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Research Article

Boundary Layer Effects for the Nonlinear Evolution Equations with the Vanishing Diffusion Limit

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We study an initial-boundary value problem for a nonlinear evolution system with damping and diffusion; our main purpose is to investigate the boundary layer effects when the vanishing diffusion limit $\alpha \to 0^+$, especially for the mixed boundary conditions; we prove that the thickness of layer is of the order $O(\alpha)$. Furthermore, the corresponding convergence rates are also obtained.

1. Introduction and Main Results

The main objective of this article is to study the nonlinear evolution equations with damping and diffusion on the domain $[0,1]\times[0,\infty)$:

$$\begin{aligned}\psi_t^\alpha &= -(k-1)\alpha\psi^\alpha - k\alpha\theta_x^\alpha + \alpha\psi_{xx}^\alpha,\\ \theta_t^\alpha &= -(1-\beta)\theta^\alpha + \mu\alpha\psi_x^\alpha + 2\psi^\alpha\theta_x^\alpha + \beta\theta_{xx}^\alpha,\end{aligned} \tag{1}$$
$$0 < x < 1,\ t > 0$$

with initial data

$$(\psi^\alpha,\theta^\alpha)(x,0) = (\psi_0,\theta_0)(x), \quad 0 \le x \le 1, \tag{2}$$

and the mixed boundary conditions, that is, Neumann-Dirichlet boundary conditions

$$(\psi_x^\alpha,\theta^\alpha)(0,t) = (\psi_x^\alpha,\theta^\alpha)(1,t) = (0,0), \quad t \ge 0, \tag{3}$$

which implies that

$$(\psi_{xt}^\alpha,\theta_t^\alpha)(0,t) = (\psi_{xt}^\alpha,\theta_t^\alpha)(1,t) = (0,0), \quad t \ge 0, \tag{4}$$

where k, α, β, and μ are positive constants with $k > 1$ and $0 < \beta < 1$. The limit problem of the vanishing parameter $\alpha \to 0^+$; we have

$$\begin{aligned}\psi_t^0 &= 0,\\ \theta_t^0 &= -(1-\beta)\theta^0 + 2\psi^0\theta_x^0 + \beta\theta_{xx}^0,\end{aligned} \tag{5}$$
$$0 < x < 1,\ t > 0$$

with the initial conditions

$$(\psi^0,\theta^0)(x,0) = (\phi_1(x),\phi_2(x)), \quad 0 \le x \le 1, \tag{6}$$

and the Dirichlet boundary conditions

$$\theta^0(0,t) = \theta^0(1,t) = (0,0), \quad t \ge 0, \tag{7}$$

which implies that

$$\begin{aligned}\theta_t^0(0,t) &= \theta_t^0(1,t) = 0,\\ \left(2\psi^0\theta_x^0 + \beta\theta_{xx}^0\right)(0,t) &= \left(2\psi^0\theta_x^0 + \beta\theta_{xx}^0\right)(1,t) = 0,\end{aligned} \tag{8}$$
$$t \ge 0.$$

We expect to prove that the solution $(\psi^\alpha,\theta^\alpha)$ of (1)–(3) converges to the solution (ψ^0,θ^0) of the limit problem (5)–(7), as the diffusion parameter $\alpha \to 0^+$, and get the boundary layer thickness.

The nonlinear interaction between ellipticity and dissipation is involved quite broadly in physical and mechanical systems, such as Rayleigh-Benard problem, superposed fluids, Taylor-Couette instability, dynamic phase transitions, and fluid flow down an inclined plane. [1–3]. As these systems are usually quite complicated, they are far from being well understood. System (1) was originally proposed by Hsieh in [2] to

investigate the nonlinear interaction between ellipticity and dissipation. We also refer to [1–3] for the physical background of (1). This study is expected to yield insights into physical systems with similar mechanism, such as the Ginzburg-Landau equation and the Kuramoto-Sivashinsky equation [1, 3, 4]; it may also help us, by comparison, to understand better the nonlinear interaction between other instabilities and dissipation.

System (1) has been extensively studied by several authors in different contexts [2, 3, 5–9], such as the well-posedness problems, the nonlinear stability problem, and optimal decay rate of the solutions. However, all the results above need to assume that all parameters are fixed constants. Another interesting problem is the zero diffusion limit: that is, consider the limit problem of solution sequences when one or more of parameters vanishes for the corresponding Cauchy problem or initial-boundary value problem [5–7, 9]. It should be emphasized that flows often move in bounded domains with constraints from boundaries in real world; in the bounded domains case, the boundary effect requires a careful mathematical analysis. Ruan and Zhu [5] consider the initial-boundary value problem (1) with the zero Dirichlet boundary conditions when diffusion parameter $\beta \to 0^+$ and they show that the boundary layer thickness $\delta(\beta)$ is of the order $O(\beta^\gamma)$ with $0 < \gamma < 1/2$. Subsequently, Peng et al. [6] consider the initial-boundary value problem (1) with the Dirichlet-Neumann boundary conditions when diffusion parameter $\beta \to 0^+$ and they show that the boundary layer thickness $\delta(\beta)$ is of the order $O(\beta^\gamma)$ with $0 < \gamma < 3/4$. For the parameter $\alpha \to 0^+$, Chen and Zhu in [10] consider the Cauchy problem for (1); Peng in [7] study the initial-boundary value problem for (1) with the zero Dirichlet boundary conditions and obtain the thickness of layer of the order $O(\alpha^{1/2})$. However, for the asymptotic limit $\alpha \to 0^+$, all these results are restricted to the boundary conditions to the Dirichlet boundary conditions. By [6], which motivates our investigation in this paper, we concentrate our efforts to investigate the boundary layer effect and the convergence rates on the initial-boundary value (1)–(3) with the Dirichlet-Neumann boundary conditions when diffusion parameter $\alpha \to 0^+$, and the solutions of the mixed boundary conditions usually exhibit different behaviors and much rich phenomena comparing with the general boundary conditions.

For later presentation, next we state the notation as follows.

Notations. Throughout this paper, we denote positive constant independent of α by C. And the constant C may vary from line to line. $L^2 = L^2([0,1])$ and $L^\infty = L^\infty([0,1])$ denote the usual Lebesgue space in $[0,1]$ with its norms $\|f\|_{L^2([0,1])} = \|f\| = (\int_0^1 |f(x)|^2 dx)^{1/2}$ and $\|f\|_{L^\infty} = \sup_{x\in[0,1]}|f(x)|$. $H^l([0,1])$ denotes the usual l-th order Sobolev space with its norm $\|f\|_{H^l([0,1])} = \|f\|_l = (\sum_{i=0}^{l} \|\partial_x^i f\|^2)^{1/2}$. For simplicity, $\|f(\cdot,t)\|_{L^2}$, $\|f(\cdot,t)\|_{L^\infty}$ and $\|f\|_l$, respectively.

Before we state the main results, let us recall the definition of boundary layer thickness (BL-thickness) in [11] as follows.

