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Research Article

Evolution of Electoral Preferences for a Regime of Three Political Parties

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In this article, we use a discrete system to study the opinion dynamics regarding the electoral preferences of a nontendentious group of agents. To measure the level of preference, a continuous opinion space is used, in which the preference (opinion) can evolve from any political option, to any other; for a regime of three parties, a circle is the convenient space. To model a nonbiased society, new agents are considered. Besides their opinion, they have a new attribute: an individual iterative monoparametric map that imitates a process of internal reflection, allowing them to update their opinion in their own way. These iterative maps introduce six fixed points on the opinion space; the points' stability depends on the sign of the parameter. When the latter is positive, three attractors are identified with political options, while the repulsors are identified with the antioptions (preferences diametrically opposed to each political choice). In this new model, pairs of agents interact only if their respective opinions are alike; a positive number called confidence bound is introduced with this purpose; if opinions are similar, they update their opinion considering each other's opinion, while if they are not alike, each agent updates her opinion considering only her individual map. In addition, agents give a certain level of trust (weight) to other agent's opinions; this results in a positive stochastic matrix of weights which models the social network. The model can be reduced to a pair of coupled nonlinear difference equations, making extracting analytical results possible: a theorem on the conditions governing the existence of consensus in this new artificial society. Some numerical simulations are provided, exemplifying the analytical results.

1. Introduction

The arising of opinion in society has become a very interesting research topic in the last decades. For example, Ding and Mo [1] have studied how noise, from different sources, like TV, newspaper, and so on, contributes to eliminate the disagreement in a social group; Chen *et al.* [2] have considered how the social similarity influences the opinion dynamics between individuals; D'Aniello *et al.* [3] have used a fuzzy consensus approach to investigate decision-making in a group; Lu *et al.* [4] have investigated the impact of the community structure on the convergence time to reach a consensus in a social group. A recent survey on some opinion dynamics models and their applications is found in [5].

In particular, the opinion dynamics about political options has been studied broadly, using models borrowed from statistical physics, such as the Ising and Potts ones [6, 7], in which agents are forced to choose between a finite set of options, no matter how much they agree with them. The set of options, naturally, is discrete as the corresponding opinion space for these models too, allowing tendentious agents. An alternative would be the use of a continuous opinion space, in which agents can show their level of agreement regarding the options; this space must also allow opinions to evolve from one option to another, options being equidistant from each other; these requirements suggest that an adequate opinion space, for n-options, is an $n-2$-sphere. For example, a circle is an appropriate opinion space for three options, while a sphere is suitable for four options. These ideas were used by Medina *et al.* [8] to study the consensus formation on a circle for a regime of three political parties; they used an iterative

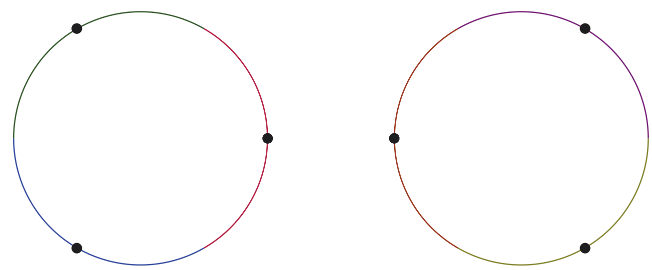

Figure 1: The figure shows attractors described by Theorem 2 and their respective attraction basins for system (7). Attractors with $0 < \kappa < 2/3$ are the associated electoral preferences. Attractors with $-2/3 < \kappa < 0$, also called antioptions, are completely opposite to existing options. Agents attracted to one antioption radically refuse the one option remaining half trusting to the other options.

monoparametric map which introduced six fixed points on the opinion space. The stability of these points is governed by the sign of the map's parameter, the personal parameter; if it is positive, the three attractors are identified with the political options, while the remaining three repellers are called the (political) antioptions. An agent having a negative personal parameter rejects all the available political options, being attracted to the antioptions. Since there is an alternating sequence of attractors and repellers on the circle, options and antioptions are diametrically opposed; see Figure 1. Hence an agent whose opinion is attracted to one antioption rejects the diametrically opposed political option, while she (or he) is insecure about the remaining two options. So in contrast, with the discrete option models, these agents are not forced to pick an option. In this sense, the iterative map allows agents to update their opinion taking into account their postures with respect to the choices, emulating an internal reflection process in them. The above agents interact by pairs updating their opinion considering their internal reflection processes and their partners' opinions with some weight (a subjective trust given to each other's opinion); if their opinions are not alike, they update their opinions considering only their own internal reflection processes. A positive number, the confidence bound, is used as a threshold to separate between these two cases.

Although agents, in Medina *et al.* [8] model, have different personal parameters, the same iterative map was used for the opinion updating. However, in a more realistic society, each person forms her opinion differently from others; hence it has sense to consider a different updating rule for each agent. So, in this work, we propose to give each agent a new attribute, a different iterative map that allows them to update their opinion in their own manner. Then we investigate the conditions under which a consensus arises, in this new artificial society, under these new updating rules. Finally, some numerical experiments are performed to validate the analytical results.

