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Research Article

New Qualitative Results for Solutions of Functional Differential Equations of Second Order

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In this paper, we are concerned with the existence of periodic solutions, stability of zero solution, asymptotic stability of zero solution, square integrability of the first derivative of solutions, and boundedness of solutions of nonlinear functional differential equations of second order by the second method of Lyapunov. We obtain sufficient conditions guaranteeing the existence of periodic solutions, stability of zero solution, asymptotic stability of zero solution, square integrability of the first derivative of solutions, and boundedness of solutions of the equations considered. We give an example for illustrations by MATLAB-Simulink, which shows the behaviors of the orbits. The findings of this paper extend and improve some results that can be found in the literature.

1. Introduction

Differential equations of second order with and without delay(s) can find a wide range of applications in atomic energy, biology, chemistry, control theory, economy, engineering technique fields, information theory, medicine, physics, population dynamics, and so forth (see Burton [1], El'sgol'ts [2], Hale [3], Krasovskii [4], Smith [5], and Yoshizawa [6]). During investigations, we would naturally be inclined to compute the solutions of differential equations of second order with and without delay(s) explicitly or numerically. However, as we know from practice, there are very few such equations, for example, linear equations with constant coefficients, but without delay(s), for which this can be effectively done. Further, it should be noted that finding analytical or explicit solutions of differential equations of second order with delay(s) is more difficult, even if, to the best of our knowledge, there is no general method in the literature to find the explicit solutions of those equations. In addition, most of the times, it is impossible to find analytical solutions for those equations. The problem therefore is to find convenient techniques that will be useful in obtaining some qualitative information such as stability, instability, convergence, global existence, integrability, boundedness of solutions, existence of periodic solutions, and so forth about the elusive solutions of ordinary or delay differential equations.

From the past till now, various methods have been constructed and are still discussed in order to investigate the various qualitative behaviors of solutions of ordinary or delay differential equations without solving those equations.

However, here, we would only like to summarize some works that can be found in the literature and methods used during the investigations of the existence of the periodic solutions, stability, asymptotic stability, square integrability, and boundedness of solutions of ordinary and functional differential equations of second order.

Yoshizawa [6] considered the following nonlinear differential equation of second order with constant delay:

$$x'' + \phi\left(t, x'\right) + f\left(x\left(t-\tau\right)\right) = p\left(t\right) \tag{1}$$

and he investigated the existence of ω−periodic solutions of this equation by using the second method of Lyapunov.

Zhao at al. [7] obtained sufficient conditions for the existence of ω−periodic solutions of the below nonlinear differential equation of second order with constant delay

$$x'' + ax' + g\left(x\left(t-\tau\right)\right) = p\left(t\right) \tag{2}$$

by the second method of Lyapunov.

Cong [8] considered a class of nonlinear differential equations of second order in the form

$$\frac{d}{dt}\left[p(t)\frac{dx}{dt}\right]+f(t,x)=0. \tag{3}$$

Cong [8] proved that there exists a unique 2π−periodic solution of those differential equations under Landesman–Lazer type conditions by applying the Leray–Schauder principle.

Guo and Xu [9] studied the existence of periodic solutions of a differential equation of second order with a deviating argument by means of Mawhin's continuation theorem. In [9], a new result on the existence of periodic solutions is obtained.

Ji and Dong [10] discussed the existence and uniqueness of periodic solutions for a class of nonlinear differential equations of second order by using a comparison theorem and Leray-Schauder degree theory. The results obtained in [10] generalize and refine a recent work that can be found in the literature.

Tian and Zeng [11] studied the existence of periodic solutions to the second-order functional differential equation

$$\begin{aligned} x''(t)+f(t,x(t),x(t-\tau(t)))\left(x'(t)\right)^n+a(t)x^2(t) \\ +b(t)x(t)=p(t),\quad n\geq 2, \end{aligned} \tag{4}$$

by applying Mawhin's continuation theorem of coincidence degree theory. In [11], some new results on the existence of at least two periodic solutions to this equation are obtained.

Li and Li [12] obtained existence results of positive ω−periodic solutions for the following functional differential equation of second order with multiple variable delays:

$$\begin{aligned} u''(t)+a(t)u(t) \\ =f\left(t,u(t),u\left(t-\tau_1(t)\right),\ldots,u\left(t-\tau_n(t)\right)\right). \end{aligned} \tag{5}$$

In [12], the existence conditions concern the first eigenvalue of the associated linear periodic boundary problem and the discussion is based on the fixed-point index theory in cones.

Li and Zhang [13] established several criteria for the existence, multiplicity, and nonexistence of positive periodic solutions of the following system

$$x''+A(t)x=f(t,x) \tag{6}$$

by combining some new properties of Green's function together with Krasnoselskii's fixed-point theorem on the compression and expression of cones.

Zu [14] studied periodic solutions for the following nonlinear second-order ordinary differential equation:

$$x''=f\left(t,x,x'\right). \tag{7}$$

By constructing upper and lower boundaries and using Leray-Schauder degree theory, the author presented a result about the existence and uniqueness of a periodic solution for the above second-order ordinary differential equation with some assumptions.

Tunç and Yazgan [15] took into consideration the following nonlinear differential equation of second order with multiple fixed delays:

$$\begin{aligned} x''+\left[f\left(x,x'\right)+g\left(x,x'\right)x'\right]x'+h(x) \\ +\sum_{i=1}^{n}g_i\left(x\left(t-\tau_i\right)\right)=p(t) \end{aligned} \tag{8}$$

and they obtained the sufficient conditions for the existence of periodic solutions of this delay equation by the second method of Lyapunov.

Ma and Lu [16] showed the existence of positive T−periodic solutions of the below second-order functional differential equation:

$$u''(t)-p^2u(t)+\lambda g(t)f(u(t-\tau(t)))=0. \tag{9}$$

The approach in [16] is based on global bifurcation theorem.

Jia and Shao [17] established sufficient conditions for the existence and uniqueness of periodic solutions of an ordinary differential equation of second order by applying Mawhin's continuation theorem of coincidence degree theory.

Ardjouni and Djoudi ([18, 19]) discussed the existence of periodic and positive periodic solutions for a class of nonlinear neutral differential equations of second order with variable delays by Burton-Krasnoselskii's hybrid fixed-point theorem.

Similarly, Lü et al. [20] and Tian [21] investigated the existence of multiple positive periodic solutions for certain ordinary differential equations of second order and a delay differential equation of second order, respectively. In addition, Zhang and Wang [22] studied the existence of periodic solutions for a class of second-order functional differential equations with deviating arguments by using the abstract continuation theorem of k−set contractive operator and some analysis techniques.

Zhou [23] considered the existence of periodic solutions for a class of semilinear second-order differential equations of the form

$$-x''+f\left(t,x,x'\right)x'+e(t)g(x)=h(t). \tag{10}$$

By applying the viscosity solutions method and the classical upper-lower solutions method, as well as the Leray-Schauder fixed-point principle, the author established the existence of periodic solutions. The result of Zhou [23] improves and generalizes many results on the ropes mechanics equations in the existing literature.

Wei [24] proved the existence and uniqueness of periodic solutions for second-order ordinary differential equation

$$x''=f\left(t,x,x'\right) \tag{11}$$

under some assumptions on the function f. The proofs in [24] are based on Schauder's fixed-point theorem.

Finally, more recently, Zhu and Li [25] discussed the existence of periodic solutions for the below differential equation of second order with multiple delays

$$\begin{aligned} -u''(t) \\ =f\left(t,u(t),u\left(t-\tau_1\right),u\left(t-\tau_2\right),\ldots,u\left(t-\tau_n\right)\right) \end{aligned} \tag{12}$$

by using the monotone iterative method of upper and lower solutions.