Definition 1. A positive function $\delta(\alpha)$ is called a boundary layer thickness for problem (1)–(3) with vanishing diffusion α, if $\delta(\alpha) \to 0$ as $\alpha \to 0$, and

$$\lim_{\alpha\to 0} \left\|\theta^\alpha - \theta^0\right\|_{L^\infty(0,T;L^\infty[0,1])} = 0, \tag{9}$$

$$\lim_{\alpha\to 0} \left\|\psi^\alpha - \psi^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} = 0, \tag{10}$$

$$\liminf_{\alpha\to 0} \left\|\psi^\alpha - \psi^0\right\|_{L^\infty(0,T;L^\infty[0,1])} > 0, \tag{11}$$

where $0 < \delta = \delta(\beta) < 1$, $(\psi^\alpha, \theta^\alpha)$, and (ψ^0, θ^0) is the solution to problem (1)–(3) and the limit problem (5)–(7), respectively.

Our first purpose is to show that the initial-boundary value problem (1)–(3) admits a unique global smooth solution $(\psi^\alpha, \theta^\alpha)$.

Theorem 2. *Suppose that the initial data satisfy the conditions:* $(\psi_0, \theta_0) \in H^2$, $(\psi_0, \theta_0)(0) = (\psi_0, \theta_0)(1) = (0,0)$ *and* $\|\psi_0\|_1 + \|\theta_0\|_1$ *is sufficiently small, then there exists a unique solution* $(\psi^\alpha, \theta^\alpha)$ *to the initial-boundary value problem (1)–(3) satisfying*

$$\begin{aligned} &\theta^\alpha \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^2\right), \\ &\theta_t^\alpha \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^1\right), \end{aligned} \tag{12}$$

where the norms are all uniform in α.

Then our second result shows that the initial-boundary value problem (5)–(7) admits a unique global smooth solution (ψ^0, θ^0).

Theorem 3. *Suppose that the initial data satisfy the conditions:* $(\phi_1, \phi_2) \in H^2$, $\phi_2(0) = \phi_2(1) = 0 \le \varepsilon$, *and* ε *is sufficiently small, then there exists a unique solution* (ψ^0, θ^0) *to the initial-boundary value problem (5)–(7) satisfying*

$$\begin{aligned} &\psi^0 \in L^\infty\left(0,T;H^2\right), \\ &\theta^0 \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^2\right), \\ &\theta_t^0 \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H^1\right). \end{aligned} \tag{13}$$

Furthermore, we give the convergence rates and boundary layer thickness.

Theorem 4. *Under the assumptions of Theorems 2 and 3, suppose that*

$$\begin{aligned} &\left\|\psi_0(x) - \phi_1(x)\right\|_1^2 \le C\alpha, \\ &\left\|\theta_0(x) - \phi_2(x)\right\|_1^2 \le C\alpha; \end{aligned} \tag{14}$$

then any function $\delta(\alpha)$ satisfying the conditions $\delta(\alpha) \to 0$ and $\alpha/\delta(\alpha) \to 0$ as $\alpha \to 0^+$ is a BL-thickness such that

$$\begin{aligned} \left\|\theta^\alpha - \theta^0\right\|_{L^\infty(0,T;L^\infty[0,1])} &\le C\alpha, \\ \left\|\psi^\alpha - \psi^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} &\le C\frac{\alpha}{\delta(\alpha)}, \\ \liminf_{\alpha\to 0}\left\|\psi^\alpha - \psi^0\right\|_{L^\infty(0,T;L^\infty[0,1])} &> 0. \end{aligned} \tag{15}$$

Consequently,

$$\begin{aligned} \lim_{\alpha\to 0}\left\|\theta^\alpha - \theta^0\right\|_{L^\infty(0,T;L^\infty[0,1])} &= 0, \\ \lim_{\alpha\to 0}\left\|\psi^\alpha - \psi^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} &= 0. \end{aligned} \tag{16}$$

The plan of this paper is as follows. Section 2 is devoted to the global existence results on the initial-boundary value problem (1)–(3) and the local existence for the limit problem (5)–(7). Section 3 details convergence rates and the BL-thickness as the diffusion parameter $\alpha \to 0^+$ for the mixed boundary conditions.

2. Proof of Theorems 2 and 3

2.1. Proof of Theorem 2. In this subsection, we shall prove Theorem 2 by adapting the elaborate nonlinear energy method. In the following, we devote ourselves to the a priori estimate of solution $(\psi^\alpha(x,t), \theta^\alpha(x,t))$ of (1)–(3) under the a priori assumption

$$N_1(t) = \sup_{0<t<T}\left\|\left(\psi^\alpha, \theta^\alpha\right)(t)\right\|_1^2 \le \varepsilon_1^2, \tag{17}$$

where ε_1 is a positive constant satisfying $0 < \varepsilon_1 \le 1$, independent of α.

By Sobolev inequality, we have

$$\left\|\left(\psi^\alpha, \theta^\alpha\right)(t)\right\|_{L^\infty} \le \varepsilon_1. \tag{18}$$

2.1.1. L^2-Estimates

Lemma 5. *Assume that the conditions in Theorem 2 hold. Then a positive constant C exists independent of α, such that*

$$\begin{aligned} &\frac{1}{2}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]dx + \int_0^t\int_0^1\left[(k-1)\alpha\left(\psi^\alpha\right)^2\right. \\ &\quad \left. + (1-\beta)\left(\theta^\alpha\right)^2 + \left(\psi_x^\alpha\right)^2 + \left(\theta_x^\alpha\right)^2\right]dx\,d\tau \\ &\quad \le C. \end{aligned} \tag{19}$$

Proof. Multiplying the first and the second equations of (1) by ψ^α and θ^α, respectively, then integrating the resulting equations over $(0,1)$, using integration by parts and the boundary conditions (3), and then adding them, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]dx \\ &\quad + \int_0^1\left[(k-1)\alpha\left(\psi^\alpha\right)^2 + (1-\beta)\left(\theta^\alpha\right)^2\right]dx \\ &\quad + \alpha\int_0^1\left(\psi_x^\alpha\right)^2dx + \beta\int_0^1\left(\theta_x^\alpha\right)^2dx \\ &\quad = (k+\mu)\alpha\int_0^1\psi_x^\alpha\theta^\alpha dx - \int_0^1\psi_x^\alpha\left(\theta^\alpha\right)^2dx. \end{aligned} \tag{20}$$

Integrating (20) over $(0,t)$, using Cauchy-Schwarz inequality and (18), we obtain for any $p_1 > 0$

$$\begin{aligned} &\frac{1}{2}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]dx \\ &\quad + \int_0^t\int_0^1\left[(k-1)\alpha\left(\psi^\alpha\right)^2 + (1-\beta)\left(\theta^\alpha\right)^2\right]dx\,d\tau \\ &\quad + \alpha\int_0^t\int_0^1\left(\psi_x^\alpha\right)^2dx\,d\tau + \beta\int_0^1\left(\theta_x^\alpha\right)^2dx \\ &\quad = \frac{1}{2}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]_{t=0}dx \\ &\quad + (k+\mu)\alpha\int_0^t\int_0^1\psi_x^\alpha\theta^\alpha dx\,d\tau \\ &\quad - \int_0^t\int_0^1\psi_x^\alpha\left(\theta^\alpha\right)^2dx\,d\tau \\ &\quad = \frac{1}{2}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]_{t=0}dx \\ &\quad + \int_0^t\left((k+\mu)\alpha + \left\|\theta^\alpha(t)\right\|_{L^\infty}\right)\int_0^1\left|\psi_x^\alpha\theta^\alpha\right|dx\,d\tau \\ &\quad \le \frac{1}{2}\int_0^1\left[\left(\psi_0\right)^2 + \left(\theta_0\right)^2\right]dx + p_1\int_0^t\int_0^1\left(\psi_x^\alpha\right)^2dx\,d\tau \\ &\quad + \frac{\left((k+\mu)\alpha + C\varepsilon_1\right)^2}{4p_1}\int_0^t\int_0^1\left(\theta^\alpha\right)^2dx\,d\tau. \end{aligned} \tag{21}$$