This work is sectioned as follows: Section 2 presents a new opinion dynamics model on the circle. Section 3, Main Results, gives a theorem stating the conditions under which the consensus emerges in the social network. Section 4 shows that some simulations with two different opinion updating rules are given. A section of concluding remarks and two appendixes end this communication.

2. Opinion Dynamics Modeled on a Circle

Through the work, the opinion space X is a circle, and it could be identified with the segment $[0, 2\pi]$ modulus 2π in radians or, equivalently, as the segment $[0, 360]$ modulo 360 in degrees. On it, the opinion updating rules establish three attractors identified with the political options and three antioptions. Options and antioptions are distributed in an alternating sequence; see Figure 1. The antioption opposed to an option corresponds to a total rejection of the latter. The personal parameter space is denoted by K; it is identified with the segment $[-\pi, \pi]$ in radians or $[-180, 180]$ in degrees. A set with an even number N of interacting agents is considered. Such interactions take place on discrete times t_n, for each $n \in \mathbb{Z}^+ \cup 0$, so $(t_n)_{n=0}^{\infty}$ is a nonbounded sequence of positive real numbers such that $t_0 = 0$. Each agent is characterized by three attributes: an opinion $x_n^i \in X$; a constant personal parameter $\kappa^i \in K$ that allows acceptance or rejection of the presented options (the attracting nature of options and antioptions depends on the sign of κ^i; see below); and an opinion updating function $\Xi_i : X \times K \longrightarrow \mathbb{R}$. Like in [8, 9], the social network describing the agent population is governed by the stochastic absolute weights matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix} \quad (1)$$

where its elements are positive real numbers fulfilling $0 < c_{ii} \leq 1$ and $0 \leq c_{ij} < 1$, $\forall i, j \in \{1, 2, \ldots, N\}$, in such a way that the sum of each row is equal to one:

$$\sum_{j=1}^{N} c_{ij} = 1. \quad (2)$$

From a social point of view, a possible interpretation to c_{ij} could be the credibility given by agent i to the other agents' opinions in the social network, including his own opinion. The first inequality above $0 < c_{ii} \leq 1$ establishes that each agent must give a nonzero credibility to its own opinion. Since the matrix rows must add to one, the second inequality

$0 \leq c_{ij} < 1$ establishes that the credibility given to other agent's opinion must be less than one.

Since the interactions between the agents are pairwise, it is necessary to reevaluate the mutual trust between the agents. Hence we use relative weights,

$$a_{ii} = \frac{c_{ii}}{c_{ii} + c_{ij}}, \qquad a_{ij} = \frac{c_{ij}}{c_{ii} + c_{ij}}. \tag{3}$$

The relative weights fulfill these properties: $a_{ii} + a_{ij} = 1$, $0 < a_{ii} \leq 1$ and $0 \leq a_{ij} < 1$.

On the same manner as well-known models of Deffuant *et al.* [10, 11] and Hegsellman-Krause [12], we consider a confidence bound $\varepsilon > 0$ that determines a criterion to set a limit to agents interaction; i.e., the opinions belonging to agents i and j are affine if and only if $|x_n^i - x_n^j| < \varepsilon$.

Now we formulate the opinion updating rules on the circle. It is important to distinguish six special points on the opinion space: three options and three antioptions. To do that, we consider iterative monoparametric maps that introduce these six fixed points. As a construction element of the dynamic rules, we use the following functions:

$$\Xi_i(x, \kappa) = x - \kappa^i f_i(x), \tag{4}$$

where

$$f_i(x) = \sin(3x)\, g_i(\sin(3x)), \tag{5}$$

where $g_i : [-1, 1] \longrightarrow (0, 1]$ is a C^1 class differential function such that $g_i(0) = 1$, $\forall i \in \{1, 2, \ldots, N\}$; the sine function, $\sin(3x)$, introduces the fixed points, $p = s\pi/3$ for $s \in \mathbb{Z}$; in the iterative map $x_{n+1} = \Xi(x_n, \kappa^i)$:

$$p = \Xi(p, \kappa). \tag{6}$$

Instead of the sine function in (5), it would be possible to use the cosine function with the same purpose. However, this would only move the fixed points on the circle.

Medina *et al.* [8] consider the same updating opinion rule for all agents with $g_i(x) = 1$. Now, we propose to use a different function $g_i(x)$ for each agent, with the purpose to give them a different way to update their opinions; thus the functions $\Xi_i(x^i, \kappa^i)$ model internal reflection processes for opinion updating in agents; they also provide the knowledge about the options and permit them to keep an attitude regarding these options; see Appendix A. In this sense, the new model introduces a more heterogeneous and less tendentious agent population.

On the recursive model, as initial conditions, a personal parameter and a random opinion are assigned to each agent; aside the stochastic matrix C is generated too. On the n temporal step, from the agents population, $N/2$-couples are randomly chosen to interact according to these rules:

(i) When a pair of agents have similar opinions, $|x_i - x_j| < \epsilon$, they update their opinions considering their respective internal reflection processes and considering each other's opinion, with some relative weights:

$$x_{n+1}^i = a_{ii}\Xi_i\left(x_n^i, \kappa^i\right) + a_{ij}x_n^j, \qquad x_{n+1}^j = a_{ji}x_n^i + a_{jj}\Xi_j\left(x_n^j, \kappa^j\right). \tag{7}$$

(ii) But if they do not have similar opinions, $|x_i - x_j| > \epsilon$, every agent in the pair uses only her iterative map to update her opinion, without considering other's opinion, consequently, using $a_{ii} = a_{jj} = 1$ and $a_{ij} = a_{ji} = 0$ in (7).

