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Research Article

Periodic Measures of Mean-Field Stochastic Predator-Prey System

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This paper discusses the dynamics of the mean-field stochastic predator-prey system. We prove the existence and pathwise uniqueness of the solution for stochastic predator-prey systems in the mean-field limit. Then we show that the solution of the mean-field equation is a periodic measure. Finally, we study the fluctuations of the periodic in distribution processes when the white noise converges to zero.

1. Introduction

The predator-prey equations is one of the most famous population models

$$\begin{aligned} \dot{x}(t) &= x(t)\, r\left(1-\frac{x(t)}{K}\right)-\frac{x(t)\, y(t)}{a+x^2(t)}, \\ \dot{y}(t) &= y(t)\left(\frac{\mu x(t)}{a+x^2(t)}-D\right), \end{aligned} \tag{1}$$

where $x(t)$ denotes the prey population density and $y(t)$ denotes the predator population density. The parameters r, K, a, μ, and D are positive real numbers. By the results in Ruan and Xiao [1], they discuss all kinds of bifurcation phenomena. Recently, system (1) was studied extensively that it exhibits complex dynamical phenomena, including bifurcation, stability, and attractive [2–8].

However, population systems in the real world are very often subject to environmental noise [9–15]. According to the Markov jump approach, a classical stochastic predator-prey model can be described by

$$\begin{aligned} dx(t) &= x(t)\, r\left(1-\frac{x(t)}{K}-\frac{y(t)}{r\left(a+x^2(t)\right)}\right)dt \\ &\quad + g_1(x(t))\, dW(t), \\ dy(t) &= y(t)\left(\frac{\mu x(t)}{a+x^2(t)}-D\right)dt + g_2(y(t))\, dW(t), \end{aligned} \tag{2}$$

where W_t is independent Brownian motions. Biologic i in population satisfies the following equation, predator-prey model:

$$\begin{aligned} &dx_{i,N}(t) \\ &\quad = x_{i,N}(t)\, r\left(1-\frac{x_{i,N}(t)}{K}-\frac{y_{i,N}(t)}{r\left(a+x_{i,N}^2(t)\right)}\right)dt \\ &\qquad + g_1\left(x_{i,N}(t)\right) dW_i(t) \\ &\qquad + D_1\left(\frac{1}{N-1}\sum_{j=1, j\neq i}^{N} x_{j,N}(t)-x_{i,N}(t)\right)dt, \\ &dy_{i,N}(t)(t) \\ &\quad = y_{i,N}(t)(t)\left(\frac{\mu x_{i,N}(t)}{a+x_{i,N}^2(t)}-D\right) \\ &\qquad + g_2\left(y_{i,N}(t)\right) dW_i(t) \\ &\qquad + D_2\left(\frac{1}{N-1}\sum_{j=1, j\neq i}^{N} y_{j,N}(t)-y_{i,N}(t)\right)dt, \end{aligned} \tag{3}$$

where $x_{i,N}$ and $y_{i,N}$ denote the population density of x and y in the ith out of N population, D_1 and D_2 are nonnegative real

number modelling the diffusion between the prey population density and the predator population density, and W_i ($i = 1, 2, \ldots, n$) are independent Brownian motions.

Under regularity conditions, for any fixed k, $((x_{1,N}, y_{1,N}), (x_{2,N}, y_{2,N}), \ldots, (x_{k,N}, y_{k,N}))$ converge in law when $N \longrightarrow \infty$, and then system (3) becomes

$$\begin{aligned} dx(t) &= x(t) r\left(1 - \frac{x(t)}{K} - \frac{y(t)}{r(a + x^2(t))}\right) dt \\ &\quad + g_1(x(t)) dW(t) + D_1(Ex(t) - x(t)) dt, \\ dy(t) &= y(t)\left(\frac{\mu x(t)}{a + x^2(t)} - D\right) dt + g_2(y(t)) dW(t) \\ &\quad + D_2(Ey(t) - y(t)) dt, \\ \mathfrak{L}(x(0), y(0)) &= \mu. \end{aligned} \tag{4}$$

where $g_1(x(t))$ and $g_2(x(t))$ are Noise intensity functions and D_1 and D_1 are real number. According to the mathematical approach [16–20], these systems can appear very standardized. However, many real world problems process the nature of mixing randomness and periodicity, e.g., due to change of temperatures on earth, harvesting seasons, seasonal economic data, individual lifecycle, and seasonal effects of weather [15]. Biological populations are very often subject to random perturbations that come in a more-or-less periodic way. A number of random periodic results have been studied in the literature [15, 16, 21–24], but none of them covers (4). And my method can also be extended to other noise, for example, the telephone noise, Markovian switching, and Lévy jumps [25–27].

In the paper, we investigate the dynamics of mean-field stochastic predator-prey system. First, based on martingale approach and Vlassov-Limits, we prove the existence and uniqueness of the solution for mean-field stochastic predator-prey systems, then, by Tihonov's fixed point theorem and martingale techniques, we prove that the solution of the stochastic predator-prey systems in the mean-field limit is a strictly periodic law under some suitable assumptions. Finally, we study the fluctuations of the periodic in distribution processes when the white noise converges to zero.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space. Suppose that r, K, a, μ, and D are positive constants, g_1 and g_2 satisfy a Lipschitz condition with constant L, and g_1 and g_2 are bounded,

$$\begin{aligned} g_1(0) &= 0, \\ g_2(0) &= 0, \\ D_1, D_2 &\geq 0. \end{aligned} \tag{5}$$

Definition 1 (see [28]). Let $\tau > 0$ be fixed. A random H-valued process $\{x(t) : t \in \mathfrak{R}\}$ is called a d-periodic (in distribution) with period τ if

$$\begin{aligned} &\forall n \in \mathbb{N},\ \forall \{t_1, t_2, \ldots, t_n\} \in \mathfrak{R}, \\ &\text{and } \forall \{B_1, B_2, \ldots, B_n\} \in \mathcal{B}(H), \\ &P\left(\bigcap_{k=1}^{n} \{\omega_k : x(\omega, t_k + \tau) \in B_k\}\right) \\ &\quad = P\left(\bigcap_{k=1}^{n} \{\omega_k : x(\omega, t_k) \in B_k\}\right), \end{aligned} \tag{6}$$

where $B(H)$ is the Borel a-algebra in H.

Let $w := \{w(t) : t \in \mathfrak{R}\}$ be an H-valued Wiener process with $P\{w(0) = 0\} = 1$. Note that, for any $\{z_1, z_2\} \in H$ and $0 < s < t$,

$$E\left((w(t), z_1), \overline{(w(s), z_2)}\right) = s(z_2, Wz_1) \tag{7}$$

with a nuclear operator W and

$$E\|w(t) - w(s)\|^2 = |t - s| \operatorname{tr} W, \quad \{t, s\} \in \mathfrak{R}, \tag{8}$$

where $\mathcal{F}_t : \sigma(w(v) - w(u) : u \leq v \leq t),\ t \in \mathfrak{R}$.

Theorem 2 (see [1]). *If $\mu^2 > (16/3)aD^2$ and $(u + \sqrt{\mu^2 - 4aD^2})/2D > K > (2u - \sqrt{\mu^2 - 4aD^2})/2D$, then system (1) has three equilibria: two hyperbolic saddles $(0, 0)$ and $(K, 0)$ and an unstable focus (or node) $((u - \sqrt{\mu^2 - 4aD^2})/2D, r(1 - (u - \sqrt{\mu^2 - 4aD^2})/2KD)(a + ((u - \sqrt{\mu^2 - 4aD^2})/2D)^2))$ in the interior of the first quadrant. Moreover, system (1) has a unique limit cycle, which is stable.*

3. Existence and Uniqueness

In the section, under some suitable assumptions, we prove the existence and uniqueness of the solution for mean-field stochastic predator-prey systems.

Theorem 3. *For every $N \geq 2$, $N \in \mathbb{N}$, let ν denote a probability measure on $[0, \infty) \times [0, \infty)$ such that*

$$\begin{aligned} &\iint y^4 d\nu(x, y) < \infty \\ &\textit{and } \iint \exp(\gamma x^2) d\nu(x, y) < \infty \\ &\qquad \textit{for some } \gamma > 0. \end{aligned} \tag{9}$$

(i) Then there exists a unique global strong solution of system (3).

(ii) Then there exists a unique nonnegative solution of system (3) satisfying $\mathfrak{L}(x(0), y(0)) = \mu$, $r \leq 1, \mu \leq 1$, and $\int_0^t (Ex(s) + Ey(s)) ds < \infty$ for all $t \geq 0$.

Proof. (i) To show that this solution is global, we prove that it does not explode in finite time.