If we choose $p_1 > 0$ such that

$$1 - \beta - \frac{\left((k+\mu)\alpha + C\varepsilon_1\right)^2}{4p_1} > 0, \quad \alpha - p_1 > 0, \tag{22}$$

then we have

$$\begin{aligned} &\frac{1}{2}\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2\right]dx \\ &\quad + \int_0^t\int_0^1\left[\left(\psi^\alpha\right)^2 + \left(\theta^\alpha\right)^2 + \left(\psi_x^\alpha\right)^2 + \left(\theta_x^\alpha\right)^2\right]dx\,d\tau \\ &\quad \le C\left\|\left(\psi_0, \theta_0\right)\right\|^2. \end{aligned} \tag{23}$$

□

2.1.2. High-Order Energy Estimates

Lemma 6. *Assume that the conditions in Theorem 2 hold. Then a positive constant C exists independent of α, such that*

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2\right]dx\\&+\int_0^t\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2+\left(\psi_{xx}^\alpha\right)^2+\beta\left(\theta_{xx}^\alpha\right)^2\right]dx\,d\tau\\&\le C.\end{aligned}\tag{24}$$

Proof. Multiplying the first and the second equations of (1) by $-\psi_{xx}^\alpha$ and $-\theta_{xx}^\alpha$, respectively, then integrating the resulting equations over $(0,1)$, using integration by parts and the boundary conditions (3), and then adding them, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2\right]dx\\&\quad+\int_0^1\left[(k-1)\,\alpha\left(\psi_x^\alpha\right)^2+(1-\beta)\left(\theta_x^\alpha\right)^2\right]dx\,d\tau\\&\quad+\alpha\int_0^1\left(\psi_{xx}^\alpha\right)^2dx+\beta\int_0^1\left(\theta_{xx}^\alpha\right)^2dx\\&=(k+\mu)\,\alpha\int_0^1\psi_{xx}^\alpha\theta_x^\alpha dx+\int_0^1\psi_x^\alpha\left(\theta_x^\alpha\right)^2dx.\end{aligned}\tag{25}$$

Integrating (25) over $(0,t)$, using Cauchy-Schwarz inequality and (17)-(18), we obtain for any $p_1>0$

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2\right]dx\\&+\int_0^t\int_0^1\left[(k-1)\,\alpha\left(\psi_x^\alpha\right)^2+(1-\beta)\left(\theta_x^\alpha\right)^2\right]dx\\&+\alpha\int_0^t\int_0^1\left(\psi_{xx}^\alpha\right)^2dx\,d\tau+\beta\int_0^t\int_0^1\left(\theta_{xx}^\alpha\right)^2dx\,d\tau\\&=\frac{1}{2}\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2\right]_{t=0}dx\\&+(k+\mu)\,\alpha\int_0^t\int_0^1\psi_{xx}^\alpha\theta_x^\alpha dx\,d\tau\\&+\int_0^t\int_0^1\psi_x^\alpha\left(\theta_x^\alpha\right)^2dx\,d\tau\\&\le\frac{1}{2}\int_0^1\left[\left(\psi_{0x}\right)^2+\left(\theta_{0x}\right)^2\right]dx\\&+p_1\int_0^t\int_0^1\left(\psi_{xx}^\alpha\right)^2dx\,d\tau\\&+\frac{\left((k+\mu)^2\alpha^2+C\varepsilon_1\right)^2}{4p_1}\int_0^t\int_0^1\left(\theta_x^\beta\right)^2dx\,d\tau.\end{aligned}\tag{26}$$

If we choose $p_1>0$ such that

$$1-\beta-\frac{\left((k+\mu)^2\alpha^2+C\varepsilon_1\right)^2}{4p_1}>0,\quad\alpha-p_1>0,\tag{27}$$

then we have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2\right]dx\\&+\int_0^t\int_0^1\left[\left(\psi_x^\alpha\right)^2+\left(\theta_x^\alpha\right)^2+\alpha\left(\psi_{xx}^\alpha\right)^2+\beta\left(\theta_{xx}^\alpha\right)^2\right]dx\,d\tau\\&\le C\left\|(\psi_0,\theta_0)\right\|_1^2.\end{aligned}\tag{28}$$

□

Lemma 7. *Let the assumptions of Theorem 2 hold. Then a positive constant C exists independent of α, such that*

$$\begin{aligned}&\int_0^1\left[\left(\psi_t^\alpha\right)^2+\left(\theta_t^\alpha\right)^2\right]dx\\&\quad+\int_0^t\int_0^1\left[\left(\psi_{xt}^\alpha\right)^2dx\,d\tau+\left(\theta_{xt}^\alpha\right)^2\right]dx\,d\tau\le C.\end{aligned}\tag{29}$$

Proof. Differentiating (1) with respect to t, we get

$$\begin{aligned}\psi_{tt}^\alpha&=-(k-1)\,\alpha\psi_t^\alpha-k\alpha\theta_{xt}^\alpha+\alpha\psi_{xxt}^\alpha,\\\theta_{tt}^\alpha&=-(1-\beta)\,\theta_t^\alpha+\mu\alpha\psi_{xt}^\alpha+2\left(\psi^\alpha\theta_x^\alpha\right)_t+\beta\left(\theta_{xxt}^\alpha\right).\end{aligned}\tag{30}$$

Multiplying the first and the second equations of (1) by $-\psi_t^\alpha$ and $-\theta_t^\alpha$, respectively, then integrating the resulting equations over $(0,1)$, using integration by parts and the boundary conditions (3), and then adding them, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left(\psi_t^\alpha\right)^2+\left(\theta_t^\alpha\right)^2\right]dx\\&+\int_0^t\int_0^1(k-1)\,\alpha\left(\psi_t^\alpha\right)^2dx\,d\tau\\&+(1-\beta)\int_0^t\int_0^1\left(\theta_t^\alpha\right)^2dx\,d\tau+\alpha\int_0^t\int_0^1\left(\psi_{xt}^\alpha\right)^2dx\,d\tau\\&+\beta\int_0^1\left(\theta_{xt}^\alpha\right)^2dx\,d\tau=(k+\mu)\,\alpha\int_0^t\int_0^1\psi_{xt}^\alpha\theta_t^\alpha dx\,d\tau\\&+2\int_0^t\int_0^1\psi_t^\alpha\theta_x^\alpha\theta_t^\alpha dx\,d\tau-\int_0^t\int_0^1\psi_x^\alpha\left(\theta_t^\alpha\right)^2dx\,d\tau\\&=\frac{1}{2}\int_0^1\left[\left(\psi_t^\alpha\right)^2+\left(\theta_t^\alpha\right)^2\right]_{t=0}dx+\sum_{i=1}^3J_i.\end{aligned}\tag{31}$$