3. Main Results

Now, we present the conditions for consensus formation in the artificial society described by the before enunciated model. It is said that model (7) exhibits a strong consensus if the opinions of agents evolve to a stable fixed point.

To simplify our notations and reduce model (7) to a system of only two nonlinear difference equations, we perform the following change of variables:

$$\begin{aligned} x^i &\longrightarrow x, \\ \kappa^i &\longrightarrow \kappa_x, \\ x^j &\longrightarrow y, \\ \kappa^j &\longrightarrow \kappa_y, \\ a_{ii} &\longrightarrow \alpha, \\ a_{ij} &\longrightarrow \beta, \\ a_{ji} &\longrightarrow \gamma, \\ a_{jj} &\longrightarrow \delta, \\ g_i\left(x_n^i\right) &\longrightarrow g_{(x)}\left(x_n\right) \\ g_j\left(x_n^j\right) &\longrightarrow g_{(y)}\left(y_n\right), \\ f_i\left(x_n^i\right) &\longrightarrow f_{(x)}\left(x_n\right) \\ f_j\left(x_n^j\right) &\longrightarrow f_{(y)}\left(y_n\right), \\ \Xi_i\left(x_n^i, \kappa^i\right) &\longrightarrow \Xi_{(x)}\left(x_n, \kappa_x\right) \\ \Xi_j\left(x_n^j, \kappa^j\right) &\longrightarrow \Xi_{(y)}\left(y, \kappa_y\right). \end{aligned} \tag{8}$$

The model becomes into

$$x_{n+1} = \alpha\Xi_{(x)}\left(x_n, \kappa_x\right) + \beta y_n, \qquad y_{n+1} = \gamma x_n + \delta\Xi_{(y)}\left(y_n, \kappa_y\right), \tag{9}$$

where $\alpha + \beta = 1$ and $\gamma + \delta = 1$.

Analyzing the fixed points of the previous equations (9), we establish conditions for the existence and stability of the fixed points of the complete system (7), analogously as the model in [8].

Theorem 1. *The point (p, p) is a common fixed point of (7), if and only if $p = s\pi/3$ for some $s \in \mathbb{Z}$.*

Proof. Assuming that (p, p) is a fixed point (9), by direct substitution results,

$$\begin{aligned} p &= \alpha\left[p - \kappa_x \sin(3p)\, g_i(\sin(3p))\right] + \beta p, \\ p &= \gamma p + \delta\left[p - \kappa_y \sin(3p)\, g_j(\sin(3p))\right]. \end{aligned} \tag{10}$$

By definition $g_i(x) \neq 0$, then $\sin(3p) = 0$ or $p = s\pi/3$ for some $s \in \mathbb{Z}$. Conversely, a direct substitution shows that the point (p, p) is a solution for (9), when p is an entire multiple of $\pi/3$, like it is asked. □

Note that these fixed points do not depend neither on the relative weights nor on personal parameters, so they are common to all pairs of (9) and they are fixed points of the full set of (7). Of course, (9) have other fixed points. However, when one considers the full system (7), fixed points from one pair of equations are not necessarily the same as another pair, so they may not be fixed points of the complete system. Some examples are presented in Appendix B.

For convenience, the index z denotes indexes x or y. Thus, for example, κ_z represents either κ_x or κ_y.

Theorem 2. *Let $p = s\pi/3$ for some $s \in \mathbb{Z}$ and $\eta_z = 1 - \kappa_z f'_{(z)}(p) = 1 - 3(-1)^s\kappa_z$ such that $|\eta_z| < 1$ and $\kappa_x, \kappa_y \in K$. Then (p, p) is a stable fixed point for (9) if*

(1) *$0 < \kappa_x, \kappa_y < 2/3$ for s even,*

(2) *$-2/3 < \kappa_x, \kappa_y < 0$ for s odd.*

Proof. For the linearized system (9), the Jacobian matrix is

$$M = \begin{pmatrix} \alpha\eta_x & \beta \\ \gamma & \delta\eta_y \end{pmatrix} \tag{11}$$

Under these considerations, the spectral radius $\rho(M)$ of M satisfies

$$\begin{aligned} \rho(M) \le \|M\|_\infty &= \max\left\{\alpha\left|\eta_x\right| + \beta, \gamma + \delta\left|\eta_y\right|\right\} \\ &< \max\{\alpha + \beta, \gamma + \delta\} = 1. \end{aligned} \tag{12}$$

Since all eigenvalues of M are complex numbers with modulus less than 1, then (p, p) is a stable fixed point; see [13, 14].

Theorem 2 gives the necessary and sufficient conditions for the existence of a consensus between a pair of agents; now Theorem 3 establishes the necessary and sufficient conditions for the existence of a consensus among all agents in the social web.