Let

$$z_{i,N} = x_{i,N} + y_{i,N}. \tag{10}$$

From (3), we have

$$\begin{aligned} dz_{i,N} = {} & x_{i,N}(t)\, r\left(1 - \frac{x_{i,N}(t)}{K} - \frac{y_{i,N}(t)}{r\left(a + x_{i,N}^2(t)\right)}\right) dt \\ & + g_1\left(x_{i,N}(t)\right) dW_i(t) \\ & + D_1\left(\frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_{j,N}(t) - x_{i,N}(t)\right) dt \\ & + y_{i,N}(t)(t)\left(\frac{\mu x_{i,N}(t)}{a + x_{i,N}^2(t)} - D\right) \\ & + g_2\left(y_{i,N}(t)\right) dW_i(t) \\ & + D_2\left(\frac{1}{N-1}\sum_{j=1,j\neq i}^{N} y_{j,N}(t) - y_{i,N}(t)\right) dt. \end{aligned} \tag{11}$$

For $m \in \mathbb{N}$, define $H_m^{1,2} : \mathfrak{R}^2 \longrightarrow [0,\infty)$ by

$$\begin{aligned} & H_m^1(x,y) \\ & = \begin{cases} \dfrac{xy}{a+x^2} + rx^2 & \text{if } |x|, |y| \le m \\ \text{Lipschitz, bounded, nonnegative} & \text{otherwise} \end{cases} \\ & H_m^2(x,y) \\ & = \begin{cases} \dfrac{\mu xy}{a+x^2} & \text{if } |x|, |y| \le m \\ \text{Lipschitz, bounded, nonnegative} & \text{otherwise} \end{cases} \end{aligned} \tag{12}$$

and $g_{i,m}$ $(i = 1, 2)$ by

$$g_{i,m}(x) = \begin{cases} g_i(x) & \text{if } 0 \le x \le m \\ \text{Lipschitz, bounded,} & \text{otherwise.} \end{cases} \tag{13}$$

Applying Itô's formula, we have

$$\begin{aligned} d\left(z_{i,N}^{(m)}(t)\right)^2 \le {} & 2z_{i,N}^{(m)} x_{i,N}^{(m)}(t)\, r dt \\ & + 2z_{i,N}^{(m)} g_1\left(x_{i,N}^{(m)}(t)\right) dW_i(t) \\ & + 2D_1 z_{i,N}^{(m)} \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_{j,N}^{(m)}(t)\, dt \\ & + g_1^2\left(x_{i,N}^{(m)}(t)\right) dt \\ & + 2z_{i,N}^{(m)} g_2\left(y_{i,N}(t)\right) dW_i(t) \\ & + 2D_2 z_{i,N}^{(m)} \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} y_{j,N}^{(m)}(t)\, dt \\ & + g_2^2\left(y_{i,N}^{(m)}(t)\right) dt \\ \le {} & 2r\left(z_{i,N}^{(m)}\right)^2 dt \\ & + 2z_{i,N}^{(m)} g_1\left(x_{i,N}^{(m)}(t)\right) dW_i(t) \\ & + 2D_1 \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} z_{j,N}^{(m)}(t)\, z_{i,N}^{(m)}(t)\, dt \\ & + g_1^2\left(x_{i,N}^{(m)}(t)\right) dtr \\ & + 2z_{i,N}^{(m)} g_2\left(y_{i,N}(t)\right) dW_i(t) \\ & + 2D_2 \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} z_{j,N}^{(m)} z_{i,N}^{(m)}(t)\, dt \\ & + g_2^2\left(y_{i,N}^{(m)}(t)\right) dt. \end{aligned} \tag{14}$$

Then, we get

$$\begin{aligned} E\left(z_{i,N}^{(m)}(t)\right)^2 \le {} & E\left(z_{i,N}^{(m)}(0)\right)^2 + 2r\int_0^t E\left(z_{i,N}^{(m)}(s)\right)^2 ds \\ & + \int_0^t E\left(g_1^2\left(x_{i,N}^{(m)}(s)\right) + g_2^2\left(y_{i,N}^{(m)}(s)\right)\right) ds \\ & + 2\left(D_1 + D_2\right)\frac{1}{N-1} \\ & \cdot \int_0^t \sum_{j=1,j\neq i}^{N} E\left[z_{j,N}^{(m)}(s)\, z_{i,N}^{(m)}(s)\right] ds. \end{aligned} \tag{15}$$

Since the law of $(x_{i,N}^{(m)}(t), y_{i,N}^{(m)}(t))$ is symmetric $(i = 1, 2, \ldots, N)$, we get

$$\begin{aligned} & E\left(z_{1,N}^{(m)}(t)\right)^2 \\ & \quad \le E\left(z_{1,N}^{(m)}(0)\right)^2 + 2r\int_0^t E\left(z_{1,N}^{(m)}(s)\right)^2 ds \\ & \qquad + 2L\int_0^t E\left(z_{1,N}^{(m)}(s)\right)^2 ds \\ & \qquad + 2\left(D_1 + D_2\right)\int_0^t E\left(z_{1,N}^{(m)}(s)\right)^2 ds \\ & \quad = E\left(z_{1,N}^{(m)}(0)\right)^2 \\ & \qquad + 2\left(r + L + D_1 + D_2\right)\int_0^t E\left(z_{1,N}^{(m)}(s)\right)^2 ds \end{aligned} \tag{16}$$

Applying the Bellman inequality, we have

$$E\left(z_{1,N}^{(m)}(t)\right)^2 \le E\left(z_{1,N}^{(m)}(0)\right)^2 \times \left(1 + 2\left(r+L+D_1+D_2\right)\int_0^t e^{2(r+L+D_1+D_2)(t-s)}ds\right). \tag{17}$$

Due to $E(z_{1,N}^{(m)}(0))^2 = E(z_{1,N}(0))^2$ and from [29], then $(x_{1,N}^{(m)}(t), y_{1,N}^{(m)}(t))$ converge weakly to $(x_{1,N}(t), y_{1,N}(t))$ when $m \longrightarrow \infty$; by Fatou's lemma, we obtain that

$$\sup_{N\ge 2}\sup_{0\le t\le T} E\left(z_{1,N}(t)\right)^2 < \infty \quad \text{for all } T > 0. \tag{18}$$

Then we have

$$P\left\{x_{i,N} \ge 0,\ y_{i,N} \ge 0 \text{ for all } t \ge 0\right\} = 1. \tag{19}$$

Therefore, we prove the first assertion of the theorem.

(ii) Next, we will prove that there exists a unique nonnegative solution of system (3).

Let $\int_0^T Ex(s) + Ey(s)ds < \infty$ for all $T > 0$; $(x(t), y(t))$ denotes nonnegative real solution of (4). Set $a(t) = Ex(t)$, $b(t) = Ey(t)$, and $h_\alpha(g_1(x(t)))$ where

$$h_\alpha(x) = \begin{cases} x & \text{if } |x| \le \alpha,\ x \in \mathfrak{R} \\ \alpha & \text{if } |x| \ge \alpha,\ x \in \mathfrak{R}. \end{cases} \tag{20}$$

Then $(x_\alpha(t), y_\alpha(t))$ denote the real solution of (4) with $h_\alpha(g_1(x(t)))$ instead of $g_1(x(t))$.

Let

$$z(t) = x(t) + y(t) \quad \text{and } z_\alpha(t) = x_\alpha(t) + y_\alpha(t). \tag{21}$$

Applying Itô's formula, we get

$$\begin{aligned} z_\alpha(t) &= z_\alpha(0) \\ &+ \int_0^t x_\alpha(s) r\left(1 - \frac{x_\alpha(s)}{K} - \frac{y_\alpha(s)}{r\left(a + x_\alpha^2(s)\right)}\right)ds \\ &+ \int_0^t h_\alpha\left(g_1(x(t))\right)dW(s) \\ &+ D_1\int_0^t \left(a(s) - x_\alpha(s)\right)ds \\ &+ D_2\int_0^t \left(b(s) - y_\alpha(s)\right)ds \\ &+ \int_0^t y_\alpha(s)\left(\frac{\mu x_\alpha(s)}{a + x_\alpha^2(s)} - D\right)ds \\ &+ \int_0^t g_2\left(y_\alpha(s)\right)dW(s) \end{aligned} \tag{22}$$

and

$$Ez_\alpha(t) \le Ez_\alpha(0) + D_1\int_0^t a(s)\,ds + D_2\int_0^t b(s)\,ds. \tag{23}$$

Then, we get

$$\sup_{\alpha>0}\sup_{0\le t\le T} Ez_\alpha(t) < \infty \tag{24}$$

and

$$\begin{aligned} Ez_\alpha(t) &+ \left(D_1 - r\right)\int_0^t Ex_\alpha(s)\,ds + D_2\int_0^t Ey_\alpha(s)\,ds \\ &\le Ez_\alpha(0) + D_1\int_0^t a(s)\,ds + D_2\int_0^t b(s)\,ds. \end{aligned} \tag{25}$$

When $\alpha \longrightarrow \infty$, then $(x_\alpha(\cdot), y_\alpha(\cdot))$ converge in law to $(x(\cdot), y(\cdot))$ in $C([0,+\infty), \mathfrak{R}_+^2)$; by dominated convergence, we have $Ez(t) \le Ez(0)$.