We estimate every term as follows:

$$\begin{aligned}J_1&\le p_1\int_0^t\int_0^1\left(\psi_{xt}^2\right)dx\,d\tau\\&\quad+\frac{(k+\mu)^2\alpha^2}{4p_1}\int_0^t\int_0^1\left(\theta_t^\alpha\right)^2dx\,d\tau.\end{aligned}\tag{32}$$

Here J_2, J_3, and J_4 are estimated as follows:

$$J_2 \le C\varepsilon_1 \int_0^t \int_0^1 \left(\psi_t^\alpha\right)^2 + \left(\theta_t^\alpha\right)^2 dx\, d\tau,$$
$$J_3 \le C\varepsilon_1 \int_0^t \int_0^1 \left(\theta_t^\alpha\right)^2 dx\, d\tau. \tag{33}$$

Here we have used Cauchy-Schwarz inequality and Sobolev inequality.

Substituting (32) and (33) into (34), we have from (18) and Lemma 6

$$\int_0^1 \left[\left(\psi_t^\alpha\right)^2 + \left(\theta_t^\alpha\right)^2\right] dx + \int_0^t \int_0^1 \left[\left(\psi_t^\alpha\right)^2 + \left(\theta_t^\alpha\right)^2 + \left(\psi_{xt}^\alpha\right)^2 + \beta\left(\theta_{xt}^\alpha\right)^2\right] dx\, d\tau \le C. \tag{34}$$

By the similar method, it is easy to obtain

$$\int_0^1 \left[\left(\psi_{xx}^\alpha\right)^2 + \beta\left(\theta_{xx}^\alpha\right)^2\right] dx\, d\tau \le C. \tag{35}$$

□

Lemma 8. *Assume that the conditions in Theorem 2 hold. Then there exists a positive constant C independent of α, such that*

$$\int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx + \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau + \beta \int_0^t \int_0^1 \left(\theta_{xxx}^\alpha\right) dx\, d\tau \le C. \tag{36}$$

Proof. Differentiating the second equation of (1) with respect to x, we get

$$\theta_{xt}^\alpha = -(1-\beta)\theta_x^\alpha + \mu\alpha\psi_{xx}^\alpha + 2\left(\psi^\alpha\theta_x^\alpha\right)_x + \beta\theta_{xxx}^\alpha. \tag{37}$$

Multiplying the equation (37) by $-\theta_{xxx}^\alpha$,

$$\begin{aligned}
&\int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx + \int_0^t \int_0^1 (1-\beta)\left(\theta_{xx}^\alpha\right)^2 dx\, d\tau \\
&+ \beta \int_0^t \int_0^1 \left(\theta_{xxx}^\alpha\right) dx\, d\tau = \int_0^1 \left(\theta_{0xx}\right)^2 dx \\
&- \mu \int_0^t \int_0^1 \left[\psi_t^\alpha + (k-1)\alpha\psi^\alpha + k\alpha\theta_x^\alpha\right]\theta_{xxx}^\alpha dx\, d\tau \\
&- 2\int_0^t \int_0^1 \psi_x^\alpha\theta_x^\alpha\theta_{xxx}^\alpha dx\, d\tau \\
&- 2\int_0^t \int_0^1 \psi^\alpha\theta_{xx}^\alpha\theta_{xxx}^\alpha dx\, d\tau = \int_0^1 \left(\theta_{0xx}\right)^2 dx \\
&+ \mu k\alpha \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau \\
&+ \int_0^t \int_0^1 \mu\left[\psi_{xt}^\alpha + (k-1)\alpha\psi_x^\alpha\right]\theta_{xx}^\alpha dx\, d\tau \\
&+ 2\int_0^t \int_0^1 \psi_{xx}^\alpha\theta_x^\alpha\theta_{xx}^\alpha dx\, d\tau \\
&+ 3\int_0^t \int_0^1 \psi^\alpha\left(\theta_{xx}^\alpha\right)^2 dx\, d\tau = \int_0^1 \left(\theta_{0xx}\right)^2 dx \\
&+ \mu k\alpha \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau + \sum_{i=4}^{6} J_i.
\end{aligned} \tag{38}$$

We estimate every term as follows:

$$\begin{aligned}
J_4 &\le C\int_0^t \int_0^1 \left[\left(\psi_{xt}^\alpha\right)^2 + \left(\psi_x^\alpha\right)^2\right] dx\, d\tau \\
&\quad + p_1 \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau, \\
J_5 &\le C\varepsilon_1 \int_0^t \int_0^1 \left(\psi_{xx}^\alpha\right)^2 dx\, d\tau \\
&\quad + C\varepsilon_1 \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau, \\
J_6 &\le C\varepsilon_1 \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau.
\end{aligned} \tag{39}$$

Substituting (39) into (45), we have

$$\int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx + \int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau + \beta \int_0^t \int_0^1 \left(\theta_{xxx}^\alpha\right) dx\, d\tau \le C. \tag{40}$$

Now we notice that the priori assumption (18) can be closed. Since, under this priori assumption (18), we deduced that (19), (24), (29), and (36) hold provided that ε_1 is sufficiently small. Therefore, assumption (18) is always true provided that $\|\psi_0, \theta_0\|_{L^2}^2$ is sufficiently small. □

Lemma 9. *Assume that the conditions in Theorem 2 hold. Then a positive constant C exists independent of α, such that*

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\left(\psi_{xt}^\alpha\right)^2 dx + \left(\theta_{xt}^\alpha\right)^2\right] dx \\
&\quad + \int_0^t \int_0^1 \left[\left(\psi_{xt}^\alpha\right)^2 + \left(\theta_{xt}^\alpha\right)^2\right] dx\, d\tau \\
&\quad + \alpha \int_0^t \int_0^1 \left(\psi_{xxt}^\alpha\right)^2 dx\, d\tau + \beta \int_0^t \int_0^1 \left(\theta_{xxt}^\alpha\right)^2 dx\, d\tau \\
&\le C.
\end{aligned} \tag{41}$$