Theorem 3. *Let $\kappa^i \in K$ for each $i \in \{1, 2, \ldots, N\}$ and $p = s\pi/3$ for some $s \in \mathbb{Z}$. A strong consensus appears around opinion p in the next cases:*

(1) *if s is even and $0 < \kappa^i < 2/3$ for each $i \in \{1, 2, \ldots, N\}$;*

(2) *if s is odd and $-2/3 < \kappa^i < 0$ for each $i \in \{1, 2, \ldots, N\}$.*

□

This theorem generalizes the one reported in [8] and it could be seen that the consensus in the social network does not depend on g_i functions, nor absolute weights matrix C, nor number of agents N.

A more realistic case is when agents have personal parameters in a subset containing both positive and negative values; then system (9), in general, will have distinct attractors for different couples of agents; see Appendix B. Intuitively, it is expected that these attractors be bounded in some subset of $X \times X$; consequently the agents' opinions will evolve towards a subset $U \subsetneq X$; hence, it is said that a weak consensus is present in the social network [8, 9].

4. Some Examples

For the sake of clearance, some simulations are provided. Here we use a social web with 200 agents; it is described by a random matrix of absolute weights. Agents' starting opinions are uniform and randomly distributed on X. Aside, a uniform time interval partition is considered. The parameters and variables will be expressed in degrees. Considering the same interacting dynamics between agents and two cases for updating rule opinion (5),

$$\begin{aligned} g_1(x) &= \cos^4(x), \\ g_2(x) &= \exp\left(-x^2\right). \end{aligned} \tag{13}$$

The simulations use 100 time steps.

Example 4. In Figure 2, the personal parameters κ^i are randomly and uniformly distributed on $[0.5^\circ, 38^\circ]$. Three different confidence bounds are considered: top rank $\varepsilon = 180^\circ$; $\varepsilon = 120^\circ$ in middle rank; $\varepsilon = 60^\circ$ for bottom rank: $g_1(x)$ to the left and $g_2(x)$ to the right. When $\varepsilon = 180^\circ$ all agents interact and one cluster appears. If $\varepsilon = 120^\circ$, agents interact with about two-thirds of total population at the most; initially, two clusters appear, but the bigger one absorbs the smaller one. For $\varepsilon = 60^\circ$, agents interact with one-third of population, and three clusters arise, each one with about one-third of total population. According to Theorem 3, strong consensus arise for all these cases.

Example 5. In Figure 3, the personal parameters κ^i are randomly and uniformly distributed on $[-10^\circ, 10^\circ]$. Three different confidence bounds are considered too; top rank $\varepsilon = 180^\circ$; $\varepsilon = 120^\circ$ in middle rank; $\varepsilon = 60^\circ$ for bottom rank: $g_1(x)$ to the left and $g_2(x)$ to the right. When $\varepsilon = 180^\circ$, all agents interact and one cluster appears, but with weak consensus. If $\varepsilon = 120^\circ$, the agents interact with about two-thirds of total population at the most, and two weak consensus clusters appear. For $\varepsilon = 60^\circ$, agents interact with one-third of population, but now strong and weak consensus states coexist.