Besides, for some other related papers, one can look at the book of Yoshizawa [26], the paper of Tunç and Çinar [27], and the references that can be found in the sources mentioned above.

In fact, through the papers or books presented above, it can be seen that the second method of Lyapunov has rarely been used to investigate the existence of periodic solutions of nonlinear differential equations of second order with and without delay(s) instead of the other mentioned methods. To the best of our knowledge, the basic reason for the lack of the papers by this method is to find suitable Lyapunov function(s) or functional(s), which give(s) meaningful results. In this paper, we study the existence of the periodic solutions by defining suitable new Lyapunov functionals. This is a contribution of this paper to the subject and literature.

On the other hand, the problems of the stability, asymptotic stability, convergence, integrability, and boundedness of solutions of linear and nonlinear differential equations of second order with and without delay(s) can appear in various physical, engineering, and many other scientific models. These kinds of differential equations are significant in describing fluid mechanical, nonlinear elastic mechanical phenomena, investigation of stability and instability of geodesic on Riemannian manifolds, dynamics process in electromechanical systems of physics and engineering, and so on. Many important theoretical and applied results related to these properties of solutions of differential equations of second order with and without delay(s) can be found in the literature (see Ahmad and Rama Mohana Rao [28], Burton [18], Burton and Hering [29], El'sgol'ts [2], Hale [3], Heidel [30], Kato [31, 32], Korkmaz and Tunç [33], Krasovskii [4], Liu and Huang [34, 35], Luk [36], Malyseva [37], Muresan [38], Mustafa and Tunç [39], Napoles Valdes [40], Amano [41], Sugie et al. [42], Tunç [43–52], Tunç and Dinç [53], Tunç and Tunç [54–57], Ye et al. [58], Yoshizawa [6], Yu and Xiao [59], Yu and Zhao [60], Zhang [61], Zhou and Xiang [62], and Zhou and Jiang [63] and their references).

In the sources mentioned, the second method of Lyapunov, perturbation theory, fixed-point method or theory, iterative techniques, the variation of constants formula, and some other tools are used to investigate the mentioned qualitative behaviors of solutions of linear and nonlinear differential equations of second order with and without delay(s). Here, for the sake of brevity, we would not like to give more details about these subjects. In addition, in view of the information given above, we would like to say that it is worthwhile to continue the investigation of the qualitative properties of the solutions of nonlinear differentials of second order with delay(s).

In this paper, we consider the following functional nonlinear differential equation of second order:

$$\begin{aligned}&\left[a(t)x'\right]' + \phi\left(t, x'\right) + h\left(t, x'(t-\tau)\right) + g(x)\\&\quad + f(x(t-\tau)) = e\left(t, x, x'\right),\end{aligned} \tag{13}$$

where $t \in \mathfrak{R}^+$, $\mathfrak{R}^+ = [0, \infty)$, $\tau \in \mathfrak{R}$, $\tau > 0$, if fixed constant delay, $x \in \mathfrak{R}$, $\mathfrak{R} = (-\infty, \infty)$, $a(t)$ is a continuous differentiable positive and ω−periodic function with $\omega > 0$, $\omega \in \mathfrak{R}$, $a(t+\omega) = a(t)$; ϕ, h, and e are continuous functions according to their related arguments and ω−periodic in t. That is, $\phi(t+\omega, x') = \phi(t, x')$, $h(t+\omega, x'(t-\tau)) = h(t, x'(t-\tau))$, and $e(t+\omega, x, x') = e(t, x, x')$. Finally, $f(x)$ and $g(x)$ are continuous differentiable functions with $f(0) = g(0) = 0$.

The following system can be written from (13):

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\frac{a'(t)}{a(t)}y - \frac{1}{a(t)}\phi(t, y) - \frac{1}{a(t)}h(t, y(t-\tau))\\&\quad - \frac{1}{a(t)}g(x) - \frac{1}{a(t)}f(x)\\&\quad + \frac{1}{a(t)}\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta\\&\quad + \frac{1}{a(t)}e(t, x, y).\end{aligned} \tag{14}$$

To the best of our knowledge from the literature, we did not find any paper on the existence of periodic solutions, stability, asymptotic stability, square integrability, and boundedness of solutions of a mathematical model like (13). The purpose of this paper is to give new sufficient hypotheses, five theorems, with an example by MATLAB-Simulink on existence of periodic solutions, stability of zero solution, asymptotic stability of zero solution, square integrability of the first derivative of solutions, and boundedness of solutions of (13) by the second method of Lyapunov. By the results of this paper, we extend and improve some results that can be found in the references of this paper (see [1–63]). These are the contributions of this paper to the mentioned topics and relevant literature.

2. Existence of Periodic Solutions

We now establish our some basic assumptions.

(A) Hypotheses. We suppose that the following hypotheses hold:

$(A1)$ $a_0 \geq a(t) \geq 1$,

$a'(t) \geq 0$,

$a_0 \in \mathfrak{R}$, $t \in \mathfrak{R}^+$.

$(A2)$ $\dfrac{\phi(t, y)}{y} > \alpha a(t) > 0$ for $|y| \geq A$, $(y \neq 0)$,

$\dfrac{h(t, y(t-\tau))}{y} \geq b_0 > 0$

for $|y| \geq A$, $(y \neq 0)$, $t \in \mathfrak{R}^+$, $y \in \mathfrak{R}$,

$$\begin{aligned}
&g(0)=0,\\
&g(x)\operatorname{sgn}x\longrightarrow\infty\\
&\qquad\qquad\text{for } |x|\longrightarrow\infty,\ \left|g'(x)\right|\le L_0,\\
&f(0)=0,\\
&f(x)\operatorname{sgn}x\longrightarrow\infty\\
&\qquad\qquad\text{for } |x|\longrightarrow\infty,\ \left|f'(x)\right|\le L,\ x\in\mathfrak{R},
\end{aligned}\tag{15}$$

where α, A, b_0, L_0, L, and $b_0\in\mathfrak{R}$ are some positive real constants with $b_0\ge 1$ and $\tau<\alpha/2L$.

$$(A3)\quad \left|e(t,x,y)\right|\le\frac{\alpha a(t)\left|y\right|}{4},\quad t\in\mathfrak{R}^+,\ x,y\in\mathfrak{R}.\tag{16}$$

Our first theorem for the existence of periodic solutions of system (14) can be given below.

Theorem 1. *If hypotheses $(A1)-(A3)$ hold, then system (14) has a ω-periodic solution.*

Proof. Let $V_0=V_0(t,x_t,y_t)$ be a Lyapunov functional defined by

$$\begin{aligned}
V_0=&\frac{2}{a(t)}\int_0^x f(s)\,ds+\frac{2}{a(t)}\int_0^x g(s)\,ds+y^2\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^0\left(\int_s^0 y^2(\theta)\,d\theta\right)ds\\
&+L\int_{-\tau}^0\left(\int_s^0\left|y(\theta)\right|d\theta\right)ds.
\end{aligned}\tag{17}$$

It is obvious that the Lyapunov functional V_0 is positive definite.