Proof. Multiplying the first and the second equations of (30) by $-\psi_{xxt}^\alpha$ and $-\theta_{xxt}^\alpha$, respectively, then integrating the

resulting equations over $(0,1)$, using integration by parts and the boundary conditions (3), and then adding them, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left[\left(\psi_{xt}^{\alpha}\right)^2dx+\left(\theta_{xt}^{\alpha}\right)^2\right]dx\\
&\quad+\int_0^t\int_0^1\left[(k-1)\,\alpha\left(\psi_{xt}^{\alpha}\right)^2+(1-\beta)\left(\theta_{xt}^{\alpha}\right)^2\right]dx\,d\tau\\
&\quad+\alpha\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)dx\,d\tau+\beta\int_0^t\int_0^1\left(\theta_{xxt}^{\alpha}\right)dx\,d\tau\\
&\quad=\frac{1}{2}\int_0^1\left[\left(\psi_{xt}^{\alpha}\right)^2dx+\left(\theta_{xt}^{\alpha}\right)^2\right]_{t=0}dx\\
&\quad+k\alpha\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)\theta_{xt}^{\alpha}dx\,d\tau\\
&\quad+\mu\alpha\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)\theta_{xt}^{\alpha}dx\,d\tau\\
&\quad+2\int_0^t\int_0^1\left(\psi^{\alpha}\theta_x^{\alpha}\right)_t\left(-\theta_{xxt}^{\alpha}\right)dx\,d\tau\\
&\quad=\frac{1}{2}\int_0^1\left[\left(\psi_{xt}^{\alpha}\right)^2dx+\left(\theta_{xt}^{\alpha}\right)^2\right]_{t=0}dx\\
&\quad+(k+\mu)\,\alpha\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)\theta_{xt}^{\alpha}dx\,d\tau\\
&\quad-2\int_0^t\int_0^1\psi_t^{\alpha}\theta_x^{\alpha}\theta_{xxt}^{\alpha}dx\,d\tau\\
&\quad-2\int_0^t\int_0^1\psi^{\alpha}\theta_{xt}^{\alpha}\theta_{xxt}^{\alpha}dx\,d\tau=M\\
&\quad+2\int_0^t\int_0^1\psi_{xt}^{\alpha}\theta_x^{\alpha}\theta_{xt}^{\alpha}dx\,d\tau+2\int_0^t\int_0^1\psi_t^{\alpha}\theta_{xx}^{\alpha}\theta_{xt}^{\alpha}dx\\
&\quad+\int_0^t\int_0^1\psi_x^{\alpha}\left(\theta_{xt}^{\alpha}\right)^2dx\,d\tau=M+\sum_{i=7}^{10}J_i.
\end{aligned}\tag{42}$$

We estimate every term as follows:

$$\begin{aligned}
J_7&\le p_1\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)^2dx\,d\tau\\
&\quad+\frac{\left[(k+\mu)\,\alpha\right]^2}{4p_1}\int_0^t\int_0^1\left(\theta_{xt}\right)^2dx\,d\tau,\\
J_8&\le C\varepsilon_1\int_0^t\int_0^1\left(\psi_{xt}^{\alpha}\right)^2dx\,d\tau\\
&\quad+C\varepsilon_1\int_0^t\int_0^1\left(\theta_{xt}^{\alpha}\right)^2dx\,d\tau,\\
J_9&\le 2\int_0^t\int_0^1\left(\psi_t^{\alpha}\theta_{xt}^{\alpha}\right)^2dx\,d\tau+\frac{1}{2}\int_0^t\int_0^1\left(\theta_{xx}^{\alpha}\right)^2dx\,d\tau\\
&\le C\int_0^t\int_0^1\left[\left(\psi_t^{\alpha}\right)^2+\left(\psi_{xt}^{\alpha}\right)^2\right]dx\int_0^1\left(\theta_{xt}^{\alpha}\right)^2dx\,d\tau\\
&\quad+\frac{1}{2}\int_0^t\int_0^1\left(\theta_{xx}^{\alpha}\right)^2dx\,d\tau,\\
J_{10}&\le C\varepsilon_1\int_0^t\int_0^1\left(\theta_{xt}^{\alpha}\right)^2dx\,d\tau.
\end{aligned}\tag{43}$$

Substituting (43) into (45), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left[\left(\psi_{xt}^{\alpha}\right)^2dx+\left(\theta_{xt}^{\alpha}\right)^2\right]dx\\
&\quad+\int_0^t\int_0^1\left[\left(\psi_{xt}^{\alpha}\right)^2+\left(\theta_{xt}^{\alpha}\right)^2\right]dx\,d\tau\\
&\quad+\alpha\int_0^t\int_0^1\left(\psi_{xxt}^{\alpha}\right)^2dx\,d\tau+\beta\int_0^t\int_0^1\left(\theta_{xxt}^{\alpha}\right)^2dx\,d\tau\\
&\le M.
\end{aligned}\tag{44}$$

By the similar method, we have

$$\begin{aligned}
&\int_0^1\left[\left(\psi_{xxx}^{\alpha}\right)^2dx+\left(\theta_{xxx}^{\alpha}\right)^2\right]dx\\
&\quad+\int_0^t\int_0^1\left[\left(\psi_{xxx}^{\alpha}\right)^2+\beta\left(\theta_{xxx}^{\alpha}\right)^2\right]dx\,d\tau\le M.
\end{aligned}\tag{45}$$

This completes the proof of Lemma 9. □

Combination of Lemmas 5–9 and a well-known result on the local existence of the solutions, we can get the global existence of the solutions; this proves Theorem 2.

2.2. Proof of Theorem 3. Suppose a priori assumption:

$$A_2(T)=\sup_{0<t<T}\left\|\left(\psi_0,\theta_0\right)(t)\right\|_3^2\le\varepsilon_2,\tag{46}$$

where $0<\varepsilon_2\ll 1$. We obtain the following energy estimates.

Lemma 10. *Suppose that the initial data satisfy the conditions:* $\phi_1\in H^2$, $\phi_2\in H^2$, $\phi_2(0)=\phi_2(1)=0$, *and* $\|\phi_1\|_{L^2}+\|\phi_1\|_{L^2}\le\varepsilon$, *and* ε *is sufficiently small. Then there exists a unique solution* (ψ^0,θ^0) *to the initial-boundary value problem (5)–(7) satisfying*

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left(\theta^0\right)^2dx+\int_0^t\int_0^1\left[\left(\theta^0\right)^2+\left(\theta_x^0\right)^2\right]dx\,d\tau\\
&\quad\le C\left\|\phi_2\right\|^2,
\end{aligned}\tag{47}$$

$$\begin{aligned}
&\int_0^1\left(\theta_x^0\right)^2dx+\int_0^t\int_0^1\left(\theta_x^0\right)^2dx\,d\tau\\
&\quad+\beta\int_0^t\int_0^1\left(\theta_{xx}^0\right)^2dx\,d\tau\le C\left\|\phi_2\right\|_1^2,
\end{aligned}\tag{48}$$

$$\begin{aligned}
&\int_0^1\left[\left(\theta_t^0\right)^2+\left(\theta_{xx}^0\right)^2\right]dx\\
&\quad+\int_0^t\int_0^1\left[\left(\theta_t^0\right)^2+\left(\theta_{xt}^0\right)^2\right]dx\,d\tau\\
&\quad\le C\left(\left\|\phi_1\right\|^2+\left\|\phi_2\right\|_2^2\right).
\end{aligned}\tag{49}$$