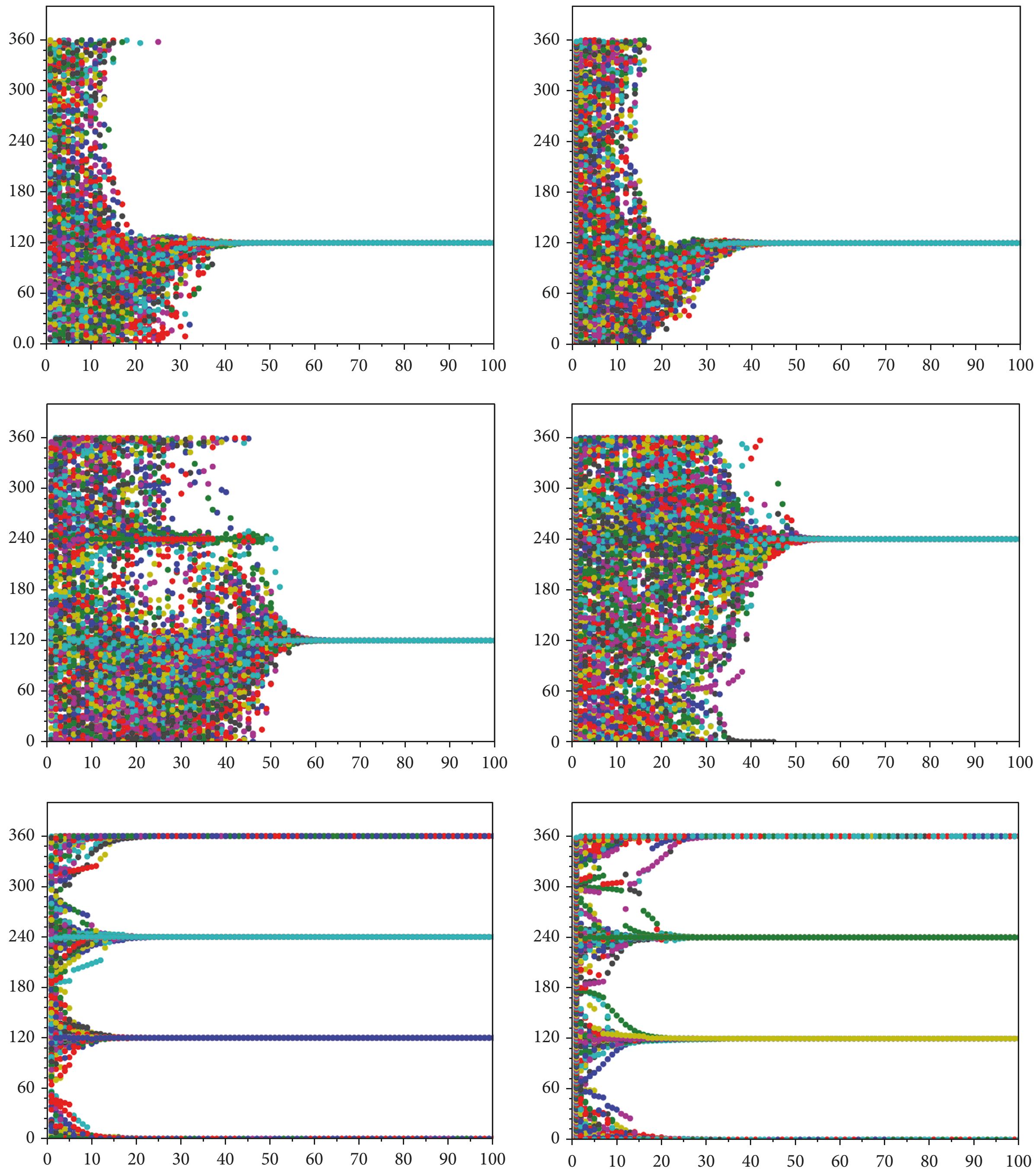

Figure 2: The confidence bounds are $\epsilon = 180^\circ$ (top row); $\epsilon = 120^\circ$ (middle row), and $\epsilon = 60$ (bottom row). $\kappa^i \in [0.5^\circ, 38^\circ]$. Function $g_1(x)$ is used on left column and $g_2(x)$ on the right. In the top row, a single cluster appears. In the middle row, initially two clusters appear, which finally merge. In the lower row, three clusters, with about one-third of the population, appear. Strong consensus appears always, in all three cases, according to Theorem 3.

5. Conclusions

In this paper, a new model for opinion dynamics of political preferences about three parties options has been presented; it generalizes the model in Medina *et al.* [8] to include new individual updating rules: a set of monoparametric iterative maps, which emulate different internal reflection processes in agents. For some positive values of the personal parameter κ (map's parameter), these rules introduce three equidistant attractors on a circular opinion space, which correspond to three choices, that can be interpreted with a different political party. For some negative values of the personal parameter, choices become repulsors, so agents with a negative personal parameter reject the political options. The value of the