To prove the uniqueness of solution, it will prove that there are some $\varepsilon > 0$ and some $\widetilde{\gamma} > 0$ such that

$$\sup_{0\le t\le 1} E\exp\left(\delta x^2\right) < \infty \quad \text{and } \sup_{0\le t\le 1} Ez^4(t) < \infty \tag{26}$$

By iteration method, next, we prove the pathwise uniqueness on $[0,\infty)$.

Firstly. Applying Itô's formula to x, we get

$$\begin{aligned} x^n(t) &= x^n(0) \\ &+ \int_0^t nrx^n(s)\left(1 - \frac{x(s)}{K} - \frac{y(s)}{r\left(a + x^2(s)\right)}\right)ds \\ &+ \int_0^t nx^{n-1}(s)\,g_1(x(s))\,dW(s) \\ &+ nD_1\int_0^t x^{n-1}(s)\left(Ex(s) - x(s)\right)ds \\ &+ \frac{n(n-1)}{2}\int_0^t x^{n-2}(s)\,g_2^2(x(s))\,ds. \end{aligned} \tag{27}$$

Let $\tau_\alpha := \inf\{t \ge 0 : z(t) \ge \alpha\}$, $B := \sup_{x\ge 0} g_2^1(x)$, and

$$x^{\alpha,n}(t) = x^n\left(t \wedge \tau_\alpha\right). \tag{28}$$

Then, we have

$$\begin{aligned} Ex^{\alpha,n}(t) &\le Ex^n(0) + E\int_0^{t\wedge\tau_\alpha} nrx^n(s)\,ds \\ &+ \frac{n(n-1)B}{2}E\int_0^{t\wedge\tau_\alpha} x^{n-2}(s)\,ds \\ &+ nD_1E\int_0^{t\wedge\tau_\alpha} x^{n-1}(s)\,Ez(0)\,ds. \end{aligned} \tag{29}$$

For all $1 \le k \le n-1$, by $\sup_{0\le t\le 1} Ex^k < \infty$ and Fatou's theorem, when $\alpha \longrightarrow \infty$, we have $\sup_{0\le t\le 1} Ex^n < \infty$, and

$$\begin{aligned} Ex^{2n}(t) \le{}& Ex^{2n}(0) + 2nr\int_0^t x^{2n}(s)\,ds \\ &+ n(2n-1)B\int_0^t Ex^{2n-2}(s)\,ds \\ &+ 2nD_1\int_0^t x^{2n-1}(s)\,Ez(0)\,ds. \end{aligned} \tag{30}$$

Since $x^{2n-1} \le x^{2n}+1$, it shows that (for $n \ge 1$)

$$\begin{aligned} Ex^{2n}(t) \le{}& Ex^{2n}(0) + 2nD_1 z(0)\,T \\ &+ 2n\left(r + D_1 Ez(0)\right)\int_0^t Ex^{2n}(s)\,ds \\ &+ n(2n-1)B\int_0^t Ex^{2n-2}(s)\,ds. \end{aligned} \tag{31}$$

Let $r > 0$ so that

$$E\left(x_0^2 + r\right)^n \ge Ex^{2n}(0) + 2nD_1 z(0)\,T. \tag{32}$$

Then from (31), we obtain that

$$\begin{aligned} Ex^{2n}(t) \le{}& E\left(x_0^2 + r\right)^n \\ &+ 2n\left(r + D_1 Ez(0)\right)\int_0^t Ex^{2n}(s)\,ds \\ &+ n(2n-1)B\int_0^t Ex^{2n-2}(s)\,ds. \end{aligned} \tag{33}$$

Now, we consider the equation

$$dX = \left(r + D_1 Ez(0)\right)X dt + B^{1/2} dw_t. \tag{34}$$

By the Itô formula, we have

$$\begin{aligned} EX^{2n}(t) ={}& E\left(X_0^2 + r\right)^n \\ &+ 2n\left(r + D_1 Ez(0)\right)\int_0^t EX^{2n}(s)\,ds \\ &+ n(2n-1)B\int_0^t EX^{2n-2}(s)\,ds. \end{aligned} \tag{35}$$

By (34) and (35), as $n = 1$, we infer that

$$\begin{aligned} Ex^2(t) \le{}& E\left(x_0^2 + r\right)^n + Bt \\ &+ 2\left(r + D_1 Ez(0)\right)\int_0^t Ex^2(s)\,ds. \end{aligned} \tag{36}$$

$$\begin{aligned} EX^2(t) \le{}& E\left(X_0^2 + r\right)^n + Bt \\ &+ 2\left(r + D_1 Ez(0)\right)\int_0^t EX^2(s)\,ds. \end{aligned} \tag{37}$$

By Gronwall's inequality, we get

$$\begin{aligned} Ex^2(t) \le{}& E\left(x_0^2 + r\right)^n + Bt + 2\left(r + D_1 Ez(0)\right) \\ &\cdot \int_0^t e^{2(r+D_1 Ez(0))(t-s)}\left(E\left(x_0^2 + r\right)^n + Bs\right)ds \\ ={}& EX^2(t). \end{aligned} \tag{38}$$

From (33) and (35), by Gronwall's inequality, we get

$$Ex_t^{2n} \le EX_t^{2n}, \quad n \ge 1,\ 0 \le t \le T. \tag{39}$$

Hence, if for some $\delta > 0$, $Ee^{\delta X_t^2} < \infty$, then $Ee^{\delta x_t^2} \le Ee^{\delta X_t^2} < \infty$.

Note that if $Ee^{\varepsilon x_0^2} < \infty$ for some ε, then we have

$$Ee^{\varepsilon X_0^2} = e^{\varepsilon r} Ee^{\varepsilon x_0^2} < \infty, \tag{40}$$

and therefore there exists $\delta = \delta(T) > 0$ such that

$$\sup_{0\le t\le 1} Ee^{\delta X^2} < \infty. \tag{41}$$

Secondly. To prove $\sup_{0\le t\le 1} Ez^4 < \infty$, defining $\tau_\alpha = \inf\{t \ge 0 : z(t) \ge \alpha\}$, then, for $i = 2, 3, 4$ and $0 \le t \le 1$,

$$\begin{aligned} z^i(t) &= z^i(0) \\ &\quad + i\int_0^t z^{i-1} x(s)\, r\left(1 - \frac{x(s)}{K} - \frac{y(s)}{r\left(a + x^2(s)\right)}\right)ds \\ &\quad + i\int_0^t z^{i-1} g_1(x(t))\,dW(s) \\ &\quad + D_1 i\int_0^t z^{i-1}\left(Ex(s) - x(s)\right)ds \\ &\quad + D_2 i\int_0^t z^{i-1}\left(Ey(s) - y(s)\right)ds \\ &\quad + i\int_0^t z^{i-1} y(s)\left(\frac{\mu x_\alpha(s)}{a + x^2(s)} - D\right)ds \\ &\quad + \int_0^t g_2(y(s))\,dW(s) \\ &\quad + \frac{1}{2} i(i-1)\int_0^t z^{i-2}\left(g_1^2(x(s)) + g_2^2(y(s))\right)ds, \end{aligned} \tag{42}$$

and for $z^{(\alpha,i)} = z^i(t \wedge \tau_\alpha)$ and for all $k \le i-1$, let $\sup_{0\le t\le 1} Ez^k$, then we have

$$\begin{aligned} Ez^{(\alpha,i)}(t) &= Ez^{(\alpha,i)}(0) \\ &\quad + 4\int_0^t z^{(\alpha,i-1)}(s)\, x(s)\left(r + \left(D_1 + D_2\right)Ez(0)\right)ds \\ &\quad + 6L^2\int_0^t z^{(\alpha,i)}(s)\,ds \end{aligned}$$

$$\begin{aligned}
&\le Ez^{(\alpha,i)}(0) \\
&\quad + 4\int_0^t z^{(\alpha,i)}(s)\left(r + \left(D_1 + D_2\right)Ez(0)\right)ds \\
&\quad + 6L^2\int_0^t z^{(\alpha,i)}(s)\,ds \\
&= Ez^{(\alpha,i)}(0) \\
&\quad + \int_0^t z^{(\alpha,i)}(s)\left(4r + 6L + 4\left(D_1 + D_2\right)Ez(0)\right)ds.
\end{aligned} \tag{43}$$