Proof. Multiplying the second equation of (5) by θ^0, integrating the resulting equation over $(0,t)\times(0,1)$, using integration by part and the boundary conditions (7), and then adding them, we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\theta^0\right)^2 dx + (1-\beta)\int_0^t\int_0^1 \left(\theta^0\right)^2 dx\,d\tau \\
&\quad + \beta\int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau = \frac{1}{2}\int_0^1 \phi_2^2 dx \\
&\quad + 2\int_0^t\int_0^1 \psi^0\theta_x^0\theta^2 dx\,d\tau = C\left\|\phi_2\right\|^2 \\
&\quad + 2\left\|\psi^0\right\|_{L^\infty}\int_0^t\int_0^1 \left|\theta^0\theta_x^0\right| dx\,d\tau = C\left\|\phi_2\right\|^2 \\
&\quad + 2\varepsilon_2\int_0^t\int_0^1 \left(\theta^0\right)^2 dx\,d\tau + C\varepsilon\int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau.
\end{aligned} \tag{50}$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\theta^0\right)^2 dx + \int_0^t\int_0^1 \left[\left(\theta^0\right)^2 + \left(\theta_x^0\right)^2\right] dx\,d\tau \\
&\quad \le C\left\|\phi_2\right\|^2.
\end{aligned} \tag{51}$$

Differentiating the second equation of (5) with respect to x, we get

$$\theta_{xt}^0 = -(1-\beta)\theta_x^0 + 2\left(\psi^0\theta_x^0\right)_x + \beta\theta_{xxx}^0. \tag{52}$$

Multiplying (52) by θ_x^0, using integration by part and the boundary conditions, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\theta_x^0\right)^2 dx + (1-\beta)\int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau \\
&\quad + \beta\int_0^t\int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\,d\tau = \frac{1}{2}\int_0^1 \left(\theta_x^0\right)^2\Big|_{t=0} dx \\
&\quad + 2\int_0^t\int_0^1 \psi_x^0\left(\theta_x^0\right)^2 dx\,d\tau \\
&\quad + 2\int_0^t\int_0^1 \left(\psi^0\theta_{xx}^0\right)\theta_x^0 dx\,d\tau \le \frac{1}{2}\int_0^1 \left(\phi_{2x}(x)\right) dx \\
&\quad + 2\left\|\psi_x^0\right\|_{L^\infty}\int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau + 2\left\|\psi^0\right\|_{L^\infty} \\
&\quad \cdot\left(\int_0^t\int_0^1 \left(\theta_{xx}^0\right)^2 dx\,d\tau + \int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau\right) \\
&\quad \le C\left\|\phi_2\right\|_1^2.
\end{aligned} \tag{53}$$

Therefore, we have

$$\begin{aligned}
&\int_0^1 \left(\theta_x^0\right)^2 dx + \int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau \\
&\qquad + \beta\int_0^t\int_0^1 \left(\theta_{xx}^0\right)^2 dx\,d\tau \le C\left\|\phi_2\right\|_1^2.
\end{aligned} \tag{54}$$

Moreover, if we differentiate the second equation of (5) with respect to t, we have

$$\theta_{tt}^0 = -(1-\beta)\theta_t^0 + 2\left(\psi^0\theta_x^0\right)_t + \beta\left(\theta_{xx}^0\right)_t, \tag{55}$$

and then multiplying (55) by θ_t^0 and integrating the resulting equation over $(0,t)\times(0,1)$, using integration by part and the boundary conditions (7), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\theta_t^0\right)^2 dx + (1-\beta)\int_0^t\int_0^1 \left(\theta_t^0\right)^2 dx\,d\tau \\
&\quad + \beta\int_0^t\int_0^1 \left(\theta_{xt}^0\right)^2 dx\,d\tau = \frac{1}{2}\int_0^1 \theta_{0t} dx \\
&\quad + 2\int_0^t\int_0^1 \left(\psi^0\theta_x^0\right)_t\theta_t^0 \le \frac{1}{2}\left\|\phi_{2t}\right\|^2 \\
&\quad + 2\int_0^t\int_0^1 \left(\psi_t^0\theta_x^0 + \psi^0\theta_{xt}^0\right) dx\,d\tau = \frac{1}{2}C \\
&\quad + 2\left\|\psi_t^0\right\|_{L^\infty}\int_0^t\int_0^1 \left(\theta_x^0\right)^2 dx\,d\tau \\
&\quad + 2\left\|\psi^0\right\|_{L^\infty}\int_0^t\int_0^1 \left(\theta_{xt}^0\right)^2 dx\,d\tau;
\end{aligned} \tag{56}$$

then we have

$$\int_0^1 \left(\theta_t^0\right)^2 dx \le C. \tag{57}$$

Utilizing the above estimates and the similar produces, by the assumption of ε, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left(\theta_t^0\right)^2 dx + (1-\beta)\int_0^t\int_0^1 \left(\theta_t^0\right)^2 dx\,d\tau \\
&\qquad + \beta\int_0^t\int_0^1 \left(\theta_{xt}^0\right)^2 dx\,d\tau \le C\left(\left\|\phi_1\right\|_2^2 + \left\|\phi_2\right\|_2^2\right).
\end{aligned} \tag{58}$$

Therefore,

$$\begin{aligned}
&\int_0^1 \left(\theta_{xx}^0\right)^2 dx \le C\int_0^1 \left[\left(\theta_t^0\right)^2 + \left(\theta^0\right)^2 + \left(\psi^0\theta_x^0\right)^2\right] dx \\
&\quad \le C\int_0^1 \left[\left(\theta_t^0\right)^2 + \left(\theta^0\right)^2 + \left(\psi^0\right)^2 + \left(\theta_x^0\right)^2\right] dx \\
&\quad \le C\left(\left\|\phi_1\right\|_2^2 + \left\|\phi_2\right\|_2^2\right).
\end{aligned} \tag{59}$$

Here we have used (47), (48), (58), and (59). The proof of Lemma 10 is finished. □

Finally, based on Lemma 10, we can prove Theorem 3.

3. Convergence Rates and BL-Thickness

In this section, we turn another interesting problem, which is concerned with L^2 convergence rates of the vanishing diffusion viscosity parameter $\alpha \to 0^+$ and the BL-thickness. That is, we will give the proof of Theorem 4, and it suffices to show the following two lemmas.

Lemma 11. *Assume that the conditions in Theorem 4 hold. Then we have*

$$\begin{aligned}&\int_0^1 \left[\left(\psi^\alpha-\psi^0\right)^2+\left(\theta^\alpha-\theta^0\right)^2\right]dx\\&\quad+\int_0^t\int_0^1\left[\alpha\left(\psi^\alpha-\psi^0\right)^2+\left(\theta^\alpha-\theta^0\right)^2\right.\\&\quad\left.+\left(\theta^\alpha-\theta^0\right)_x^2\right]dx\,d\tau\le C\alpha,\\&\int_0^1\left(\theta^\alpha-\theta^0\right)_x^2dx+\int_0^t\int_0^1\left(\theta^\alpha-\theta^0\right)_t^2dx\,d\tau\le C\alpha,\end{aligned}\tag{60}$$

where C is a positive constant, independent of α.

