)\phi(i_1)\xi(i_1)x_1c_2(i_1) + (1+\beta)(\xi(i_1))^2(c_2(i_1))^2 - \varphi(i_1)(x_1)^2 - (c_2(i_1))^2\xi(i_1) + \beta\gamma^2\right]\omega.\end{aligned}\tag{D.2}$$

In (D.2), in view of $c_1(i_1)(1-(1+\beta)\xi(i_1))=1$, we derive

$$\begin{aligned}
&\phi(i_1)x_1-\xi(i_1)\beta c_1(i_1)\gamma+\xi(i_1)c_2(i_1)\\
&\quad+(1+\beta)\xi(i_1)c_1(i_1)\phi(i_1)x_1\\
&\quad+(1+\beta)(\xi(i_1))^2c_1(i_1)c_2(i_1)\\
&\quad-c_1(i_1)c_2(i_1)\xi(i_1)\\
&\quad=\phi(i_1)x_1[1+(1+\beta)\xi(i_1)c_1(i_1)]\\
&\quad+\xi(i_1)[c_2(i_1)-\beta\gamma c_1(i_1)]\\
&\quad+[(1+\beta)\xi(i_1)-1]\xi(i_1)c_1(i_1)c_2(i_1)\\
&\quad=\phi(i_1)x_1c_1(i_1)+\xi(i_1)[c_2(i_1)-\beta\gamma c_1(i_1)]\\
&\quad-\xi(i_1)c_2(i_1)=(\phi(i_1)x_1-\beta\gamma\xi(i_1))c_1(i_1)\\
&\quad=\frac{\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)},\\
&[2\xi(i_1)+((1+\beta)\xi(i_1)-1)\xi(i_1)c_1(i_1)]\frac{c_1(i_1)}{4\omega}\\
&\quad=\frac{\xi(i_1)}{1-(1+\beta)\xi(i_1)}\frac{1}{4\omega}.
\end{aligned}\tag{D.3}$$

Moreover, according to

$$\begin{aligned}
c_1(i_1)&=\frac{1}{1-(1+\beta)\xi(i_1)},\\
c_2(i_1)&=c_1(i_1)[(1+\beta)\phi(i_1)x_1-\beta\gamma],
\end{aligned}\tag{D.4}$$

the coefficient of ω can be simplified as

$$\begin{aligned}
&(1+\beta)(\phi(i_1))^2(x_1)^2-2\beta\gamma(\phi(i_1)x_1\\
&\quad+\xi(i_1)c_2(i_1))+2(1+\beta)\phi(i_1)\xi(i_1)x_1c_2(i_1)\\
&\quad+(1+\beta)(\xi(i_1))^2(c_2(i_1))^2-\varphi(i_1)(x_1)^2\\
&\quad-(c_2(i_1))^2\xi(i_1)+\beta\gamma^2=(1+\beta)(\phi(i_1))^2(x_1)^2\\
&\quad-2\beta\gamma\phi(i_1)x_1+2[(1+\beta)\phi(i_1)x_1-\beta\gamma]\xi(i_1)\\
&\quad\cdot c_2(i_1)-(1-(1+\beta)\xi(i_1))\xi(i_1)(c_2(i_1))^2\\
&\quad-\varphi(i_1)(x_1)^2+\beta\gamma^2=(1+\beta)(\phi(i_1))^2(x_1)^2\\
&\quad-2\beta\gamma\phi(i_1)x_1+2((1+\beta)\phi(i_1)x_1-\beta\gamma)\xi(i_1)\\
&\quad\cdot c_2(i_1)-((1+\beta)\phi(i_1)x_1-\beta\gamma)\xi(i_1)c_2(i_1)\\
&\quad-\varphi(i_1)(x_1)^2+\beta\gamma^2=(1+\beta)(\phi(i_1))^2(x_1)^2\\
&\quad-\varphi(i_1)(x_1)^2-2\beta\phi(i_1)x_1\gamma+((1+\beta)\phi(i_1)x_1\\
&\quad-\beta\gamma)^2\xi(i_1)c_1(i_1)+\beta\gamma^2=(1+\beta)(\phi(i_1))^2(x_1)^2\\
&\quad-\varphi(i_1)(x_1)^2-2\beta\phi(i_1)x_1\gamma\\
&\quad+\left[(1+\beta)^2(\phi(i_1))^2(x_1)^2-2\beta\gamma(1+\beta)\phi(i_1)x_1\right.\\
&\quad\left.+\beta^2\gamma^2\right]\xi(i_1)c_1(i_1)+\beta\gamma^2=\beta(\beta\xi(i_1)c_1(i_1)+1)\\
&\quad\cdot\gamma^2-2\beta\phi(i_1)(1+(1+\beta)\xi(i_1)c_1(i_1))x_1\gamma\\
&\quad+\left[(1+\beta)(\phi(i_1))^2(1+(1+\beta)\xi(i_1)c_1(i_1))\right.\\
&\quad\left.-\varphi(i_1)\right](x_1)^2=\beta(1-\xi(i_1))c_1(i_1)\gamma^2\\
&\quad-2\beta\phi(i_1)c_1(i_1)x_1\gamma+\left[(1+\beta)(\phi(i_1))^2\right.\\
&\quad\left.-(1-(1+\beta)\xi(i_1))\varphi(i_1)\right]c_1(i_1)(x_1)^2\\
&\quad=\frac{y(\gamma)}{1-(1+\beta)\xi(i_1)},
\end{aligned}\tag{D.5}$$