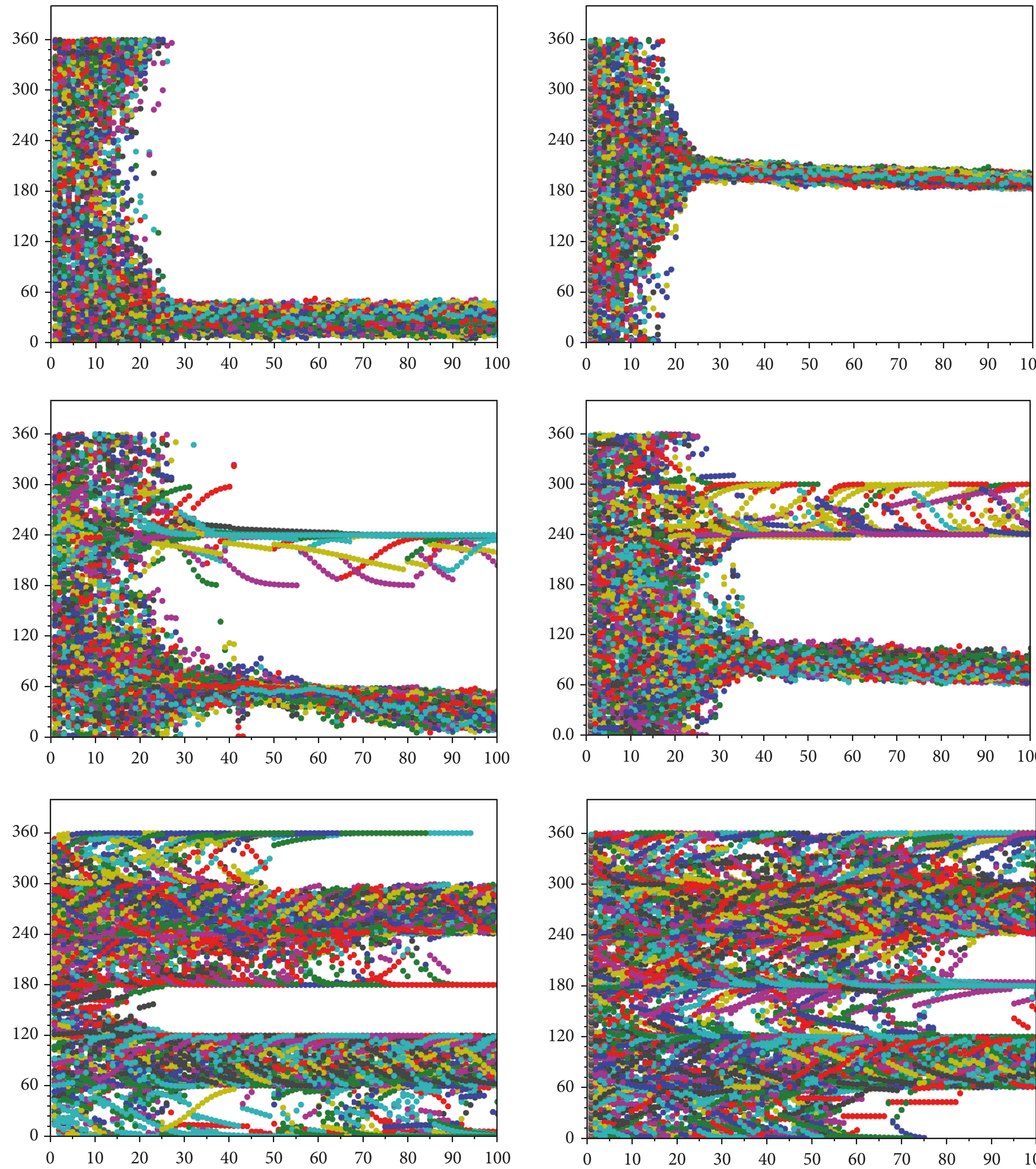

Figure 3: The confidence bounds are $\epsilon = 180^\circ$ (top row), $\epsilon = 120^\circ$ (middle row), and $\epsilon = 60$ (bottom row). $\kappa^i \in [-10^\circ, 10^\circ]$. Function $g_1(x)$ is used on left column and $g_2(x)$ on the right. In the top row, a weak consensus appears throughout the social group. In the middle row, there are two clusters, one grouping most of the opinions and another with a minority of these. In the lower row, the formation, of both weak and strong consensus, can be seen.

personal parameter may lead to a convergence to the fixed points with oscillation or without it; in the first case, the agent is interpreted as a hesitant one, whose opinion consolidates with time, while in the second case the agent is secure about her opinion. At least, for the examples presented here, the fixed points of the iterative maps can bifurcate, for other values of the personal parameter, in n-cycles and even lead to chaos (see Appendix A, and Figures 4 and 5). The n-cycles might be interpreted as a set of n-postures that an insecure agent can hold; for example, a 2-cycle may be considered a dilemma, while an agent, with a personal parameter in the chaotic regime, is considered an erratic one. In this sense, the personal parameter introduces different behaviors and strong postures in the agents.

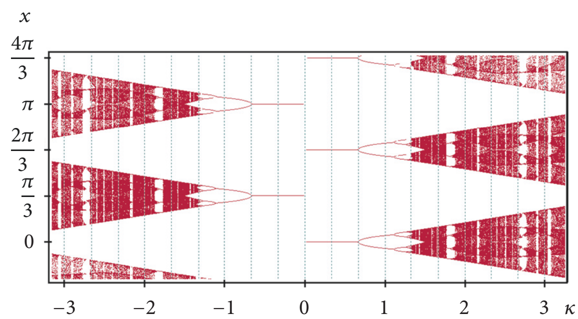

FIGURE 4: The figure shows a bifurcation diagram for the iterative map $x_{n+1} = x_n - \kappa \cos^4(\sin(3x_n)) \sin(3x_n)$; an alternating sequence of tree-like graphs goes upwards and downwards. Here are depicted the options for $0 < \kappa < 2/3$ and antioptions for $-2/3 < \kappa < 0$. Stable fixed points bifurcate in a 2-cycle for $\kappa = 2/3$; then a cascade of bifurcations appears, leading to chaos. The 2-cycle may model a dilemma, since agent's opinion jumps from one posture to another. Other n-cycles may model other doubt processes in agents.

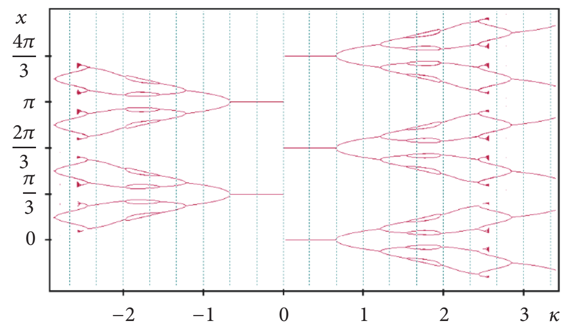

FIGURE 5: The figure shows a bifurcation diagram for the iterative map $x_{n+1} = x_n - \kappa \exp(-\sin^2(3x_n)) \sin(3x_n)$; an alternating sequence of tree-like graphs goes upwards and downwards. Stable fixed points bifurcate in different n-cycles for $|\kappa| > 2/3$. Stable fixed points bifurcate in a 2-cycle for $\kappa = 2/3$; then a cascade of bifurcations appears, leading to chaos, at least in some small regions. The 2-cycle may model a dilemma, since agent's opinion jumps from one posture to another. Other n-cycles may model other doubt processes in agents.