By Gronwall's inequality and Fatou's theorem, when $\alpha \longrightarrow \infty$, we get

$$\sup_{0\le t\le 1} Ez^4 < \infty. \tag{44}$$

Finally. Let $(x(t), y(t))$ and $(\tilde{x}(t), \tilde{y}(t))$ denote two solutions of (4) on the same space. Ones already showed that

$$\sup_{0\le t\le 1}\left(Ex^4(t) + Ey^4(t) + E\tilde{x}^4(t) + E\tilde{y}^4(t)\right) < \infty. \tag{45}$$

Set

$$\begin{aligned}
c(s) &= Ey(s) - E\tilde{y}(s), \\
d(s) &= Ex(s) - E\tilde{x}(s), \\
\overline{x}(s) &= x(s) - \tilde{x}(t), \\
\overline{y}(s) &= y(s) - \tilde{y}(t), \\
\overline{z}(s) &= z(s) - \tilde{z}(t).
\end{aligned} \tag{46}$$

By truncation technique, we get

$$\begin{aligned}
d\overline{x}^2(t) &= 2\overline{x}(t)\,d\overline{x}(t) + \left(g_1(x(t)) - g_1(\tilde{x}(t))\right)^2 dt \\
&= 2\overline{x}(t)\left(r\overline{x}(t) - \frac{rx^2(t)}{K} - \frac{y(t)x(t)}{(a + x^2(t))} + \frac{r\tilde{x}^2(t)}{K}\right. \\
&\quad \left. + \frac{\tilde{x}(t)\tilde{y}(t)}{(a + \tilde{x}^2(t))}\right)dt + 2\overline{x}(t)\left(g_1(x(t))\right. \\
&\quad \left. - g_1(\tilde{x}(t))\right)dW(t) + 2\overline{x}(t)\left(D_1 d(t)\right. \\
&\quad \left. - D_1\overline{x}(t)\right)dt + \left(g_1(x(t)) - g_1(\tilde{x}(t))\right)^2 dt,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
E\overline{x}^2(t) &\le E2\overline{x}(t)\left(r\overline{x}(t) - \frac{rx^2(t)}{K} - \frac{y(t)x(t)}{(a + x^2(t))}\right. \\
&\quad \left. + \frac{r\tilde{x}^2(t)}{K} + \frac{\tilde{x}(t)\tilde{y}(t)}{(a + \tilde{x}^2(t))}\right)dt + 2D_1\int_0^t d^2(s)\,ds \\
&\quad - 2D_1\int_0^t E\overline{x}^2(s)\,ds + L^2\int_0^t E\overline{x}^2(s)\,ds.
\end{aligned} \tag{48}$$

Since

$$d^2(s) \le E\overline{x}^2(t) \tag{49}$$

then we have

$$\begin{aligned}
E\overline{x}^2(t) &\le \int_0^t 2E\overline{x}(s)\left(\frac{r\tilde{x}^2(s)}{K} - \frac{rx^2(s)}{K}\right. \\
&\quad \left. + \frac{\tilde{x}(s)\tilde{y}(s)}{(a + \tilde{x}^2(s))} - \frac{y(t)x(s)}{(a + x^2(s))}\right)ds + \left(L^2\right. \\
&\quad \left. + 2r\right)\int_0^t E\overline{x}^2(s)\,ds \le \int_0^t 2E\overline{x}(s)\frac{M^2(s)}{a}ds + \left(L^2\right. \\
&\quad \left. + 2r\right)\int_0^t E\overline{x}^2(s)\,ds,
\end{aligned} \tag{50}$$

where $M(s) = \max\{\tilde{x}(s), \tilde{y}(s)\}$. Furthermore, we have

$$\begin{aligned}
d\overline{z}^2(t) &\le 2r\overline{z}\left[\overline{x}(s) + D_1 d(t) + D_2 c(t) - D_1\overline{x}(t)\right. \\
&\quad \left. - D_2\overline{y}(t)\right]dt + 2\overline{z}\left(g_1(x) - g_1(\tilde{x})\right)dW(t) \\
&\quad + 2\overline{z}\left(g_2(y) - g_2(\tilde{y})\right)dW(t) \\
&\quad + \left[\left(g_1(x) - g_1(\tilde{x})\right)^2 + \left(g_2(y) - g_2(\tilde{y})\right)^2\right]dt.
\end{aligned} \tag{51}$$

Therefore, we have

$$\begin{aligned}
E\overline{z}^2(t) &\le \left(2r + 1 + 2D_1 + 2D_2 + K^2\right)\int_0^t E\overline{x}^2(s)\,ds \\
&\quad + \left(1 + 2D_1 + 2D_2 + K^2\right)\int_0^t E\overline{y}^2(s)\,ds.
\end{aligned} \tag{52}$$

Let

$$H(s) = E\overline{x}^2(s) + E\overline{y}^2(s), \tag{53}$$

we get

$$\begin{aligned}
H(t) &= E\left(\overline{z}(t) - \overline{x}(t)\right)^2 + E\overline{x}^2(s) \\
&\le 2E\overline{z}^2(t) + 3E\overline{x}^2(s) \\
&\le k_1\int_0^t H(s)\,ds + \frac{6}{a}\int_0^t E\tilde{y}(s)M^2(s)\,ds,
\end{aligned} \tag{54}$$

where

$$k_1 = 10r + 2 + 4D_1 + 4D_2 + 2K^2 + 3L^2. \tag{55}$$

For $n > 0$, $s \le 1$, let $q = 1 + 1/n$ and $p = n + 1$. Then $1/p + 1/q = 1$. By Gronwall's inequality, we have

$$\begin{aligned}
EM^2(s)\overline{x}(s) &= EM^2(s)\overline{x}^{(q-1)/q}(s)\overline{x}^{1/q}(s) \\
&\le E\left(M^{2p}(s)\overline{x}(s)\right)^{1/p}\left(E\overline{x}(s)\right)^{1/q} \\
&\le E\left(M^{4p}(s)\right)^{1/2p}\left(E\overline{x}^2(s)\right)^{1/2p}\left(E\overline{x}(s)\right)^{1/q} \\
&\le k_2\alpha_p\left(E\overline{x}^2(s)\right)^{1/2p},
\end{aligned} \tag{56}$$

where

$$\alpha_p = \sup_{0\le s\le 1} E\left(M^{4p}\right)^{1/2p},$$
$$k_2 = \sup_{0\le s\le 1} (E\overline{x}(s))^{1/q} + 1 < \infty. \tag{57}$$

By Gronwall's inequality, for $t \in [0,1]$, we showed that

$$H(t) \le k_1 \int_0^t H(s)\,ds + \frac{6k_2\alpha_p}{a}\int_0^t (H(s))^{n/(n+1)}\,ds. \tag{58}$$

$D(\cdot)$ is continuous, and by Hölder's inequality, we get

$$0 \le H(t) \le \left[\left(k_1 + \frac{6k_2\alpha_{n+1}}{a}\right)\frac{1}{n+1}t\right]^{n+1} \quad \text{for } 0 \le t \le \varepsilon_1. \tag{59}$$

For some $\varepsilon_1 \ge \varepsilon_2 > 0$, it is easily seen that there is a sequence $(n_k)_{k\in\mathbb{N}} \longrightarrow \infty (k \longrightarrow \infty)$ such that $\sup H(t) \longrightarrow 0$ when $n_k \longrightarrow \infty$ uniformly in $t \in [0,\varepsilon_2]$ iff

$$\liminf_{n\longrightarrow\infty}\left(k_1 + \frac{6k_2\alpha_{n+1}}{a}\right)\frac{1}{n+1} < \infty \tag{60}$$

and

$$\liminf_{n\longrightarrow\infty}\frac{1}{n}\sup_{0\le s\le\varepsilon_2}\left(EM^{2n}(s)\right)^{1/n} < \infty. \tag{61}$$

By Chebychev's inequality, we have, for $s \in [0,\varepsilon_2]$,

$$P\{M(s) \ge \sigma\} \le P\{x(s) \ge \sigma\} + P\{\tilde{x}(s) \ge \sigma\} \le k_3 e^{-\delta\sigma^2}, \tag{62}$$

where

$$k_3 = \sup_{s\in[0,\varepsilon_2]} E\exp\left(\delta x^2(t)\right) + \sup_{s\in[0,\varepsilon_2]} E\exp\left(\delta\tilde{x}^2(t)\right) < \infty. \tag{63}$$

Hence, we have, for $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{n}\left(EM^{2n}(s)\right)^{1/n} &= \frac{1}{n}\left(\int_0^\infty P\left\{M^{2n}(s) \ge \sigma\right\}d\sigma\right)^{1/n}\\ &\le \frac{1}{n}\left(k_3\int_0^\infty e^{-\delta\sigma^{1/n}}d\sigma\right)^{1/n}\\ &= \frac{1}{n}\left(k_3\int_0^\infty e^{-\delta u}nu^{n-1}du\right)^{1/n}\\ &= \frac{1}{n}\left(k_3 n!\delta^{-n}\right)^{1/n}.\end{aligned} \tag{64}$$

Since $n! < n^n$ for $n \in \mathbb{N}$, we get

$$\frac{1}{n}\left(EM^{2n}(s)\right)^{1/n} \le k_3^{1/n}\delta^{-1}. \tag{65}$$

In summary, we proved the theorem on $[0,\infty)$. □

Theorem 4. *Suppose that Theorem 3 is satisfied. When $N \longrightarrow \infty$, then*

$$\left((x_{1,N}, y_{1,N}), (x_{2,N}, y_{2,N}), \ldots, (x_{k,N}, y_{k,N})\right) \tag{66}$$

converge to k independent copies of solutions of (4) in $C([0,\infty), \mathfrak{R}^{2k})$.