By calculating the time derivative of the Lyapunov functional V_0 with respect to t along system (14) and by usage of hypotheses $(A1)-(A3)$ of Theorem 1, we have

$$\begin{aligned}
\frac{d}{dt}V_0\le&-\frac{2a'(t)}{a(t)}y^2-\frac{2}{a(t)}\frac{\phi(t,y)}{y}y^2\\
&-\frac{2}{a(t)}\frac{h(t,y(t-\tau))}{y}y^2\\
&+\frac{2}{a(t)}y\int_{-\tau}^0 f'(x(t+\theta))\,y(t+\theta)\,d\theta+\frac{\alpha}{2}y^2\\
&+L\tau\left|y\right|-L\int_{-\tau}^0\left|y(t+\theta)\right|d\theta\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t)\,d\theta-\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t+\theta)\,d\theta.
\end{aligned}\tag{18}$$

It is clear that

$$-\frac{2a'(t)}{a(t)}\le 0\tag{19}$$

by hypothesis $(A1)$ of Theorem 1. Then, from (18), we can write

$$\begin{aligned}
\frac{dV_0}{dt}\le&-2\alpha y^2\\
&+\frac{2}{a(t)}\int_{-\tau}^0\left|f'(x(t+\theta))\right|\left|y(t)\right|\left|y(t+\theta)\right|d\theta\\
&+\frac{\alpha}{2}y^2+L\tau\left|y\right|-L\int_{-\tau}^0\left|y(t+\theta)\right|d\theta\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t)\,d\theta-\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t+\theta)\,d\theta\\
\le&-2\alpha y^2+2\int_{-\tau}^0 L\left|y(t)\right|\left|y(t+\theta)\right|d\theta+\frac{\alpha}{2}y^2\\
&+L\tau\left|y\right|-L\int_{-\tau}^0\left|y(t+\theta)\right|d\theta\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t)\,d\theta-\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t+\theta)\,d\theta
\end{aligned}\tag{20}$$

by hypotheses $(A1)-(A3)$ of Theorem 1.

Let $L<\alpha/2\tau$. Then, we have

$$\begin{aligned}
\frac{dV_0}{dt}\le&-\int_{-\tau}^0\left[\frac{\alpha}{2\tau}y^2(t)-\frac{\alpha}{\tau}\left|y(t)\right|\left|y(t+\theta)\right|\right.\\
&\left.+\frac{\alpha}{2\tau}y^2(t+\theta)\right]d\theta-\frac{3}{2}\alpha y^2+\frac{\alpha}{2}y^2+L\tau\left|y\right|\\
&-L\int_{-\tau}^0\left|y(t+\theta)\right|d\theta+\frac{\alpha}{2\tau}\int_{-\tau}^0 y^2(t)\,d\theta=-\frac{\alpha}{2\tau}\\
&\cdot\int_{-\tau}^0\left[\left|y(t)\right|-\left|y(t+\theta)\right|\right]^2d\theta-\alpha y^2+L\tau\left|y\right|\\
&-L\int_{-\tau}^0\left|y(t+\theta)\right|d\theta
\end{aligned}\tag{21}$$

by the last inequality. If $\tau<\alpha/2L$ and $|y(t)|\ge c=\max\{A, 2L\tau/\alpha+1\}$, then we can obtain

$$\frac{dV_0}{dt}\le-\frac{1}{2}\alpha y^2.\tag{22}$$

Case I. We assume that $|y(t)|\le c$, $c\in\mathfrak{R}$, $c>0$, and B_0 is a positive constant. Then, it is clear that

$$\frac{2\left|y\right|\left|\phi(t,y)\right|}{a(t)}+\frac{\left|\phi(t,y)\right|}{a(t)}+\frac{\left|h(t,y(t-\tau))\right|}{a(t)}\le B_0.\tag{23}$$

Suppose that $x(t)\ge d$, $d\in\mathfrak{R}$, $d>0$ and define the Lyapunov functional

$$V(t,x_t,y_t)=V_0(t,x_t,y_t)+y(t).\tag{24}$$

Then, the time derivative of functional V along system (14) is given by

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt}V_0 + \frac{d}{dt}y(t)\\ &= \frac{d}{dt}V_0 - \frac{a'(t)}{a(t)}y - \frac{1}{a(t)}\phi(t,y)\\ &\quad - \frac{1}{a(t)}\left(h\left(t, y(t-\tau)\right)\right) - \frac{1}{a(t)}g(x)\\ &\quad - \frac{1}{a(t)}f(x)\\ &\quad + \frac{1}{a(t)}\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta\\ &\quad + \frac{e(t,x,y)}{a(t)}.\end{aligned} \tag{25}$$

In view of hypotheses $(A1)-(A3)$ of Theorem 1, we have

$$\begin{aligned}\frac{dV}{dt} &\le -\frac{2a'(t)}{a(t)}y^2 - \frac{2}{a(t)}\phi(t,y)\, y\\ &\quad - \frac{2}{a(t)}h\left(t, y(t-\tau)\right) y\\ &\quad + \frac{2}{a(t)}y\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta\\ &\quad + \frac{2}{a(t)}e(t,x,y)\, y + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta\\ &\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta + L\tau|y|\\ &\quad - L\int_{-\tau}^{0}|y(t+\theta)|\, d\theta - \frac{a'(t)}{a(t)}y - \frac{1}{a(t)}\phi(t,y)\\ &\quad - \frac{1}{a(t)}\left(h\left(t, y(t-\tau)\right)\right) - \frac{1}{a(t)}g(x)\\ &\quad - \frac{1}{a(t)}f(x)\\ &\quad + \frac{1}{a(t)}\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta\\ &\quad + \frac{e(t,x,y)}{a(t)}.\end{aligned} \tag{26}$$

By hypotheses $(A1)-(A3)$ of Theorem 1, inequality (23), and the last estimate, we obtain

$$\begin{aligned}\frac{dV}{dt} &\le \left[\frac{2|y|\,|\phi(t,y)|}{a(t)} + \frac{1}{a(t)}|\phi(t,y)|\right.\\ &\quad \left. + \frac{1}{a(t)}\left|h\left(t, y(t-\tau)\right)\right|\right] + \frac{\alpha}{2}c^2\\ &\quad - \left[\frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta - 2L\int_{-\tau}^{0}|y(t)|\,|y(t+\theta)|\, d\theta\right.\\ &\quad \left. + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta\right] + \alpha c^2 + L\tau|y|\\ &\quad - L\int_{-\tau}^{0}|y(t+\theta)|\, d\theta - \frac{a'(t)}{a(t)}y - \frac{1}{a(t)}g(x)\\ &\quad - \frac{1}{a(t)}f(x) + L\int_{-\tau}^{0}|y(t+\theta)|\, d\theta + \frac{|e(t,x,y)|}{a(t)}\\ &\le B_0 + \frac{\alpha c^2}{2} - \frac{\alpha}{2\tau}\int_{-\tau}^{0}\left[|y(t)| - |y(t+\theta)|\right]^2 d\theta\\ &\quad + \alpha c^2 + L\tau c + a_0 c + \frac{\alpha c}{4} - \frac{1}{a(t)}g(x) - \frac{1}{a(t)}f(x).\end{aligned} \tag{27}$$

Since $a_0 \ge a(t) \ge 1$, it follows that

$$\frac{dV}{dt} \le B_0 + \frac{\alpha c^2}{2} + \alpha c^2 + a_0 c + \frac{3\alpha c}{4} - \left[g(x) + f(x)\right]. \tag{28}$$

In view of the hypotheses $f(x)\operatorname{sgn} x \longrightarrow \infty$ for $|x| \longrightarrow \infty$ and $g(x)\operatorname{sgn} x \longrightarrow \infty$ for $|x| \longrightarrow \infty$, it is obvious that

$$\frac{dV}{dt} \le -1 \tag{29}$$

when $|x| \longrightarrow \infty$.

Case II. We assume that

$$\begin{aligned}&|y(t)| \le c,\\ &x(t) \le -d,\\ &c, d \in \mathfrak{R},\ c > 0,\ d > 0.\end{aligned} \tag{30}$$

It is known that the function $a(t)$ is bounded and $|y| \le c$. Then, inequality (23) holds.