where

$$\begin{aligned}
y(\gamma)&=\beta(1-\xi(i_1))\gamma^2-2\beta\phi(i_1)x_1\gamma\\
&\quad+\left[(1+\beta)(\phi(i_1))^2-(1-(1+\beta)\xi(i_1))\varphi(i_1)\right]\\
&\quad\cdot(x_1)^2.
\end{aligned}\tag{D.6}$$

□

E. Proof of Lemma 9

Proof. Referring to (42), $\gamma<[\mathrm{E}_{i,x_1}(W^{\tilde{u}}_{T+1})|_{\lambda^*}]|_{\omega^*}$ is equivalent to

$$\gamma<\frac{\sqrt{y(\gamma)\xi(i_1)}+\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}.\tag{E.1}$$

Since $1-(1+\beta)\xi(i_1)>0$, the above formula is equivalent to

$$(1-\xi(i_1))\gamma<\phi(i_1)x_1+\sqrt{y(\gamma)\xi(i_1)},\tag{E.2}$$

which will be proved to be equivalent to $\gamma\le\phi(i_1)x_1/(1-\xi(i_1))$. On one hand, when $\gamma\le\phi(i_1)x_1/(1-\xi(i_1))$, that is, $(1-\xi(i_1))\gamma\le\phi(i_1)x_1$, (E.2) holds naturally. On the other hand, when (E.2) holds, we want to obtain $\gamma\le\phi(i_1)x_1/(1-\xi(i_1))$. Otherwise, if $\gamma>\phi(i_1)x_1/(1-\xi(i_1))$, that is, $(1-\xi(i_1))\gamma-\phi(i_1)x_1>0$, then (E.2) can be equivalently written as

$$[(1-\xi(i_1))\gamma-\phi(i_1)x_1]^2<y(\gamma)\xi(i_1),\tag{E.3}$$

which is simplified to

$$\begin{aligned}
&(1-\xi(i_1))\gamma^2-2\phi(i_1)x_1\gamma+(\phi(i_1))^2(x_1)^2\\
&\quad+\xi(i_1)\varphi(i_1)(x_1)^2<0.
\end{aligned}\tag{E.4}$$

Let $f(\gamma) = (1-\xi(i_1))\gamma^2 - 2\phi(i_1)x_1\gamma + (\phi(i_1))^2(x_1)^2 + \xi(i_1)\varphi(i_1)(x_1)^2$ and it is easy to have

$$\begin{aligned}\Delta &= -4\xi(i_1)\left[\varphi(i_1)-(\phi(i_1))^2-\xi(i_1)\varphi(i_1)\right](x_1)^2\\ &\leq 0\end{aligned}\tag{E.5}$$

according to Lemma 6. This together with $1-\xi(i_1) > 0$ implies $f(\gamma)\geq 0$, which conflicts with $f(\gamma)<0$. Therefore, $\gamma > \phi(i_1)x_1/(1-\xi(i_1))$ does not hold. Now we can claim that $\gamma < [\mathrm{E}_{i,x_1}(W_{T+1}^{\tilde{u}})|_{\lambda^*}]|_{\omega^*}$ is equivalent to $\gamma \leq \phi(i_1)x_1/(1-\xi(i_1))$. Noting that

$$y_1 < \frac{\phi(i_1)x_1}{1-\xi(i_1)} < y_2;\tag{E.6}$$

thus (44) is reduced to $\gamma \in \{\gamma \mid \gamma < y_1\}$. □

F. Proof of Theorem 11

Proof. Because $y(\gamma)$ is decreasing in the interval $\gamma \in (-\infty, y_1)$, referring to (42) and (45), it is clear that $\mathrm{E}_{i,x_1}(W_{T+1}^{u^*})$ and $u_n^*(i,x_n)$ are decreasing functions with respect to $\gamma \in (-\infty, y_1)$. In addition, since

$$\gamma < y_1 < \frac{\phi(i_1)x_1}{1-\xi(i_1)},\tag{F.1}$$

together with $y(\gamma)>0$, we know that $\mathrm{Var}_{i,x_1}(W_{T+1}^{u^*})$ in (46) is also decreasing along with $\gamma\in(-\infty,y_1)$. □