Proof. Let P_N be the law of $(1/N)\sum_{i=1}^N \varepsilon_{x_{i,N(\cdot)}, x_{i,N(\cdot)}}$, where ν_a is the measure

$$\nu_a = \begin{cases} 1 & a \in A \\ 0 & \text{otherwise.}\end{cases} \tag{67}$$

Then $\{P_N, N \in \mathbb{N}, N \ge 2\}$ is relatively compact and $(\widetilde{P}_N = \mathscr{L}(x_{1,N}, y_{1,N}))_{N=2,3,\ldots}$ is a tight family. So, we have

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\sup_{N\ge 2}\widetilde{P}_N\\ &\quad\cdot\left\{\sup_{0\le s<t\le T, t-s<\epsilon}|x(t) - x(s)| + \left|y(t) - y(s)\right| > \rho\right\}\\ &= 0.\end{aligned} \tag{68}$$

Let

$$\begin{aligned}\overline{x}_N &= \frac{1}{N}\sum_{i=1}^N x_{i,N}(t),\\ \overline{y}_N &= \frac{1}{N}\sum_{i=1}^N y_{i,N}(t),\\ \text{and } \overline{z}(t) &= \overline{x}(t) + \overline{y}(t).\end{aligned} \tag{69}$$

Next, we will prove that

$$\overline{z}(t) + \int_0^t \overline{z}(s)\,ds \tag{70}$$

is a submartingale. It is easy to see that

$$\begin{aligned}\overline{z}(t) \le \overline{z}(0) &+ \int_0^t\left(\frac{rK}{4} - D\overline{y}_N(s)\right)ds\\ &+ \int_0^t \frac{1}{N}\sum_{i=1}^N g_1\left(x_{i,N}(t)\right)dW(s)\\ &+ \int_0^t \frac{1}{N}\sum_{i=1}^N g_2\left(y_{i,N}(s)\right)dW(s).\end{aligned} \tag{71}$$

By martingale theory, we get

$$\overline{z}(t) - \overline{z}(0) - \int_0^t\left(\frac{rK}{4} - D\overline{y}_N(s)\right)ds \tag{72}$$

is a martingale. Therefore, we have

$$\overline{z}(t) + D\int_0^t \overline{z}_N(s)\,ds \ge 0 \tag{73}$$

is a submartingale. By martingale inequality [29], we get

$$\begin{aligned} &P_N\left\{\sup_{0\le t\le T}\overline{z}(t)\ge\alpha>0\right\}\\ &\le P_N\left\{\sup_{0\le t\le T}\left(\overline{z}(t)+D\int_0^t\overline{z}_N(s)\right)\ge\alpha>0\right\}\\ &\le\frac{1}{\alpha}E\left(\overline{z}(T)+D\int_0^T\overline{z}_N(s)\right)\\ &\le\frac{1}{\alpha}\left((DT+1)\overline{z}(0)+\frac{rK}{4}\left(T+\frac{T^2}{2}\right)\right). \end{aligned}\tag{74}$$

Furthermore, we have that

$$\begin{aligned} &z_{1,N}(t)-z_{1,N}(0)-\int_0^t\left(\frac{rK}{4}-Dy_{1,N}(s)\right)ds\\ &\quad+D_1\left(\frac{1}{N-1}\sum_{i=2}^N x_{i,N}-x_{1,N}\right)\\ &\quad+D_2\left(\frac{1}{N-1}\sum_{i=2}^N y_{i,N}-y_{1,N}\right) \end{aligned}\tag{75}$$

is a martingale, then we get that

$$z_{1,N}(t)+D(1+D_1+D_1)\int_0^t z_{1,N}(s)\,ds\ge0\tag{76}$$

is a submartingale. Furthermore, we have

$$\begin{aligned} &P_N\left\{\sup_{0\le t\le T}\overline{z}_{1,n}(t)\ge\alpha\right\}\\ &\le\frac{1}{\alpha}\left((TD(1+D_1+D_1)+1)\overline{z}_{1,N}(0)\right.\\ &\quad\left.+\frac{rK}{4}\left(T+\frac{T^2}{2}+D(1+D_1+D_1)\right)\right). \end{aligned}\tag{77}$$

So, for every $T>0$, it is easy to see that

$$\lim_{\alpha\uparrow\infty}\sup_{N\le2}P_N\left\{\sup_{0\le t\le T}\overline{z}(t)\ge\alpha\right\}=0\tag{78}$$

and

$$\lim_{\alpha\uparrow\infty}\sup_{N\le2}P_N\left\{\sup_{0\le t\le T}\overline{z}_{1,N}(t)\ge\alpha\right\}=0.\tag{79}$$

Let

$$\tau_{\alpha,N}=\inf\left\{t\ge0\mid\overline{z}(t)\ge\alpha\text{ or }\overline{z}_{1,N}(t)\ge\alpha\right\}\tag{80}$$

and $\tau_{\alpha,N}=\infty$ if there does not exist such t. For any $T>0$, $\rho,>0$, $\sigma>0$ and $\alpha>0$, it is easy to see that

$$\begin{aligned} &\overline{P}_N\left\{\sup_{0\le s<t\le T,t-s<\sigma}|x(t)-x(s)|+|y(t)-y(s)|>\rho\right\}\\ &\le P_N\{\tau_{\alpha,N}\le T\}+P_N\left\{\sup_{0\le s<t\le T,t-s<\sigma}\left|x_{1,N}(t\wedge\tau_{\alpha,N})\right.\right.\\ &\quad\left.-x_{1,N}(s\wedge\tau_{\alpha,N})\right|+\left|y_{1,N}(t\wedge\tau_{\alpha,N})-y_{1,N}(s\right.\\ &\quad\left.\left.\wedge\tau_{\alpha,N})\right|>\rho\right\}. \end{aligned}\tag{81}$$

Let $f\in\mathscr{C}_0^\infty(\mathfrak{R}^2)$, $p\in\mathbb{N}$; it is easy to see that $\overline{g}_1,\overline{g}_2,\ldots,\overline{g}_p$ are continuous bounded function from $\mathfrak{R}^2$ to $\mathfrak{R}$ and set $0\le s_p<s_{p-1}<\ldots<s_1\le s<t$. Let

$$\begin{aligned} M_0=\left\{\nu\in M:\sup_{0\le u<t}\int(x(\omega,u)+y(\omega,u))\,d\nu(\omega)\right.\\ \left.<\infty\right\} \end{aligned}\tag{82}$$

and for $\nu\in M_0$

$$\begin{aligned} F(\nu)=\Big\langle\nu,\Big(&f(x(t),y(t))-f(x(s),y(s))\\ &-\int_s^t\int\mathscr{A}f(x(\cdot,u),y(\cdot,u),x(\omega,u),y(\omega,u))\,\nu(d\omega)\,du\Big)\\ &\cdot\prod_{j=1}^p\overline{g}_j(x(\cdot,s_j),y(\cdot,s_j))\Big\rangle \end{aligned}\tag{83}$$

where

$$\begin{aligned} \mathscr{A}f(x,y,x',y')=&\frac{\partial f}{\partial x}(x,y)\\ &\cdot\left(rx\left(1-\frac{x}{K}-\frac{y}{r(a+x^2)}\right)+D_1(x'-x)\right)\\ &+\frac{\partial f}{\partial y}(x,y)\left(y\left(\frac{\mu x}{a+x^2}-D\right)+D_2(y'-y)\right)+\frac{1}{2}\\ &\cdot g_1^2(x)\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{1}{2}g_2^2(y)\frac{\partial^2 f}{\partial y^2}(x,y). \end{aligned}\tag{84}$$