So, the key point of the present model is a new agent who is characterized by three attributes: opinion x, personal parameter κ, and an iterative map to update her own opinion differently as her interlocutors. This kind of agent becomes a fundamental piece of a more heterogeneous artificial society.

The presented model, also, considers pairwise interactions among agents and uses the confidence bound ϵ like Hegselmann & Krause and Deffuant & Weisbuch classical models to set a limit for dissimilar agents opinion case and random absolute weights to describe the subjective trust given to agents' opinions, defining an artificial social web.

The main interest of this work was to study consensus arising in this new artificial society; two classes of consensus were considered: a strong one which is the convergence of the opinions to a fixed point and a weak one which is the convergence of the opinions to a subset of the opinion space. Theorem 3 provides the necessary and sufficient conditions for the arising of a strong consensus in this artificial heterogeneous society.

Finally two sets of simulations were performed; in each set, a different iterative map and three different confidence bounds were used. In each simulation, were used either positive different personal parameters or mixed (positive and negative) personal parameters, random absolute weights, or random initial opinions, but the same iterative map for all agents. However, simulations show practically the same behaviors and convergence times as found in [8]; some of them show the emergence of strong consensus or the existence of weak consensus regions.

As perspectives of future research work, the model could be used to study the opinion dynamic of the electoral preferences regarding three political parties when a charismatic political actor (a very persuasive agent) is present. It is also possible to generalize the model to opinion spaces with four or more options, spheres of two or more dimensions. These latter models would be useful to study the situation when there are four or more political parties.

Appendix

A. On the Role of the Personal Parameter

From a mathematical point of view, iterative maps present a vast range of behaviors; some of them can be visualized in the corresponding bifurcation diagrams; see Marotto [13]. Some of these mathematical behaviors are used to model how agents update their own opinion: with certainty, if the opinion converges without oscillating; with doubt, if the opinion converges oscillating.

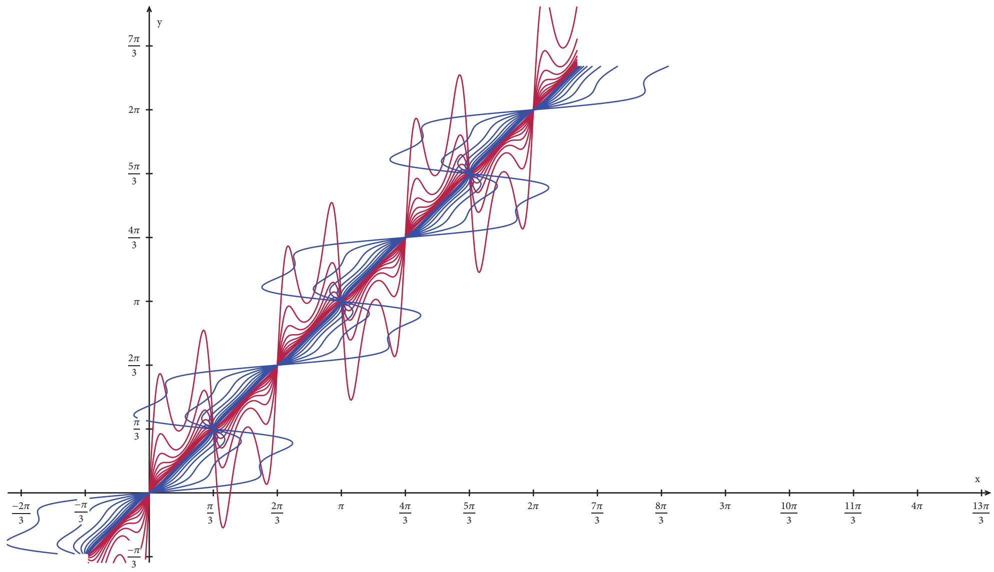

Figure 6: The figure shows a parametric plot of families of curves for system (9) for $\Xi_{(x)} = x - \kappa_x g_1(\sin(3x))\sin(3x)$, $\kappa_x = 0.66$, and different relatives weights $\alpha = 0.1, 0.2, \ldots, 0.9$ (red lines), $\Xi_{(y)} = y - \kappa_y g_2(\sin(3y))\sin(3y)$, $\kappa_y = 0.4$ and different relative weights $\delta = 0.1, 0.2, \ldots, 0.9$ (blue lines). The intersections correspond to different fixed points of different pairs of equations. Notice that only those fixed points of the form (p, p) are common to all equations in system (7). Other fixed points depend on parameters' values.

The iterative map $x_{n+1} = \Xi(x_n, \kappa)$ converges to one of the fixed points $p = s\pi/3$ without oscillation if

$$0 < \left.\frac{d\Xi(x,\kappa)}{dx}\right|_p < 1; \qquad \text{(A.1)}$$

this happens for $0 < \kappa < 1/3$ and $s = 0, 2, 4$ or $-1/3 < \kappa < 0$ and $s = 1, 3, 5$.