We define a Lyapunov functional by

$$V(t, x_t, y_t) = V_0(t, x_t, y_t) - y(t). \tag{31}$$

The time derivative of the Lyapunov functional V along system (14) implies that

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt}V_0 - \frac{d}{dt}y(t)\\ &= \frac{dV_0}{dt} + \frac{a'(t)}{a(t)}y + \frac{1}{a(t)}\phi(t,y)\\ &\quad + \frac{1}{a(t)}h\left(t, y(t-\tau)\right) + \frac{1}{a(t)}g(x)\\ &\quad + \frac{1}{a(t)}f(x)\\ &\quad - \frac{1}{a(t)}\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta\\ &\quad - \frac{e(t,x,y)}{a(t)}.\end{aligned} \tag{32}$$

Hence, using inequality (18), we obtain

$$\begin{aligned}\frac{d}{dt}V \le & -\frac{2a'(t)}{a(t)}y^2 - \frac{2}{a(t)}\phi(t,y)\,y \\ & - \frac{2}{a(t)}h(t,y(t-\tau))\,y \\ & + \frac{2}{a(t)}y\int_{-\tau}^{0} f'(x(t+\theta))\,y(t+\theta)\,d\theta \\ & + \frac{2}{a(t)}e(t,x,y)\,y + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta \\ & - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\,d\theta + L\tau|y| \\ & - L\int_{-\tau}^{0}|y(t+\theta)|\,d\theta + \frac{a'(t)}{a(t)}y + \frac{1}{a(t)}\phi(t,y) \\ & + \frac{1}{a(t)}(h(t,y(t-\tau))) + \frac{1}{a(t)}g(x) \\ & + \frac{1}{a(t)}f(x) \\ & - \frac{1}{a(t)}\int_{-\tau}^{0} f'(x(t+\theta))\,y(t+\theta)\,d\theta \\ & - \frac{e(t,x,y)}{a(t)}.\end{aligned} \tag{33}$$

After that, in view of hypotheses of Theorem 1, it follows that

$$\begin{aligned}\frac{d}{dt}V \le & -\frac{2}{a(t)}\phi(t,y)\,y + 2L\int_{-\tau}^{0}|y(t)|\,|y(t+\theta)|\,d\theta \\ & + \frac{\alpha}{2}c^2 + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta \\ & - \left[\frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\,d\theta\right] \\ & + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta + L\tau|y| \\ & - L\int_{-\tau}^{0}|y(t+\theta)|\,d\theta + \frac{a'(t)}{a(t)}y \\ & + \frac{1}{a(t)}|\phi(t,y)| + \frac{1}{a(t)}|h(t,y(t-\tau))| \\ & + \frac{1}{a(t)}g(x) + \frac{1}{a(t)}f(x) \\ & + \frac{1}{a(t)}\int_{-\tau}^{0}|f'(x(t+\theta))|\,|y(t+\theta)|\,d\theta + \frac{\alpha c}{4}.\end{aligned} \tag{34}$$

By taking into consideration the hypotheses of Theorem 1, it can be obtained that

$$\begin{aligned}\frac{d}{dt}V \le & B_0 + \frac{\alpha c^2}{2} - \frac{\alpha}{2\tau}\int_{-\tau}^{0}\left[|y(t)| - |y(t+\theta)|\right]^2 d\theta \\ & + \alpha c^2 + L\tau c + \beta c + \frac{\alpha c}{4} + \frac{1}{a(t)}g(x) \\ & + \frac{1}{a(t)}f(x) \\ \le & B_0 + \frac{\alpha c^2}{2} + \alpha c^2 + \beta c + \frac{3\alpha c}{4} + g(x) + f(x).\end{aligned} \tag{35}$$

Since $f(x)\operatorname{sgn} x \longrightarrow \infty$ for $|x| \longrightarrow \infty$ and $g(x)\operatorname{sgn} x \longrightarrow \infty$ for $|x| \longrightarrow \infty$, it can be easily concluded that

$$\frac{dV}{dt} \le -1. \tag{36}$$

Case III. Let

$$\begin{aligned}&|x(t)| \le d, \\ &y(t) \ge c, \\ &c, d \in \mathfrak{R},\ c > 0,\ d > 0.\end{aligned} \tag{37}$$

We now define a Lyapunov functional by

$$V(t, x_t, y_t) = V_0(t, x_t, y_t) + \frac{c}{d}x(t). \tag{38}$$

Then, calculating the time derivative of the functional V with respect to t, using inequality (18) and hypotheses $(A1)-(A3)$ of Theorem 1, we have

$$\begin{aligned}\frac{d}{dt}V = & \frac{d}{dt}V_0 + \frac{c}{d}y(t) \\ \le & -\frac{2a'(t)}{a(t)}y^2 - \frac{2}{a(t)}\frac{\phi(t,y)}{y}y^2 \\ & - \frac{2}{a(t)}\frac{h(t,y(t-\tau))}{y}y^2 \\ & + \frac{2}{a(t)}y\int_{-\tau}^{0} f'(x(t+\theta))\,y(t+\theta)\,d\theta \\ & + \frac{2}{a(t)}e(t,x,y)\,y + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta \\ & - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\,d\theta + L\tau|y| \\ & - L\int_{-\tau}^{0}|y(t+\theta)|\,d\theta + \frac{c}{d}y \\ \le & -2\alpha y^2 + 2L\int_{-\tau}^{0}|y(t)|\,|y(t+\theta)|\,d\theta + \frac{\alpha}{2}y^2 \\ & + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta \\ & - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\,d\theta - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\,d\theta \\ & + \frac{\alpha}{2}|y| + \frac{c}{d}y\end{aligned}$$

$$\begin{aligned}
&\le -2\alpha y^2 + \frac{\alpha}{2}y^2 \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0}\left[|y(t)| - |y(t+\theta)|\right]^2 d\theta + \alpha y^2 \\
&\quad + \frac{\alpha}{2}|y| + \frac{c}{d}y \\
&\le -2\alpha y^2 + \frac{\alpha}{2}y^2 + \alpha y^2 + \frac{\alpha}{2}|y| + \frac{c}{d}y \\
&\le -\frac{1}{2}\alpha y^2 + \frac{c}{d}y \le -\frac{1}{4}\alpha y^2.
\end{aligned} \tag{39}$$

Case IV. Let

$$\begin{aligned}
&|x(t)| \le d, \\
&y(t) \le -c, \\
&\qquad c, d \in \mathfrak{R},\ c > 0,\ d > 0.
\end{aligned} \tag{40}$$

We define a Lyapunov functional by

$$V(t, x_t, y_t) = V_0(t, x_t, y_t) - \frac{c}{d}x(t). \tag{41}$$

Like before, we can obtain

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}V_0 - \frac{c}{d}y(t) \\
&\le -\frac{2a'(t)}{a(t)}y^2 - \frac{2}{a(t)}\frac{\phi(t,y)}{y}y^2 \\
&\quad - \frac{2}{a(t)}\frac{h(t, y(t-\tau))}{y}y^2 \\
&\quad + \frac{2}{a(t)}y\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta \\
&\quad + \frac{2}{a(t)}e(t,x,y)\, y + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta + L\tau|y| \\
&\quad - L\int_{-\tau}^{0}|y(t+\theta)|\, d\theta - \frac{c}{d}y \\
&\le -2\alpha y^2 + 2L\int_{-\tau}^{0}|y(t)|\,|y(t+\theta)|\, d\theta + \frac{\alpha}{2}y^2 \\
&\quad + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta \\
&\quad - \frac{c}{d}y \\
&\le -2\alpha y^2 + \frac{\alpha}{2}y^2 \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0}\left[|y(t)| - |y(t+\theta)|\right]^2 d\theta + \alpha y^2 \\
&\quad + \frac{\alpha}{2}|y| - \frac{c}{d}y \le -2\alpha y^2 + \frac{\alpha}{2}y^2 + \alpha y^2 - \frac{c}{d}y \\
&\le -\frac{1}{2}\alpha y^2 - \frac{c}{d}y \le -\frac{1}{4}\alpha y^2
\end{aligned} \tag{42}$$

by (18) and hypotheses $(A1) - (A3)$ of Theorem 1.