Defining

$$\begin{aligned} \overline{x}(u)&=\frac{1}{N_k}\sum_{i=1}^{N_k}x_{i,N_k}(u)\\ \text{and }\overline{y}(u)&=\frac{1}{N_k}\sum_{i=1}^{N_k}y_{i,N_k}(u) \end{aligned}\tag{85}$$

where $(N_k)_{k=1,2,\ldots}$ are a sequence such that $P_{N_k} \underset{k\longrightarrow\infty}{\longrightarrow} P_\infty$. By (78), we have $P_\infty(M_0) = 1$. Furthermore

$$\begin{aligned}\int_{M_0} F(\nu)\, dP_{N_k}(\nu) = E_{N_k}\Bigg(&\frac{1}{N_k}\sum_{i=1}^{N_k}\Big(f(x_i(t), y_i(t)) \\ &- f(x_i(s), y_i(s)) \\ &- \int_s^t \mathscr{A} f(x_i(u), y_i(u), \overline{x}(u), \overline{y}(u))\, du\Big) \\ &\cdot \prod_{j=1}^{p} \overline{g}_j(x_i(s_j), y_i(s_j))\Bigg)^2 = E_{N_k}\frac{N_k-1}{N_k} \\ &\cdot \prod_{i=1}^{N_k}\Bigg(\Big(f(x_i(t), y_i(t)) - f(x_i(s), y_i(s)) \\ &- \int_s^t \mathscr{A} f(x_i(u), y_i(u), \overline{x}(u), \overline{y}(u))\, du\Big) \\ &\cdot \prod_{j=1}^{p} \overline{g}_j(x_i(s_j), y_i(s_j))\Bigg)^2 + \frac{1}{N_k} \\ &\cdot E_{N_k}\Bigg(\Big(f(x_1(t), y_1(t)) - f(x_1(s), y_1(s)) \\ &- \int_s^t \mathscr{A} f(x_1(u), y_1(u), \overline{x}(u), \overline{y}(u))\, du\Big) \\ &\cdot \prod_{j=1}^{p} \overline{g}_j(x_1(s_j), y_1(s_j))\Bigg)^2.\end{aligned} \tag{86}$$

When $k \longrightarrow \infty$, the last term is $o(1)$. Then, for $1 \le i \le N_k$, we have that

$$\begin{aligned}\Theta_i(\tau) = {} & f(x_i(t), y_i(t)) - f(x_i(s), y_i(s)) \\ & - \int_0^\tau \mathscr{A} f(x_i(u), y_i(u), \overline{x}(u), \overline{y}(u))\, du \\ & + \int_0^\tau \frac{\partial f}{\partial x}(x_i(u), y_i(u)) \\ & \cdot D_1\left(\overline{x}(u) - \frac{1}{N_k-1}\sum_{\substack{j=1\\ j\neq i}}^{N_k} x_j(u)\right)\end{aligned} \tag{87}$$

are P_{N_k}-martingales and $\langle \Theta_i, \Theta_j\rangle$ for $j \neq i$.

Using

$$\overline{x}(u) - \frac{1}{N_k-1}\sum_{j=2}^{N_k} x_j(u) = \frac{1}{N_k-1}\left(x_1(u) - \overline{x}(u)\right), \tag{88}$$

for all $N \ge 2$, we get

$$\begin{aligned} E_{N_k}\left(\overline{x}(u) + \overline{y}(u)\right)^2 &< \infty, \\ E_{N_k}\left(x_1(u) + y_2(u)\right)^2 &< \infty,\end{aligned} \tag{89}$$

then we have

$$\lim_{k\longrightarrow\infty}\int_{M_0} F(\nu)\, dP_{N_k}(\nu) = 0. \tag{90}$$

Next, we prove that

$$\lim_{k\longrightarrow\infty}\int_{M} F(\nu)\, dP_{\infty}(\nu) = 0. \tag{91}$$

For $\alpha > 0$, define $h_\alpha : [0,\infty] \longrightarrow \infty$ by

$$h_\alpha = \begin{cases} x & x \le \alpha \\ \alpha & x \ge \alpha. \end{cases} \tag{92}$$

We will show that

$$\begin{aligned} &(a)\ \lim_{k\longrightarrow\infty}\int_{M_0} F^2(\nu)\, dP_{N_k}(\nu) = 0 \\ &(b)\ \lim_{\alpha\longrightarrow\infty}\sup_k \left|\int_{M_0} F_\alpha^2(\nu)\, dP_{N_k}(\nu) - \int_{M_0} F^2(\nu)\, dP_{N_k}(\nu)\right| \\ &\quad = 0 \\ &(b)\ \lim_{k\longrightarrow\infty}\int_{M} F_\alpha^2(\nu)\, dP_{N_k}(\nu) = \int_{M} F_\alpha^2(\nu)\, dP_{\infty}(\nu) \\ &(d)\ \lim_{\alpha\longrightarrow\infty}\int_{M} F_\alpha^2(\nu)\, dP_{\infty}(\nu) \ge \int_{M} F^2(\nu)\, dP_{\infty}(\nu).\end{aligned} \tag{93}$$

It is easy to see that we already proved (a). By the definition of weak convergence, (c) is proven directly.

By Fatou's theory, for $\nu \in M_0$, then

$$\begin{aligned} F_\alpha(\nu) = F(\nu) + \Bigg\langle \nu, \Bigg(&\int_s^t \int \frac{\partial f}{\partial x}(x(\cdot,u), x(\cdot,u)) \\ &\cdot D_1(x(\omega,u) - h_\alpha(x(\omega,u))) \\ &+ \frac{\partial f}{\partial y}(x(\cdot,u), x(\cdot,u)) \\ &\cdot D_2(y(\omega,u) - h_\alpha(y(\omega,u)))\, d\nu(\omega)\, du\Bigg) \\ &\cdot \prod_{j=1}^{P} \overline{g}_j(x(\cdot,s_j), x(\cdot,s_j))\Bigg\rangle,\end{aligned} \tag{94}$$

and it is easy to see that (d) is proved. By similar way, we can prove (b).

Then (a)-(c) give

$$\lim_{\alpha\longrightarrow\infty}\int_{M} F_\alpha^2(\nu)\, dP_{\infty}(\nu) = 0. \tag{95}$$

So (d) implies

$$\int_M F^2(\nu)\, dP_\infty(\nu) = 0 \tag{96}$$

and therefore $F(\nu) = 0$ P_∞-a.s.

By the law of large numbers, we have proved that P_∞-a.s. the projection of ν at $t = 0$ is equal to μ. By Lemma 3.1 in [30], we proved the assertion of the Theorem 4. □

4. Periodic Distribution

In the section, under some suitable assumptions, we prove that the solution of mean-field stochastic predator-prey systems possesses a strictly periodic distribution.

Lemma 5. *Suppose that $T > 0$, $b > 0$, $\overline{b} > 0$, $\kappa_1 > 0$, $\kappa_2 > 0$. Let*

$$\overline{z}(t) = g(y(t)) + x(t) \tag{97}$$

where

$$A = \overline{b} + \frac{r}{4K}T + 1 \tag{98}$$

and

$$g(x) = \begin{cases} A & 0 \le x \le A-1 \\ x & x \ge A+1 \\ \text{arbitrary} & \text{otherwise, but such that } g \in \mathscr{C}^2[0,\infty) \text{ and } 0 \le \dfrac{dg(x)}{dx} < 1. \end{cases} \tag{99}$$

Then

$$\begin{aligned} A_n = &\sup_{Ez(0)\le\overline{b}} \sup_{E\overline{z}^n(0)\le b} \sup_{D_2\le 0} \sup_{\kappa_2 D_2+\kappa_1\ge D_1\le 0} \sup_{0\le t\le T} E\overline{z}^n(t) \\ &< \infty. \end{aligned} \tag{100}$$

for all $n \in \mathbb{N}$.

Proof. Applying Itô's formula to $\overline{z}^n(t)$, we get

$$\begin{aligned} d\overline{z}^n(t) = &n\overline{z}^{n-1}(t) \\ &\cdot \left(dx(t) + g'(y)\, dy + \frac{1}{2}g''(y)\, g_2^2(y(t))\, dt\right) \\ &+ \frac{1}{2}n(n-1)\overline{z}^{n-2}(t) \\ &\cdot \left(g_1^2(x(t)) + g'^2(y(t))\, g_2^2(y(t))\right) dt. \end{aligned} \tag{101}$$

Suppose that $E\overline{z}^n(0) \le b$ and $Ez^n(0) \le \widetilde{b}$.