The iterative map $x_{n+1} = \Xi(x_n, \kappa)$ converges to one of the fixed points $p = s\pi/3$ with oscillation if

$$0 < \left.\frac{d\Xi(x,\kappa)}{dx}\right|_p < 1; \qquad \text{(A.2)}$$

this happens for $1/3 \le \kappa \le 2/3$ and $s = 0, 2, 4$ or $-2/3 \le \kappa \le 1/3$ and $s = 1, 3, 5$.

The above fixed points bifurcate in a 2-cycle for $|\kappa| = 2/3$; for greater values of $|\kappa|$, other bifurcations occur, finally leading to a chaotic behavior; see Figures 4 and 5. In Medina *et al.* [9], n-cycles are used to model different possible doubt processes, in which agents opinions jump among n-different postures; in particular, a 2-cycle is interpreted as a dilemma. The chaotic behavior models the behavior of an agent, who changes her opinion in an erratic way.

B. Fixed Points of the Updating Rules

Fixed points (p_s, q_s) for $s = 1, \ldots, M$ of system (9) can be found solving the system

$$\begin{aligned} p &= \alpha\Xi_{(x)}\left(p, \kappa_x\right) + \beta q, \\ q &= \gamma p + \delta\Xi_{(y)}\left(q, \kappa_y\right). \end{aligned} \qquad \text{(B.1)}$$

However, due to the nonlinear nature of the system, it is difficult to obtain solutions in the explicit form: $p = p(\kappa_x, \kappa_y, \alpha, \delta)$, $q = q(\kappa_x, \kappa_y, \alpha, \delta)$. Nevertheless, it is always possible to find them graphically. For example, in Figure 6, the special case is considered when two different mappings are used $\Xi_{(x)}(x, \kappa_x) = x - \kappa_x \sin(3x)\cos(\sin(3y))^4$ (red lines) and $\Xi_{(y)}(y, \kappa_y) = y - \kappa_y \sin(3y)\exp(-\sin(3y)^2)$ (blue lines) in system (B.1) with $\kappa_x = 0.66$ and $\kappa_y = 0.4$; $\alpha = 0.1, \ldots, 0.9$ and $\delta = 0.1, \ldots, 0.9$, respectively. In Figure 7, the same situation is considered but now using $\kappa_x = -0.66$ and $\kappa_y = 0.66$. Notice that the corresponding fixed points of a concrete pair of equations correspond to intersections of only one red curve with one blue curve. Nevertheless, there are fixed points common to all equations pairs, those of the form (p, p). Since, in general, in each temporal step, different pairs of agents interact, so that different pairs of equations are used each time, consequently, the only permanent fixed points are those of Theorem 1.

Data Availability

The manuscript uses data from computational experiments.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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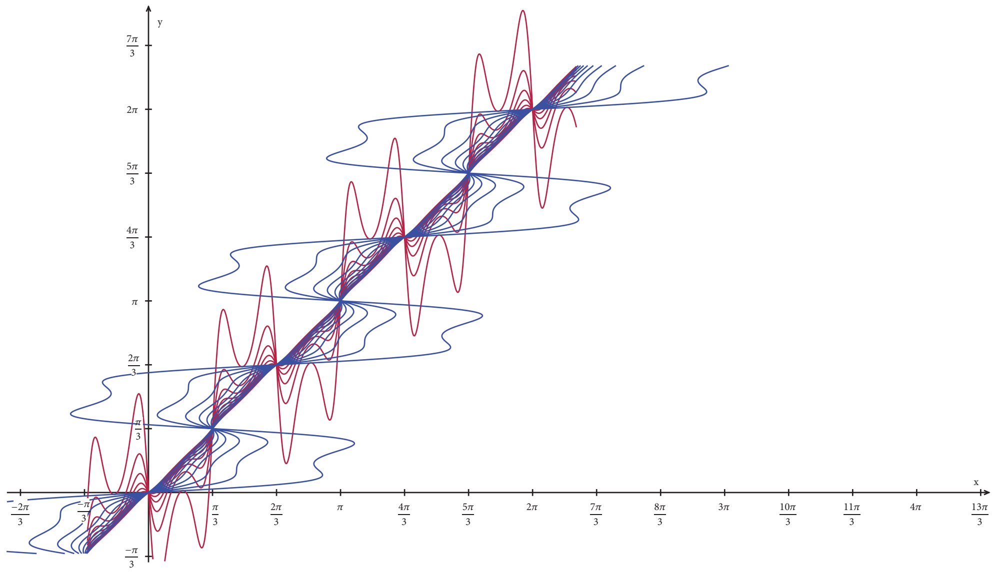

Figure 7: The figure shows a parametric plot of families of curves for system (9) for $\Xi_{(x)} = x - \kappa_x g_1(\sin(3x))\sin(3x)$, $\kappa_x = -0.66$, and different relatives weights $\alpha = 0.1, 0.2, \ldots, 0.9$ (red lines), $\Xi_{(y)} = y - \kappa_y g_2(\sin(3y))\sin(3y)$, $\kappa_y = 0.66$ and different relative weights $\delta = 0.1, 0.2, \ldots, 0.9$ (blue lines). The intersections correspond to different fixed points of different pairs of equations. Notice that only those fixed points of the form (p, p) are common to all equations in system (7); however, they are unstable. Other fixed points depend on parameters' values.

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