Case V. We suppose that $x(t) \ge d$ for $c, d \in \mathfrak{R},\ c > 0,\ d > 0,\ y(t) \ge c$ or $x(t) \le -d,\ y(t) \le -c$.

We define a Lyapunov functional V by

$$V(t, x_t, y_t) = V_0(t, x_t, y_t) + c. \tag{43}$$

Case VI. Further, in the case of $x(t) \ge d,\ y(t) \le -c$, or $x(t) \le -d,\ y(t) \ge c$, we define Lyapunov functional by

$$V(t, x_t, y_t) = V_0(t, x_t, y_t) - c. \tag{44}$$

In view of the above two cases, that is, Cases V and VI, and the hypotheses of Theorem 1, since c is a positive constant, the time derivative of functional V along system (14) leads to the following:

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}V_0 \\
&\le -\frac{2a'(t)}{a(t)}y^2 - \frac{2}{a(t)}\frac{\phi(t,y)}{y}y^2 \\
&\quad - \frac{2}{a(t)}\frac{h(t, y(t-\tau))}{y}y^2 \\
&\quad + \frac{2}{a(t)}y\int_{-\tau}^{0} f'(x(t+\theta))\, y(t+\theta)\, d\theta \\
&\quad + \frac{2}{a(t)}e(t,x,y)\, y + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta + L\tau|y| \\
&\quad - L\int_{-\tau}^{0}|y(t+\theta)|\, d\theta
\end{aligned} \tag{45}$$

so that

$$\begin{aligned}
\frac{d}{dt}V &\le -2\alpha y^2 + 2L\int_{-\tau}^{0}|y(t)|\,|y(t+\theta)|\, d\theta + \frac{\alpha}{2}y^2 \\
&\quad + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta + \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta \\
&\quad - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t)\, d\theta - \frac{\alpha}{2\tau}\int_{-\tau}^{0} y^2(t+\theta)\, d\theta \\
&\le -2\alpha y^2 + \frac{\alpha}{2}y^2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha}{2\tau}\int_{-\tau}^{0}\left[\left|y(t)\right|-\left|y(t+\theta)\right|\right]^{2}d\theta+\alpha y^{2}\\
&+\frac{\alpha}{2}\left|y\right|\leq-2\alpha y^{2}+\frac{\alpha}{2}y^{2}+\alpha y^{2}=-\frac{1}{2}\alpha y^{2}\\
&\leq-\frac{1}{4}\alpha y^{2}.
\end{aligned}
\tag{46}
$$

Hence, it is obvious that the time derivative of functional V, that is, $(d/dt)V$, is negative semidefined for the above two cases. In addition, we can see that

$$
\begin{aligned}
&\frac{2}{a(t)}\int_{0}^{x}f(s)\,ds+\frac{2}{a(t)}\int_{0}^{x}g(s)\,ds+y^{2}-c\\
&\leq 2\int_{0}^{x}f(s)\,ds+2\int_{0}^{x}g(s)\,ds+y^{2}-c\\
&\leq V(t,x_{t},y_{t})\\
&\leq 2\int_{0}^{x}f(s)\,\mathrm{ds}+2\int_{0}^{x}g(s)\,ds+c\\
&\quad+\frac{a}{2\tau}\int_{-\tau}^{0}\left(\int_{\tau}^{0}y^{2}(\theta)\,d\theta\right)ds\\
&\quad+L\int_{-\tau}^{0}\left(\int_{\tau}^{0}\left|y(\theta)\right|d\theta\right)ds.
\end{aligned}
\tag{47}
$$

Hence, "wedges" can be easily found (see Burton [1]), bounding the functional V_0 from bottom to top. Then, we can conclude that system (14) has a periodic solution of period ω (see Yoshizawa ([6, Theorem 37.2])). □

3. Stability of Solutions

Let $e(t,x,y)\equiv 0$.

(B) Hypothesis. It is assumed that the following hypothesis holds:

$$
\begin{aligned}
(A4)\quad &\phi(t,0)=0,\\
&\frac{\phi(t,y)}{y}>\alpha a(t)>0\\
&\qquad\qquad\text{for } y\in\mathfrak{R},\ (y\neq 0),\\
&h(t,0)=0,\\
&\frac{h(t,y(t-\tau))}{y}\geq b_{0}>0\\
&\qquad\qquad\text{for } t\in\mathfrak{R}^{+},\ y\in\mathfrak{R},\ (y\neq 0),\\
&g(0)=0,\\
&\frac{g(x)}{x}\geq g_{0},\\
&\qquad\qquad x\in\mathfrak{R},\ (x\neq 0),\\
&f(0)=0,\\
&\frac{f(x)}{x}\geq f_{0},\\
&\qquad\qquad(x\neq 0),\ \left|f'(x)\right|\leq M,\ x\in\mathfrak{R},
\end{aligned}
\tag{48}
$$

where α, b_0, f_0, g_0, and $M\in\mathfrak{R}$ are some positive real constants with $b_0\geq 1$ and $\tau<\alpha/2M$.

Theorem 2. *If hypotheses $(A1)$ and $(A4)$ hold, then the zero solution of system (14) is stable.*

Proof. Consider the Lyapunov functional $V_1=V_1(t,x_t,y_t)$ defined by

$$
\begin{aligned}
V_{1}=&\frac{2}{a(t)}\int_{0}^{x}f(s)\,ds+\frac{2}{a(t)}\int_{0}^{x}g(s)\,ds+y^{2}\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^{0}\left(\int_{s}^{0}y^{2}(\theta)\,d\theta\right)ds.
\end{aligned}
\tag{49}
$$

It is clear that

$$
V_{1}(t,0,0)=0
\tag{50}
$$

and

$$
\begin{aligned}
V_{1}=&\frac{2}{a(t)}\int_{0}^{x}\frac{f(s)}{s}s\,ds+\frac{2}{a(t)}\int_{0}^{x}\frac{g(s)}{s}s\,ds+y^{2}\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^{0}\left(\int_{s}^{0}y^{2}(\theta)\,d\theta\right)ds\\
\geq&\left(\frac{f_{0}+g_{0}}{a_{0}}\right)x^{2}+y^{2}=\sigma\left(x^{2}+y^{2}\right),
\end{aligned}
\tag{51}
$$

where $\sigma=\min\{(f_0+g_0)a_0^{-1},1\}$, by hypotheses $(A1)$ and $(A4)$.

Differentiating the Lyapunov functional V_1 along system (14) and considering hypotheses $(A1)$ and $(A4)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}V_{1}\leq&-\frac{2}{a(t)}\frac{\phi(t,y)}{y}y^{2}-\frac{2}{a(t)}\frac{h(t,y(t-\tau))}{y}y^{2}\\
&+\frac{2}{a(t)}y\int_{-\tau}^{0}f'(x(t+\theta))\,y(t+\theta)\,d\theta\\
&+\frac{\alpha}{2\tau}\int_{-\tau}^{0}y^{2}(t)\,d\theta-\frac{\alpha}{2\tau}\int_{-\tau}^{0}y^{2}(t+\theta)\,d\theta.
\end{aligned}
\tag{52}
$$

Hence,

$$
\begin{aligned}
\frac{dV_{1}}{dt}\leq&-\frac{3}{2}\alpha y^{2}-2\frac{b_{0}}{a_{0}}y^{2}\\
&+\frac{2}{a(t)}\int_{-\tau}^{0}\left|f'(x(t+\theta))\right|\left|y(t)\right|\left|y(t+\theta)\right|d\theta\\
&-\frac{\alpha}{2\tau}\int_{-\tau}^{0}y^{2}(t+\theta)\,d\theta.\\
\leq&-\frac{3}{2}\alpha y^{2}-2\frac{b_{0}}{a_{0}}y^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-\tau}^{0} M\left[y^{2}(t)+y^{2}(t+\theta)\right] d\theta \\
&-\frac{\alpha}{2\tau}\int_{-\tau}^{0} y^{2}(t+\theta)\, d\theta \\
\leq &-\left[\frac{3}{2}\alpha+2\frac{b_0}{a_0}-M\tau\right] y^{2} \\
&-\left[\frac{\alpha}{2\tau}-M\right]\int_{-\tau}^{0} y^{2}(t+\theta)\, d\theta
\end{aligned}
\tag{53}
$$

by hypotheses $(A1)$ and $(A4)$ of Theorem 2.