G. Proof of Theorem 12

Proof. Referring to (42), (45), and (46), it is clear that $u_n^*(i,x_n)$, $\mathrm{E}_{i,x_1}(W_{T+1}^{u^*})$, and $\mathrm{Var}_{i,x_1}(\mathrm{W}_{T+1}^{u^*})$ contain the term $y(\gamma)$. Thus, we first derive the partial derivative of $y(\gamma)$ with respect to β. By (32), we have

$$\begin{aligned}\frac{\partial y(\gamma)}{\partial\beta} &= (1-\xi(i_1))\gamma^2 - 2\phi(i_1)x_1\gamma\\ &\quad + \left((\phi(i_1))^2+\xi(i_1)\varphi(i_1)\right)(x_1)^2\\ &= (1-\xi(i_1))\left(\gamma-\frac{\phi(i_1)x_1}{1-\xi(i_1)}\right)^2\\ &\quad - \frac{(\phi(i_1))^2}{1-\xi(i_1)}(x_1)^2\\ &\quad + \left((\phi(i_1))^2+\xi(i_1)\varphi(i_1)\right)(x_1)^2\\ &= (1-\xi(i_1))\left(\gamma-\frac{\phi(i_1)x_1}{1-\xi(i_1)}\right)^2\\ &\quad + \frac{\xi(i_1)}{1-\xi(i_1)}\left[(1-\xi(i_1))\varphi(i_1)-(\phi(i_1))^2\right](x_1)^2.\end{aligned}\tag{G.1}$$

In view of Lemma 6, $\partial y(\gamma)/\partial\beta > 0$. Therefore, by (45), we know that $\sqrt{y(\gamma)}$ and $1/(1-(1+\beta)\xi(i_1))$ are increasing with respect to β. Together with

$$\begin{aligned}\frac{\partial}{\partial\beta}\frac{(1+\beta)\phi(i_1)x_1-\beta\gamma}{1-(1+\beta)\xi(i_1)} &= \frac{(\phi(i_1)x_1-\gamma)(1-(1+\beta)\xi(i_1))+\xi(i_1)\left[(1+\beta)\phi(i_1)x_1-\beta\gamma\right]}{(1-(1+\beta)\xi(i_1))^2}\\ &= \frac{\phi(i_1)x_1-(1-\xi(i_1))\gamma}{(1-(1+\beta)\xi(i_1))^2} > 0,\end{aligned}\tag{G.2}$$

we first obtain that $u_n^*(i,x_n)$ is an increasing function of β. Next, we shall prove that this conclusion holds true for $\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*})$. Referring to (42), we have

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathrm{E}_{i_1,x_1}\left(W_{T+1}^{u^*}\right) &= \frac{\partial}{\partial\beta}\frac{\sqrt{y(\gamma)\xi(i_1)}+\phi(i_1)x_1-\beta\gamma\xi(i_1)}{1-(1+\beta)\xi(i_1)}\\ &= \frac{\left(\partial\sqrt{y(\gamma)\xi(i_1)}/\partial\beta-\gamma\xi(i_1)\right)(1-(1+\beta)\xi(i_1))+\xi(i_1)\left(\sqrt{y(\gamma)\xi(i_1)}+\phi(i_1)x_1-\beta\gamma\xi(i_1)\right)}{(1-(1+\beta)\xi(i_1))^2}\\ &= \frac{(1-(1+\beta)\xi(i_1))\left(\partial\sqrt{y(\gamma)\xi(i_1)}/\partial\beta\right)+\xi(i_1)\sqrt{y(\gamma)\xi(i_1)}+\xi(i_1)(\phi(i_1)x_1-\gamma(1-\xi(i_1)))}{(1-(1+\beta)\xi(i_1))^2}.\end{aligned}\tag{G.3}$$

When $0 < \beta < 1/\xi(i_1) - 1$ and $\gamma < y_1$, together with $y_1 < \phi(i_1)x_1/(1-\xi(i_1))$ and $\partial y(\gamma)/\partial\beta > 0$, we know that $(\partial/\partial\beta)\mathrm{E}_{i_1,x_1}(W_{T+1}^{u^*}) > 0$. Finally, we aim to show that $\partial/\partial\beta\ \mathrm{Var}_{i_1,x_1}(W_{T+1}^{u^*})$ is also greater than zero. In view of (46), because $\gamma < \phi(i_1)x_1/(1-\xi(i))$, it is easy to find that $c_1(i_1)\beta(\phi(i_1)x_1/(1-\xi(i_1))-\gamma)$ is increasing along with β, leading to

$$\frac{\partial}{\partial\beta}\mathrm{Var}_{i_1,x_1}\left(W_{T+1}^{u^*}\right) > 0. \quad \text{(G.4)}$$

□

Conflicts of Interest

The authors, Chuangwei Lin and Huiling Wu, declare that there are not any conflicts of interest related to this paper.

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