Defining $\tau^{(\alpha)} = \inf\{t \ge 0 : \overline{z} \ge \alpha\}$ and

$$1_\alpha(t) = \begin{cases} 0 & \text{if } \sup_{0\le s\le t} \overline{z}(s) \ge \alpha \\ 1 & \text{otherwise} \end{cases} \tag{102}$$

and

$$\overline{z}_\alpha(t) = \overline{z}_\alpha(t \wedge \tau_\alpha). \tag{103}$$

Then, we have

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) \le \overline{z}_\alpha^n(t) = &E\overline{z}^n(0) + n\int_0^t E1_\alpha(s) \\ &\cdot \overline{z}^{n-1}(s)\left[x(s)\, r\left(1 - \frac{x(s)}{K} - \frac{y(s)}{r(a+x^2(s))}\right)\right. \\ &+ g'(y) \\ &\cdot \left(y(s)\left(\frac{\mu x(s)}{a+x^2(s)} - D\right) + D_2(Ey(s) - y(s))\right) \\ &\left. + D_1(Ex(s) - x(s)) + \frac{1}{2}g''(y(s))\, g_2^2(y(s))\right] ds \\ &+ \frac{1}{2}n(n-1)\int_0^t E1_\alpha(s)\overline{z}^{n-2}(t)\left(g_1^2(x(t))\right. \\ &\left. + g'^2(y(t))\, g_2^2(y(t))\right) ds. \end{aligned} \tag{104}$$

Case 1 $(D_2 \ge D_1)$. Substituting $D_1(Ex(s) - x(s))$ by $-D_1\overline{z}(s) + D_2 g(y(s)) + D_2 Ex(s)$, we have

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) = &E\overline{z}^n(0) - nD_1\int_0^t E1_\alpha(s)\overline{z}^n(s)\, ds \\ &+ \int_0^t R(s)\, ds + R_0(t). \end{aligned} \tag{105}$$

Solving the above equation, we have

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) &\le e^{-D_1 nt}\left(E\overline{z}^n(0) + \int_0^t e^{D_1 ns} R(s)\, ds\right) \\ &\le E\overline{z}^n(0) + \int_0^t (b_1 D_1 + b_2)\, e^{-D_1(t-s)} ds \end{aligned}$$

$$+ n(n-1)L^2 \int_0^t E1_\alpha(t)\overline{z}^n(s)e^{-D_1 n(t-s)}ds$$
$$\leq b + \frac{b_1}{n} + b_2 T + n(n-1)L^2 \int_0^t E1_\alpha(t)\overline{z}^n(s)\,ds, \tag{106}$$

where b_1 and b_2 are real number independent of D_1, D_2, but N possibly based on $b, \overline{b}$ and T, then we have

$$\begin{aligned} 0 &\geq D_1 g'(y(s))(Ey(s) - y(s)) \\ &\geq D_2 g'(y(s))(Ey(s) - y(s)), \end{aligned} \tag{107}$$

since $g'(y(s)) \neq 0$, it implies

$$Ey(s) - y(0) \leq \overline{b} + \frac{r}{4K}T - (A-1) = 0, \quad \text{for } 0 \leq s \leq T. \tag{108}$$

Using Gronwall's Inequality and Fatou's theory, when $\alpha \longrightarrow \infty$, we obtain the result.

Case 2 ($D_1 \geq D_1 > 0$ and $D_1 \leq \kappa_2 D_2 + \kappa_1 \geq D_1$). By (104),

$$\begin{aligned} 0 &= -nD_2 \int_0^t E1_\alpha(s)\overline{z}^n(s)\,ds \\ &\quad + nD_2 \int_0^t E1_\alpha(s)\overline{z}^{n-1}(s)(x(s) + g(y(s)))\,ds. \end{aligned} \tag{109}$$

Then we get

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) &= E\overline{z}^n(0) - nD_2 \int_0^t E1_\alpha(s)\overline{z}^n(s)\,ds \\ &\quad + \int_0^t \overline{R}(s)\,ds + R_0(t). \end{aligned} \tag{110}$$

As before, we have

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) &\leq E\overline{z}^n(0) + \int_0^t \left(\overline{b}_2 D_2 + b_1\right) e^{-D_2 n(t-s)}ds \\ &\quad + n(n-1)L^2 \int_0^t E1_\alpha(t)\overline{z}^n(s)\,ds, \end{aligned} \tag{111}$$

where $\overline{b}_1$ and $\overline{b}_2$ are real numbers not relying on D_1, D_2, and N but possibly relying on $\overline{b}, b, n, T, \kappa_1$, and κ_2, then we get

$$D_1(Ex(s) - x(s)) \leq \kappa_2 D_2 + \kappa_1 E(x(s)) - D_2 x(s). \tag{112}$$

Similarly the proof is as in Case 1, so we prove Case 2 directly. □

Lemma 6. *Suppose that the condition of Lemma 5 is satisfied. Let $n \in \mathbb{N} \setminus \{1\}$, $\alpha_{n-1} > 0$, and $\overline{N} > L^2(n-1)$. The following conditions hold:*

(1) For all D_1, D_2 satisfying $\overline{N} \leq D_1 \leq \kappa_2 D_2 + \kappa_1$, $D_2 \geq N$.

(2) There exists a constant $\overline{A}_n$ such that if $\infty > E\overline{z}^n(0) \geq \overline{A}_n$, $E\overline{z}^{n-1}(0) \leq \alpha_{n-1}$ and $Ez(0) \leq \overline{b}$.

Then $E\overline{z}^n(t) \leq E\overline{z}^n(0)$ for all $t \in [0, T]$.

Proof. Let $E\overline{z}^n(0) < \infty$ and $E\overline{z}^{n-1}(0) \leq \alpha_{n-1}$. By (105) and (106), we can estimate $R(s)$ and $\overline{R}(s)$ by

$$|R(s)| \leq n(n-1)L^2 E1_\alpha(s)\overline{z}^n(s) + b_2 D_1 + b_1 \tag{113}$$

and

$$\left|\overline{R}(s)\right| \leq n(n-1)L^2 E1_\alpha(s)\overline{z}^n(s) + \overline{b}_2 D_2 + \overline{b}_2, \tag{114}$$

then, for any $\alpha > 0$, we have

$$\begin{aligned} E1_\alpha(t)\overline{z}^n(t) &\leq E\overline{z}^n(0) + \int_0^t \gamma\overline{D} + \lambda \\ &\quad - n\left(\overline{D} - L^2(n-1)\right) E1_\alpha(s)\overline{z}^n(s)\,ds, \end{aligned} \tag{115}$$

where $\overline{D} = \min\{D_1, D_2\} \geq \overline{N}$ and $\gamma > 0$ and $\lambda > 0$ are real number independent of D_1, D_2 and $E\overline{z}^n(0)$ but possibly based on κ_1, κ_2, T, n, and α_{n-1}. When $\alpha \uparrow \infty$, we have

$$\begin{aligned} &E\overline{z}^n(t) \\ &\quad \leq E\overline{z}^n(0) \\ &\qquad + \int_0^t \gamma\overline{D} + \lambda - n\left(\overline{D} - L^2(n-1)\right) E\overline{z}^n(s)\,ds. \end{aligned} \tag{116}$$

Furthermore

$$\begin{aligned} &E\overline{z}^n(t) \\ &\quad \leq E\overline{z}^n(u) \\ &\qquad + \int_u^t \gamma\overline{D} + \lambda - n\left(\overline{D} - L^2(n-1)\right) E\overline{z}^n(s)\,ds. \end{aligned} \tag{117}$$

where $0 \leq u \leq t \leq T$.

Let

$$\begin{aligned} \overline{A}_n &= \frac{\gamma\overline{N} + \lambda}{n\left(\overline{N} - L^2(n-1)\right)} \\ &= \left(\frac{n}{\lambda}\left(1 - \frac{L^2(n-1)}{\overline{N}}\right)\right) + \frac{\gamma}{n\left(\overline{N} - L^2(n-1)\right)} \\ &\geq \frac{\gamma\overline{D} + \lambda}{n\left(D - L^2(n-1)\right)}, \end{aligned} \tag{118}$$

Fix $E\overline{z}^n(0) \geq \overline{A}_n$, and if there are some $t \leq T$ satisfying $E\overline{z}^n(0) \leq E\overline{z}^n(t)$. Define $u = \sup\{\tau \leq t : E\overline{z}^n(0) \geq E\overline{z}^n(\tau)\}$. Since

$$E\overline{z}^n(s+h) - E\overline{z}^n(s) \leq \left(\gamma\overline{D} + \lambda\right)h \tag{119}$$

for all $0 \leq s \leq s + h \leq T$, then $u < t$ and $E\overline{z}^n(u) \geq E\overline{z}^n(0)$; it implies contradiction (117). □

Lemma 7. *Suppose that $\kappa_1 \geq 0$, $\kappa_2 \geq 0$, $T > 0$, and $b > 0$. The following conditions hold:*

(i) There are constants $\alpha, \overline{\alpha} \geq 0$ such that, for any β_1, β_2 and $\beta = \max\{\beta_1, \beta_2\}$,

$$0 \leq D_1 \leq \kappa_2 D_2 + \kappa_1$$
$$\text{and } D_2 \geq \max\left\{0, \frac{\alpha}{2\beta_2} - D\right\}, \tag{120}$$
$$\min\left\{2r - \frac{\overline{\alpha}}{\beta_1}\right\} \geq D_1 > 0.$$

(ii) For all initial conditions satisfying $Ez^4(0) \leq b$,

$$E\left(Ex(0) - x(0)\right)^2 \leq \beta_1,$$
$$\left(Ey(0) - y(0)\right)^2 \leq \beta_2. \tag{121}$$

Then there exist constants $\Lambda \geq 0$ such that

$$\left(Ex(t) - f_1(t)\right)^2 + \left(Ey(t) - f_2(t)\right)^2 \leq \Lambda\beta, \tag{122}$$

where (f_1, f_2) is the solution of (1) with $f_1(0) = Ex(0)$, $f_2(0) = Ey(0)$.