Proof. Set $u^\alpha=\psi^\alpha-\psi^0$, $v^\alpha=\theta^\alpha-\theta^0$; then we have

$$\begin{aligned}u_t^\alpha&=-(k-1)\alpha u^\alpha-(k-1)\alpha\psi^0-k\alpha\theta_x^\alpha+\alpha\psi_{xx}^\alpha,\\v_t^\alpha&=-(1-\beta)v^\alpha+\mu\alpha\psi_x^\alpha+2\psi^\alpha v_x^\alpha+2u^\alpha\theta_x^0+\beta v_{xx}^\alpha\end{aligned}\tag{61}$$

with initial data

$$\left(u^\alpha,v^\alpha\right)(x,0)=\left(\psi_0(x)-\phi_1(x),\theta_0(x)-\phi_2(x)\right),\quad 0\le x\le 1,\tag{62}$$

and boundary condition

$$v^\alpha(0,t)=v^\alpha(1,t)=0,\tag{63}$$

which implies that

$$v_t^\alpha(0,t)=v_t^\alpha(1,t)=0,\quad t\ge 0.\tag{64}$$

Multiplying the first equation of (61) by u_α and the second equation of (61) by v_α, respectively, and then adding them, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left(u^\alpha\right)^2+\left(v^\alpha\right)^2\right]dx+\int_0^t\int_0^1\left[(k-1)\alpha\left(u^\alpha\right)^2\right.\\&\quad\left.+(1-\beta)\left(v^\alpha\right)^2+\beta\left(v_x^\alpha\right)^2\right]dx\,d\tau=\frac{1}{2}\\&\quad\cdot\int_0^1\left[\left(u^\alpha\right)^2+\left(v^\alpha\right)^2\right]_{t=0}dx-(k-1)\\&\quad\cdot\alpha\int_0^t\int_0^1\left(u^\alpha\right)\psi^0dx\,d\tau-k\alpha\int_0^t\int_0^1\theta_x^\alpha u^\alpha dx\,d\tau\\&\quad+\mu\alpha\int_0^t\int_0^1\psi_x^\alpha v^\alpha dx\,d\tau+2\int_0^t\int_0^1\psi^\alpha v_x^\alpha v^\alpha dx\,d\tau\\&\quad+2\int_0^t\int_0^1u^\alpha\theta_x^0v^\alpha dx\,d\tau+\alpha\int_0^t\int_0^1\psi_{xx}^\alpha u^\alpha dx\,d\tau=\frac{1}{2}\\&\quad\cdot\int_0^1\left[\left|\psi_0(x)-\phi_1(x)\right|^2+\left|\theta_0(x)-\phi_2(x)\right|^2\right]dx\\&\quad-(k-1)\alpha\int_0^t\int_0^1\psi^0u^\alpha dx\,d\tau\\&\quad-k\alpha\int_0^t\int_0^1\theta_x^\alpha u^\alpha dx\,d\tau+\mu\alpha\int_0^t\int_0^1\psi_x^\alpha v^\alpha dx\,d\tau\\&\quad-\int_0^t\int_0^1\psi_x^\alpha\left(v^\alpha\right)^2dx\,d\tau=\frac{1}{2}\int_0^1\left[\left|\psi_0(x)-\phi_1(x)\right|^2\right.\\&\quad\left.+\left|\theta_0(x)-\phi_2(x)\right|^2\right]dx+\sum_{i=1}^6M_i.\end{aligned}\tag{65}$$

Then we estimate every term M_1–M_6 as follows:

$$\begin{aligned}M_1&\le\frac{(k-1)\alpha}{2}\int_0^t\int_0^1\left(\psi^0\right)^2dx\,d\tau\\&\quad+\frac{(k-1)\alpha}{2}\int_0^t\int_0^1\left(u^\alpha\right)^2dx\,d\tau,\\M_2&\le k\int_0^t\int_0^1\left(\theta_x^\alpha\right)^2dx\,d\tau\\&\quad+\frac{(k-1)\alpha}{2k}\int_0^t\int_0^1\left(u^\alpha\right)^2dx\,d\tau,\\M_3&\le\frac{\mu^2\alpha^2}{4\lambda}\int_0^t\int_0^1\left(\psi_x^\alpha\right)^2dx\,d\tau\\&\quad+C\lambda\int_0^t\int_0^1\left(v^\alpha\right)^2dx\,d\tau,\\M_4+M_5&=2\int_0^t\int_0^1\psi v_x^\alpha v^\alpha dx\,d\tau\\&\quad+2\int_0^t\int_0^1u^\alpha\theta_x^0v^\alpha dx\,d\tau\\&\le C\varepsilon\int_0^t\int_0^1\left(v_x^\alpha\right)^2dx\,d\tau\\&\quad+C\varepsilon\int_0^t\int_0^1\left(v^\alpha\right)^2dx\,d\tau\\&\quad+C\delta\int_0^t\int_0^1\left(u^\alpha\right)^2dx\,d\tau\\&\quad+C\delta\int_0^t\int_0^1\left(v^\alpha\right)^2dx\,d\tau,\\M_6&\le\frac{\alpha^2}{4\lambda}\int\left(\psi_{xx}^\alpha\right)^2dx\,d\tau\\&\quad+\lambda\int_0^t\int_0^1\left(u^\alpha\right)^2dx\,d\tau.\end{aligned}\tag{66}$$

Putting (66) into (65), we obtain

$$\begin{aligned}&\int_0^1\left[\left(u^\alpha\right)^2+\left(v^\alpha\right)^2\right]dx\\&\quad+\int_0^t\int_0^1\left[\alpha\left(u^\alpha\right)^2+\left(v^\alpha\right)^2+\left(v_x^\alpha\right)^2\right]dx\,d\tau\\&\le C\alpha+(\lambda+C\delta)\int_0^t\int_0^1\left(u^\alpha\right)^2dx\,d\tau.\end{aligned}\tag{67}$$

By Gronwall's inequality, we have

$$\int_0^1 \left[(u^\alpha)^2 + (u^\alpha)^2\right] dx + \int_0^t \int_0^1 \left[\alpha (u^\alpha)^2 + (v^\alpha)^2 + (v_x^\alpha)^2\right] dx\, d\tau \le C\alpha. \tag{68}$$

Multiplying the second equation of (61) by v_t^α, we have

$$\begin{aligned} &\frac{1}{2}(1-\beta)\int_0^1 (v^\alpha)^2 dx + \frac{1}{2}\beta\int_0^1 (v_x^\alpha)^2 dx \\ &+ \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau \\ &= \frac{1}{2}\int_0^1 \left[(1-\beta)(v^\alpha)^2 + \beta (v_x^\alpha)^2\right]_{t=0} dx \\ &+ \int_0^t \int_0^1 \mu\alpha \psi_x^\alpha v_t^\alpha dx\, d\tau + 2\int_0^t \int_0^1 \psi^\alpha v_x^\alpha v_t^\alpha dx\, d\tau \\ &+ 2\int_0^t \int_0^1 u^\alpha \theta_x^0 v_t^\alpha dx\, d\tau = \sum_{i=7}^{10} M_i. \end{aligned} \tag{69}$$