Let $\tau < \alpha/2M$. Then, it follows that

$$\frac{dV_1}{dt} \leq -\left(\alpha+2a_0b_0^{-1}\right) y^{2}. \tag{54}$$

This result completes that the zero solution of system (14) is stable. □

Corollary 3. *If hypotheses $(A1)$ and $(A4)$ hold, then the zero solution of system (14) is uniformly stable.*

Theorem 4. *If hypotheses $(A1)$ and $(A4)$ hold, then the zero solution of system (14) is asymptotically stable.*

Proof. In the proof of this theorem, we benefit from the Lyapunov functional V_1 given in the proof of Theorem 2.

In the light of the hypotheses of Theorem 4, we can conclude that

$$\frac{dV_1}{dt} \leq -\left(\alpha+2a_0b_0^{-1}\right) y^{2}. \tag{55}$$

Consider now the set defined by

$$S \equiv \left\{(t,x,y) : \dot{V}_1(t,x_t,y_t)=0\right\}. \tag{56}$$

When we apply LaSalle's invariance principle, we observe that $(t,x,y) \in S$ implies that $y=0$, and hence $x=\rho$ ($\rho \neq 0$ is a constant). The last estimate and system (14), together, necessarily imply that

$$\frac{1}{a(t)}g(x)+\frac{1}{a(t)}f(x)=0. \tag{57}$$

When $x=\rho$, this equality can hold if and only if

$$g(\rho)+f(\rho)=0. \tag{58}$$

Hence,

$$
\begin{aligned}
g(\rho)+f(\rho)=0 \Longleftrightarrow \\
\rho=0,
\end{aligned}
\tag{59}
$$

which implies that $g(x)+f(x)=0 \Longleftrightarrow x=0$. Therefore, we have $x=y=0$. In fact, this result shows that the largest invariant set contained in S is $(t,0,0) \in S$. Hence, we can conclude that the zero solution of system (14) is asymptotically stable. This completes the proof of Theorem 4. □

Theorem 5. *If hypotheses $(A1)$ and $(A4)$ hold, then the first derivatives of all solutions of (13) are square-integrable; that is, $y(t) \in L^2[0,\infty)$, where $L^2[0,\infty)$ is the space of all Lebesgue square-integrable functions on $[0,\infty)$.*

Proof. Here, we also use the functional V_1 used in both theorems given just above. Notice the hypotheses of Theorem 5; we have

$$\frac{d}{dt}V_1(t,x_t,y_t) \leq -ky^2 \leq 0, \quad k=\alpha+2a_0b_0^{-1}>0. \tag{60}$$

Integrating the last inequality from t_0 to t, we have

$$V_1(t,x_t,y_t)-V_1\left(t_0,x_{t_0},y_{t_0}\right) \leq -k\int_{t_0}^{t}|y(s)|^2\, ds. \tag{61}$$

From the above discussion, it can be seen that $V_1(t,x_t,y_t)$ is positive definite and a decreasing functional. Therefore, we can say that

$$V_1\left(t_0,x_{t_0},y_{t_0}\right)=\ell, \quad \ell>0,\ \ell\in\mathfrak{R}, \tag{62}$$

and hence, it is clear that

$$
\begin{aligned}
k\int_{t_0}^{t}|y(s)|^2\, ds &\leq V_1\left(t_0,x_{t_0},y_{t_0}\right)-V_1(t,x_t,y_t) \\
&\leq V_1\left(t_0,x_{t_0},y_{t_0}\right).
\end{aligned}
\tag{63}
$$

As the result of the above inequalities, we can conclude that

$$\int_{t_0}^{\infty}|y(s)|^2\, ds \leq k^{-1}\ell. \tag{64}$$

This result is the end of the proof of Theorem 5. □

4. Boundedness of Solutions

Let $e(t,x,y) \neq 0$.

(C) Hypothesis. It is assumed that the following hypothesis holds:

$$(A5) \quad |e(t,x,y)| \leq \theta(t)|y|, \quad t\in\mathfrak{R}^+,\ x,y\in\mathfrak{R}, \tag{65}$$

where $\theta(t)$ is a nonnegative and continuous function for all $t\in\mathfrak{R}^+$ such that $\theta\in L^1[0,\infty)$, $L^1[0,\infty)$ is the space of all Lebesgue integrable functions on $[0,\infty)$.

Theorem 6. *If hypotheses $(A1)$, $(A4)$, and $(A5)$ hold, then all solutions of system (13) are bounded as $t\longrightarrow\infty$.*

Proof. Here, once again, we use the functional V_1 just given above. Notice the hypotheses of Theorem 6; we can have

$$\frac{d}{dt}V_1(t,x_t,y_t) \leq 2ye(t,x,y). \tag{66}$$

In view of hypothesis $(A5)$, we can get

$$\frac{d}{dt}V_1(t,x_t,y_t) \leq 2\theta(t)y^2 \leq 2\sigma^{-1}\theta(t)V_1(t,x_t,y_t). \tag{67}$$

Integrating the former inequality from t_0 to t, we have

$$V_1(t, x_t, y_t) - V_1(t_0, x_{t_0}, y_{t_0}) \leq 2\sigma^{-1} \int_{t_0}^{t} \theta(s) V_1(s, x_s, y_s)\, ds \tag{68}$$

so that

$$V_1(t, x_t, y_t) \leq \ell + 2\sigma^{-1} \int_{t_0}^{t} \theta(s) V_1(s, x_s, y_s)\, ds. \tag{69}$$

By the Gronwall inequality, it follows that

$$V_1(t, x_t, y_t) \leq \ell \exp\left(2\sigma^{-1} \int_{t_0}^{\infty} \theta(s)\, ds\right) < \infty. \tag{70}$$

Let

$$\ell \exp\left(2\sigma^{-1} \int_{t_0}^{\infty} \theta(s)\, ds\right) = k_1. \tag{71}$$

In addition, we also have

$$x^2 + y^2 \leq \sigma^{-1} V_1(t, x_t, y_t). \tag{72}$$

Hence, we can conclude that

$$x^2 + y^2 \leq \sigma^{-1} k_1. \tag{73}$$

This is the end of the proof of Theorem 6. □

Example 7. We consider the following second-order nonlinear differential equation with constant delay:

$$x'' + (3 + \sin t) x' + 3 + \sin t + x'^2(t-1), +2x + x(t-1) = \frac{1}{8} x' \sin t. \tag{74}$$

When we compare this equation with (13), it follows that

$$\begin{aligned} \phi(t, y) &= (3 + \sin t) y, \\ a(t) &= 1, \\ h(t, y(t-\tau)) &= 3 + \sin t + y^2(t-1), \\ f(x) &= x, \\ g(x) &= 2x \\ e(t, x, y) &= \frac{1}{8} y \sin t. \end{aligned} \tag{75}$$

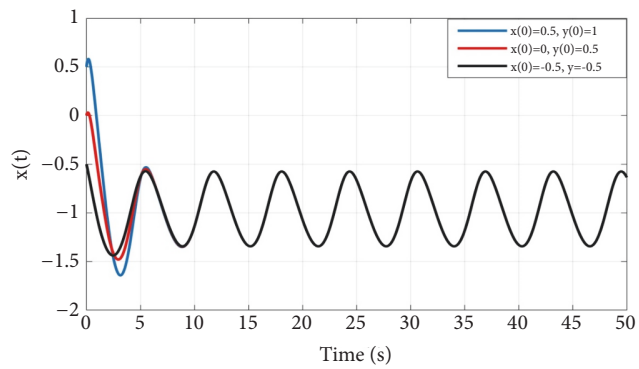

Figure 1: Trajectory of $x(t)$ for Example 7.

It is obvious that the hypotheses of Theorem 1 are satisfied:

$$\begin{aligned} \frac{\phi(t, y)}{y} &= 3 + \sin t > \alpha = 1, \\ h(t, y(t-\tau)) &= 3 + \sin^2 t + y^2(t-1) \geq 3 = b_0 > 0, \\ f(x) \operatorname{sgn} x &= x \operatorname{sgn} x \longrightarrow \infty \quad \text{for } |x| \longrightarrow \infty \\ |f'(x)| &= |x| = 1 \leq 2 = L, \\ |e(t, x, y)| &= \frac{1}{8}|y||\sin t| \leq \frac{\alpha|y|}{4} = \frac{|y|}{4}, \quad \alpha = 1, \\ g(x) \operatorname{sgn} x &= 2x \operatorname{sgn} x \longrightarrow \infty \quad \text{for } |x| \longrightarrow \infty, \\ |g'(x)| &= |2| = 2 \leq L, \\ a(t) &= 1, \\ a'(t) &= 0. \end{aligned} \tag{76}$$

Therefore, the given differential equation satisfies all hypotheses of Theorem 1. Then, there exists a 2π−periodic solution of the above delay differential equation.

The orbits of the solutions of the considered delay differential equation are shown by Figures 1 and 2.

Remark 8. The differential equation with constant delay just given above by Example 7 can be modified and the related graphs of the orbits can be drawn by MATLAB-Simulink to verify the stability of zero solution, asymptotic stability of zero solution, square integrability of the first derivative of solutions, and boundedness of solutions. Here, we omit the details.

5. Conclusion

With the help of the second method of Lyapunov, the qualitative properties such as the existence of periodic solutions, stability of zero solution, asymptotic stability of zero solution, square integrability of the first derivative of solutions, and boundedness of solutions of a class of nonlinear differential equations of second order with constant delay are investigated. On the mentioned topics, five new theorems are

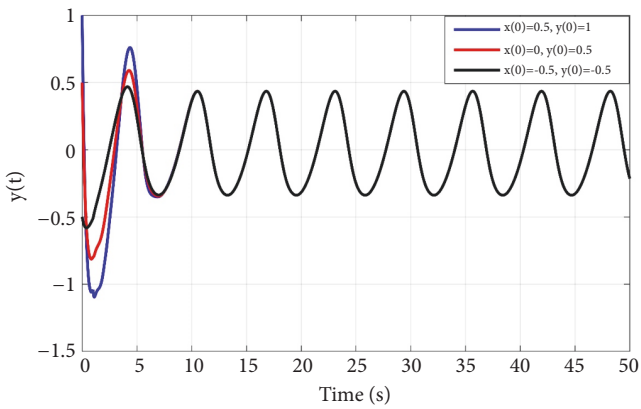

Figure 2: Trajectory of $y(t)$ for Example 7.

proven. The obtained results include and improve some results in the literature. We give an example to verify the applicability of the results by MATLAB-Simulink, which shows the behaviors of the orbits (see Figures 1 and 2).

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

The authors confirm that the paper has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. The authors further confirm that the order of authors listed in the paper has been approved by all of them.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

[1] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, Fla, USA, 1985.

[2] L. E. El'sgol'ts, *Introduction to The Theory of Differential Equations with Deviating Arguments*, Holden-Day, San Francisco, Calif, USA, 1966, Translated from the Russian by Robert J. McLaughlin.

[3] J. K. Hale, "Sufficient conditions for stability and instability of autonomous functional-differential equations," *Journal of Differential Equations*, vol. 1, pp. 452–482, 1965.

[4] N. N. Krasovskii, *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Stanford University Press, Stanford, Calif, USA, 1963.

[5] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, NY, USA, 2011.

[6] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method*, The Mathematical Society of Japan, Tokyo, Japan, 1966.

[7] Z. Jie-min, H. Ke-lei, and L. Qi-shao, "The existence of periodic solutions for a class of functional differential equations and their application," *Applied Mathematics and Mechanics-English Edition*, vol. 15, no. 1, pp. 49–59, 1994.

[8] F. Cong, "Periodic solutions for second order differential equations," *Applied Mathematics Letters*, vol. 18, no. 8, pp. 957–961, 2005.

[9] C. Guo and Y. Xu, "Existence of periodic solutions for a class of second order differential equation with deviating argument," *Applied Mathematics and Computation*, vol. 28, no. 1-2, pp. 425–433, 2008.

[10] T. Ji and W. Dong, "Existence of periodic solutions for a class of differential equations of second order," in *Applied and Computational Mathematics*, pp. 255–257, WSEAS Press, Athens, Greece, 2008.

[11] D. S. Tian and X. W. Zeng, "Existence of several periodic solutions for a class of second-order functional differential equations," *Acta Mathematica Scientia. Series A*, vol. 30, no. 2, pp. 525–530, 2010.

[12] Q. Li and Y. Li, "On the existence of positive periodic solutions for second-order functional differential equations with multiple delays," *Abstract and Applied Analysis*, vol. 2012, Article ID 929870, 13 pages, 2012.

[13] X. Li and Z. Zhang, "On the existence of positive periodic solutions of systems of second order differential equations," *Mathematische Nachrichten*, vol. 284, no. 11-12, pp. 1472–1482, 2011.

[14] J. Zu, "Existence and uniqueness of periodic solution for nonlinear second-order ordinary differential equations," *Boundary Value Problems*, vol. 2011, Article ID 192156, 11 pages, 2011.

[15] Cemil Tunç and Ramazan Yazgan, "Existence of periodic solutions to multidelay functional differential equations of second order," *Abstract and Applied Analysis*, vol. 2013, Article ID 968541, 5 pages, 2013.

[16] R. Ma and Y. Lu, "Existence of positive periodic solutions for second-order functional differential equations," *Monatshefte für Mathematik*, vol. 173, no. 1, pp. 67–81, 2014.

[17] R. Jia and J. Shao, "Existence and uniqueness of periodic solutions of second-order nonlinear differential equations," *Journal of Inequalities and Applications*, vol. 2013, article 115, 10 pages, 2013.

[18] A. Ardjouni and A. Djoudi, "Existence of positive periodic solutions for a second-order nonlinear neutral differential equation with variable delay," *Advances in Nonlinear Analysis*, vol. 2, no. 2, pp. 151–161, 2013.

[19] A. Ardjouni and A. Djoudi, "Existence of periodic solutions for a second order nonlinear neutral differential equation with variable delay," *Palestine Journal of Mathematics*, vol. 3, no. 2, pp. 191–197, 2014.

[20] Y. Lü, P. Sun, H. Cai, and P.-s. Han, "Existence of multiple positive periodic solutions for second order differential equations," *Communications in Mathematical Research*, vol. 32, no. 3, pp. 272–280, 2016.

[21] D. S. Tian, "Existence of multiple periodic solutions for a second-order delayed differential equation," *Acta Mathematicae Applicatae Sinica*, vol. 39, no. 4, pp. 537–546, 2016.

[22] L. Zhang and Q. Y. Wang, "Existence of periodic solutions for a class of second-order functional differential equations with deviating arguments," *Acta Mathematica Scientia. Series A*, vol. 29, no. 2, pp. 253–261, 2009.