We estimate every term M_7–M_{10} as follows:

$$\begin{aligned} M_7 = &\frac{1}{2}\int_0^1 \left[(1-\beta)(\theta_0(x) - \phi_2(x))^2\right. \\ &\left. + \beta (\theta_0(x) - \phi_2(x))_x^2\right] dx \le C\|\theta_0 \\ &- \phi_2(x)\|_1^2 \le C\alpha, \end{aligned} \tag{70}$$

$$\begin{aligned} M_8 \le &\frac{\mu^2\alpha^2}{2}\int_0^t \int_0^1 \left[(\phi_x^\alpha)^2 + \frac{1}{2}(v_t^\alpha)^2\right] dx\, d\tau \le C\alpha + \frac{1}{2} \\ &\cdot \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau, \end{aligned} \tag{71}$$

$$M_9 \le C\varepsilon \int_0^t \int_0^1 (v_x^\alpha)^2 dx\, d\tau + C\varepsilon \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau, \tag{72}$$

$$\begin{aligned} M_{10} &\le C\delta \int_0^t \int_0^1 (u^\alpha)^2 dx\, d\tau + C\delta (v_t^\alpha)^2 dx\, d\tau \\ &\le C\delta\alpha + C\delta \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau. \end{aligned} \tag{73}$$

Collecting (70)–(73) and putting into (69), we have

$$\frac{1}{2}\int_0^1 \left[(v^\alpha)^2 + (v_x^\alpha)^2\right] dx + \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau \le C\alpha. \tag{74}$$

Utilizing the similar step, multiplying the second equation of (61) by v_t^α, then integrating the resulting equation over $(0,t)\times(0,1)$, and then estimating every term, finally we obtain

$$\int_0^1 \left[(v^\alpha)^2 + (v_x^\alpha)^2\right] dx + \int_0^t \int_0^1 (v_t^\alpha)^2 dx\, d\tau \le C\alpha. \tag{75}$$

The following lemma will be devoted to the boundary layer thickness. □

Lemma 12. *Assume that the conditions in Theorem 2 hold. Then we have*

$$\delta \int_\delta^{1-\delta} \left|\psi_x^\alpha - \psi_x^0\right| dx \le C\alpha. \tag{76}$$

Proof. Differentiating (62) with respect to x, we get

$$u_{xt}^\alpha = -(k-1)\alpha u_x^\alpha - (k-1)\alpha\psi_x^0 - k\alpha\theta_{xx}^\alpha + \alpha\psi_{xxx}^\alpha. \tag{77}$$

Let $u_x^\alpha = \omega$; then we have

$$\omega_t = -(k-1)\alpha\omega - (k-1)\alpha\psi_x^0 - k\alpha\theta_{xx}^\alpha + \alpha\psi_{xxx}^\alpha. \tag{78}$$

To construct a cut-off function as in [12], we define φ_η, for $\eta \in (0,1)$, and ξ_δ for $\delta \in (0,1/2)$, by

$$\begin{aligned} \varphi_\eta(s) &= \sqrt{s^2 + \eta^2}, \\ \xi_\delta &= \begin{cases} x, & 0 \le x \le \delta, \\ \delta, & \delta \le x \le 1-\delta, \\ 1-x, & 1-\delta \le x \le 1. \end{cases} \end{aligned} \tag{79}$$

It is easy to check that φ_η satisfies

$$\begin{aligned} |s| \le \left|\varphi_\eta(s)\right| &\le |s| + 1, \\ \left|\varphi_\eta'(s)\right| &\le 1, \\ 0 \le s\varphi_\eta'(s) &\le \varphi_\eta(s), \\ \varphi_\eta''(s) &\ge 0, \\ s^2\varphi_\eta''(s) &\ge 0. \end{aligned} \tag{80}$$

ξ_δ satisfies $0 \le \xi_\delta \le \delta$, $\xi_\delta(1) = \xi_\delta(0) = 0$, multiplying (78) by $\varphi_\eta'(\omega)\xi_\delta$ and then integrating over

$$\begin{aligned} &\int_0^t \int_0^1 \varphi_\eta'(\omega)\xi_\delta dx - \varepsilon\int_0^1 \xi_\delta dx \\ &= -(k-1)\alpha\int_0^t \int_0^1 \omega\varphi_\eta'(\omega)\xi_\delta dx\, d\tau \\ &- (k-1)\alpha\int_0^t \int_0^1 \psi_x^0\varphi_\eta'(\omega)\xi_\delta dx\, d\tau \\ &- k\alpha\int_0^t \int_0^1 \theta_{xx}^\alpha\varphi_\eta'(\omega)\xi_\delta dx\, d\tau \\ &+ \alpha\int_0^t \int_0^1 \psi_{xxx}^\alpha\varphi_\varepsilon'(\omega)\xi_\delta dx\, d\tau = \sum_{i=1}^4 M_i. \end{aligned} \tag{81}$$

By the fact that $0 \le s\varphi_\eta'(s) \le \varphi_\eta(s)$, we have

$$M_1 \le \int_0^t \int_0^1 \varphi_\varepsilon(\omega)\, \xi_\delta dx\, d\tau. \tag{82}$$

By $\varphi_\eta'(\omega) \le 1$, $0 \le \xi_\delta \le \delta$, we obtain

$$\begin{aligned} M_2 &\le C\delta \left[\int_0^t \int_0^1 \left(\varphi_x^0\right)^2 dx\, d\tau \right]^{1/2} \le C\delta\alpha, \\ M_3 &\le C\alpha\delta \left[\int_0^t \int_0^1 \left(\theta_{xx}^\alpha\right)^2 dx\, d\tau \right]^{1/2} \le C\delta\alpha, \\ M_4 &\le C\alpha\delta \int_0^t \int_0^1 \left(\theta_{xxx}^\alpha\right)^2 dx\, d\tau \le C\alpha. \end{aligned} \tag{83}$$

Combining (82) and (83), we notice that

$$\begin{aligned} \varepsilon \int_0^1 \xi_\delta dx &\le \varepsilon\delta, \\ \int_0^t \int_0^1 \varphi_\eta'(\omega)\, \xi_\delta dx\, d\tau &\le C \int_0^t \int_0^1 \varphi_\eta(\omega)\, \xi_\delta dx\, d\tau \\ &\quad + C\alpha\delta + \epsilon\delta. \end{aligned} \tag{84}$$

By Gronwall's inequality, we have

$$\int_0^1 \varphi_\eta(\omega)\, \xi_\delta dx \le C(\varepsilon\delta + \alpha\delta). \tag{85}$$

By the definition of ξ_δ, $|z| \le \varphi_\eta(\omega)$, we have

$$\int_\delta^{1-\delta} |\omega|\, dx \le C(\varepsilon\delta + \alpha\delta) < C(\varepsilon + \alpha). \tag{86}$$

Letting $\varepsilon \to 0$,

$$\int_\delta^{1-\delta} |\omega|\, dx \le C\alpha. \tag{87}$$

Because $W^{1,1}([\delta, 1-\delta]) \hookrightarrow L^\infty([\delta, 1-\delta])$, it follows that

$$\begin{aligned} \left\|\psi^\alpha - \psi^0\right\|_{L^\infty([\delta,1-\delta])} &\le \int_0^1 \left|\psi^\alpha - \psi^0\right| dx \\ &\quad + \int_\delta^{1-\delta} \left|\psi_x^\alpha - \psi_x^0\right| dx \le C\alpha. \end{aligned} \tag{88}$$

As in [5, 6, 13], we observe that inequality (11) holds. This completes the proof of Theorem 4. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The paper was finished in collaboration among all authors. Linrui Li found the motivation of this paper and suggested the outline of the proofs. Shu Wang provided good ideas and gave some valuable suggestions for completing the paper. All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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