[23] T. F. Zhou, "Existence of periodic solutions for semi-linear second-order differential equations," *Chinese Journal of Engineering Mathematics*, vol. 32, no. 5, pp. 690–696, 2015.

[24] Y. Wei, "Existence and uniqueness of periodic solutions for second order differential equations," *Journal of Function Spaces*, vol. 2014, Article ID 246258, 5 pages, 2014.

[25] L. M. Zhu and Y. X. Li, "Existence of periodic solutions for second-order differential equations with multiple delays," *Journal of Jilin University*, vol. 55, no. 5, pp. 1077–1083, 2017.

[26] T. Yoshizawa, *Stability Theory and The Existence of Periodic Solutions and Almost Periodic Solutions*, Springer, 1975.

[27] C. Tunç and I. Çinar, "On the existence of periodic solutions to nonlinear differential equations of second order," *Differential Equations and Control Processes*, no. 3, pp. 20–25, 2008.

[28] S. Ahmad and M. R. M. Rao, *Theory of Differential Equations*, Affiliated East West Press Private, New Delhi, India, 1999.

[29] T. A. Burton and R. H. Hering, "Liapunov theory for functional-differential equations," *Rocky Mountain Journal of Mathematics*, vol. 24, no. 1, pp. 3–17, 1994.

[30] J. W. Heidel, "Global asymptotic stability of a generalized Liénard equation," *SIAM Journal on Applied Mathematics*, vol. 19, pp. 629–636, 1970.

[31] J. Kato, "On a boundedness condition for solutions of a generalized Liénard equation," *Journal of Differential Equations*, vol. 65, no. 2, pp. 269–286, 1986.

[32] J. Kato, "A simple boundedness theorem for a Liénard equation with damping," *Annales Polonici Mathematici*, vol. 51, pp. 183–188, 1990.

[33] E. Korkmaz and C. Tunç, "Convergence to non-autonomous differential equations of second order," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 1, pp. 27–30, 2015.

[34] B. Liu and L. Huang, "Boundedness of solutions for a class of retarded Liénard equation," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 422–434, 2003.

[35] B. Liu and L. Huang, "Boundedness of solutions for a class of Liénard equations with a deviating argument," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 109–112, 2008.

[36] W. S. Luk, "Some results concerning the boundedness of solutions of Lienard equations with delay," *SIAM Journal on Applied Mathematics*, vol. 30, no. 4, pp. 768–774, 1976.

[37] I. A. Malyseva, "Boundedness of solutions of a Liénard differential equation," *Differentsial'nye Uravneniya*, vol. 15, no. 8, pp. 1420–1426, 1979.

[38] M. Mureşan, "Boundedness of solutions for Liénard type equations," *The Mathematica Journal*, vol. 40(63), no. 2, pp. 243–257, 1998.

[39] O. G. Mustafa and C. Tunç, "Asymptotically linear solutions of differential equations via Lyapunov functions," *Applied Mathematics and Computation*, vol. 215, no. 8, pp. 3076–3081, 2009.

[40] J. E. Nápoles Valdés, "Boundedness and global asymptotic stability of the forced Liénard equation," *Revista de la Unión Matemática Argentina*, vol. 41, no. 4, pp. 47–59, 2000.

[41] J. Sugie and Y. Amano, "Global asymptotic stability of nonautonomous systems of Liénard type," *Journal of Mathematical Analysis and Applications*, vol. 289, no. 2, pp. 673–690, 2004.

[42] J. Sugie, D.-L. Chen, and H. Matsunaga, "On global asymptotic stability of systems of Liénard type," *Journal of Mathematical Analysis and Applications*, vol. 219, no. 1, pp. 140–164, 1998.

[43] C. Tunc, "Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order," *Iranian Journal of Science & Technology*, vol. 30, no. 2, pp. 213–221, 2006.

[44] C. Tunç, "A new boundedness theorem for a class of second order differential equations," *The Arabian Journal for Science and Engineering. Section A. Science*, vol. 33, no. 1, pp. 83–92, 2008.

[45] C. Tunç, "Some stability and boundedness results to nonlinear differential equations of Liénard type with finite delay," *Journal of Computational Analysis and Applications*, vol. 11, no. 4, pp. 711–727, 2009.

[46] C. Tunç, "Some new stability and boundedness results of solutions of Liénard type equations with a deviating argument," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 1, pp. 85–91, 2010.

[47] C. Tunç, "A note on boundedness of solutions to a class of non-autonomous differential equations of second order," *Applicable Analysis and Discrete Mathematics*, vol. 4, no. 2, pp. 361–372, 2010.

[48] C. Tunç, "New stability and boundedness results of Liénard type equations with multiple deviating arguments," *Natsionalnaya Akademiya Nauk Armenii. Izvestiya. Matematika*, vol. 45, no. 4, pp. 47–56, 2010.

[49] C. Tunç, "Boundedness results for solutions of certain nonlinear differential equations of second order," *Journal of the Indonesian Mathematical Society*, vol. 16, no. 2, pp. 115–126, 2010.

[50] C. Tunç, "Stability and boundedness of solutions of non-autonomous differential equations of second order," *Journal of Computational Analysis and Applications*, vol. 13, no. 6, pp. 1067–1074, 2011.

[51] C. Tunç, "A note on the bounded solutions to $x'' + c(t, x, x') + q(t)b(x) = f(t)$," *Applied Mathematics & Information Sciences*, vol. 8, no. 1, pp. 393–399, 2014.

[52] C. Tunç, "On the qualitative behaviors of a functional differential equation of second order," *Applications and Applied Mathematics*, vol. 12, no. 2, pp. 813–842, 2017.

[53] C. Tunç and Y. Dinç, "Qualitative properties of certain nonlinear differential systems of second order," *Journal of Taibah University for Science*, vol. 11, no. 2, pp. 359–366, 2017.

[54] C. Tunc and E. Tunc, "On the asymptotic behavior of solutions of certain second-order differential equations," *Journal of The Franklin Institute*, vol. 344, no. 5, pp. 391–398, 2007.

[55] C. Tunc and O. Tunc, "A note on certain qualitative properties of a second order linear differential system," *Applied Mathematics & Information Sciences*, vol. 9, no. 2, pp. 953–956, 2015.

[56] C. Tunç and O. Tunç, "On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order," *Journal of Advanced Research*, vol. 7, no. 1, pp. 165–168, 2016.

[57] C. Tunç and O. Tunç, "A note on the stability and boundedness of solutions to non-linear differential systems of second order," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, vol. 24, pp. 169–175, 2017.

[58] G.-R. Ye, H.-S. Ding, and X.-L. Wu, "Uniform boundedness of solutions for a class of Liénard equations," *Electronic Journal of Differential Equations*, vol. 2009, no. 97, pp. 1–5, 2009.

[59] Y. Yu and B. Xiao, "Boundedness of solutions for a class of Liénard equation with multiple deviating arguments," *Vietnam Journal of Mathematics*, vol. 37, no. 1, pp. 35–41, 2009.

[60] Y. Yu and C. Zhao, "Boundedness of solutions for a Liénard equation with multiple deviating arguments," *Electronic Journal of Differential Equations*, vol. 2009, no. 14, pp. 1–5, 2009.

[61] B. Zhang, "Boundedness and stability of solutions of the retarded Liénard equation with negative damping," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 20, no. 3, pp. 303–313, 1993.

[62] J. Zhou and L. Xiang, "On the stability and boundedness of solutions for the retarded Liénard-type equation," *Annals of Differential Equations*, vol. 15, no. 4, pp. 460–465, 1999.

[63] X.-f. Zhou and W. Jiang, "Stability and boundedness of retarded Liénard-type equation," *Chinese Quarterly Journal of Mathematics*, vol. 18, no. 1, pp. 7–12, 2003.