Proof. Let

$$\Phi(t) = x(t) - Ex(t),$$
$$\Psi(t) = y(t) - Ey(t). \tag{123}$$

Applying Itô's formula, we have

$$\begin{aligned} d\Psi^2(t) = &\Big(-2(D + D_2)\Psi^2(t) \\ &+ 2\Psi(t)\left(\frac{\mu x(t)y(t)}{a + x^2(t)} - E\left(\frac{\mu x(t)y(t)}{a + x^2(t)}\right)\right)\Big)dt \\ &+ g_2^2(y(t))dt + 2\Psi(t)g_2(y(t))dW_2. \end{aligned} \tag{124}$$

Then

$$\begin{aligned} E\Psi^2(t) = &E\Psi^2(0)e^{-2(D+D_2)t} \\ &+ \int_0^t e^{-2(D+D_2)(t-s)}\Big(2E\frac{\mu x(t)y(t)\Psi(t)}{a + x^2(t)} \\ &+ Eg_2^2(y(t))\Big)dt. \end{aligned} \tag{125}$$

By Lemma 5, we have

$$\begin{aligned} \alpha = &\sup_{Ez^4(0)\leq b}\sup_{D_2\geq 0}\sup_{0\leq D_1\leq \kappa_2 D_2+\kappa_1}\sup_{0\leq s\leq T} 2E \\ &\cdot\left|\frac{\mu x(t)y(t)\Psi(t)}{a + x^2(t)} + Eg_2^2(y(t))\right| < \infty. \end{aligned} \tag{126}$$

Therefore

$$\begin{aligned} E\Psi^2(t) \leq &E\Psi^2(0)e^{-2(D+D_2)t} \\ &+ \frac{\alpha}{2(D + D_2)}\left(1 - e^{-2(D+D_2)t}\right) \\ \leq &\max\left\{E\Psi^2(0), \frac{\alpha}{2(D + D_2)}\right\}. \end{aligned} \tag{127}$$

Next, applying Itô's formula, we get

$$\begin{aligned} d\Phi^2(t) = &\Big(-2(r - D_1)\Psi^2(t) \\ &+ 2\Phi(t)\left(E\left(\frac{\mu x(t)y(t)}{a + x^2(t)}\right) - \frac{\mu x(t)y(t)}{a + x^2(t)}\right)\Big)dt \\ &+ 2\Phi(t)\frac{r\left(Ex^2(t) - x^2(t)\right)}{K} + g_1^2(y(t))dt \\ &+ 2\Phi(t)g_1(y(t))dW_2 \leq \Big(-2(r - D_1)\Psi^2(t) \\ &+ \frac{2\mu}{a}|\Phi(t)|\left(x(t)y(t) + Ex(t)y(t)\right)\Big)dt \\ &+ \left(\frac{2r}{K}|\Phi(t)|\left(Ex^2(t) + x^2(t)\right) + g_1^2 y(t)\right)dt \\ &+ 2\Phi(t)g_1(y(t))dW_2. \end{aligned} \tag{128}$$

Then

$$\begin{aligned} E\Phi^2(t) \leq &E\Phi^2(0)e^{-2(r-D_1)t} \\ &+ \int_0^t e^{-2(r-D_1)(t-s)}E|\Phi(s)|\Big(\left(\frac{2\mu}{a} + \frac{2r}{K}\right) \\ &\cdot\left[x^2(s) + y^2(s) + Ex^2(s) + Ey^2(s)\right] \\ &+ \frac{g_1^2(y(t))}{|\Phi(s)|}\Big)ds. \end{aligned} \tag{129}$$

By Lemma 5, we also have

$$\begin{aligned} \overline{\alpha} = &\sup_{Ez^4(0)\leq b}\sup_{D_2\geq 0}\sup_{0\leq D_1<r}\sup_{0\leq s\leq T} E|\Phi(s)| \\ &\cdot\left|x^2(s) + y^2(s) + Ex^2(s) + Ey^2(s)\right| + Eg_1^2(y(t)) \\ &< \infty. \end{aligned} \tag{130}$$

Therefore

$$\begin{aligned} E\Phi^2(t) \leq &E\Phi^2(0)e^{-2(r-D_1)t} \\ &+ \frac{\overline{\alpha}}{2(r - D_1)}\left(1 - e^{-2(r-D_1)t}\right) \\ \leq &\max\left\{\Phi^2(0), \frac{\overline{\alpha}}{2(r - D_1)}\right\}. \end{aligned} \tag{131}$$

Let $E\Psi^2(0) \leq \beta$, and we have $E\Psi^2(t) \leq \beta$. Then

$$\begin{aligned} \left(Ex(t) - f_1(t)\right)^2 = &\int_0^t 2\left(Ex(s) - f_1(s)\right) \\ &\cdot\Big(r\left(Ex(s) - f_1(s)\right) + r\left(E\frac{x^2(s)}{K} - \frac{f_1^2(t)}{K}\right) \\ &- \left(E\left(\frac{x(s)y(s)}{a + x^2(s)}\right) - \frac{f_1(s)f_2(s)}{a + f_1^2(s)}\right)\Big)ds, \end{aligned}$$

$$\begin{aligned}(Ey(t)-f_2(s))^2 &= \int_0^s 2(Ey(s)-f_2(s)) \\ &\cdot\left(-D(Ey(s)-f_2(s))\right. \\ &\left.+\left(E\left(\frac{x(t)y(t)}{a+x^2(t)}\right)-\frac{f_1(t)f_2(t)}{a+f_1^2(t)}\right)\right)ds.\end{aligned} \tag{132}$$

Let

$$H(t)=(Ex(t)-f_1(t))^2+(Ey(t)-f_2(t))^2, \tag{133}$$

and we get

$$\begin{aligned}H(t) &\le \int_0^t \left(2rH(s)+\frac{2r}{K}|Ex(s)-f_1(s)|\left|E\Phi^2(s)\right.\right. \\ &\left.+2E\Phi(s)Ex(s)+E^2x(s)-f_1^2(s)\right|\Big)ds \\ &+\int_0^t \frac{2}{a}\left(|Ex(s)-f_1(s)|+|Ey(s)-f_2(s)|\right) \\ &\cdot\left(\left|\left(E\Phi^2(s)\right)^{1/2}\left(E\Psi^2(s)\right)^{1/2}+E\Phi(s)Ey(s)\right.\right. \\ &\left.\left.+E\Psi Ex(s)+Ex(s)Ey(s)\right|+|f_1(s)f_2(s)|\right)ds \\ &\le \int_0^t (2rH(s)+\Theta(s))\,ds,\end{aligned} \tag{134}$$

where

$$\begin{aligned}\Theta(t) &= \frac{2r}{K}|Ex(s)-f_1(s)|\left|E\Phi^2(s)+2E\Phi(s)Ex(s)\right. \\ &\left.+E^2x(s)-f_1^2(s)\right|+\frac{2}{a}\left(|Ex(s)-f_1(s)|+|Ey(s)\right. \\ &\left.-f_2(s)|\right)\left(\left|\left(E\Phi^2(s)\right)^{1/2}\left(E\Psi^2(s)\right)^{1/2}\right.\right. \\ &\left.+E\Phi(s)Ey(s)+E\Psi Ex(s)+Ex(s)Ey(s)\right| \\ &\left.+|f_1(s)f_2(s)|\right).\end{aligned} \tag{135}$$

By $Ez^2(0)\le b$, $f_1<\infty$, $f_2<\infty$, $Ex(t)<\infty$ and $Ey(t)<\infty$, for $\forall t\in[0,T]$, for some Λ and $0\le t\le T$, we have

$$\begin{aligned}\Theta(t) &= \frac{2r}{K}|Ex(s)-f_1(s)|\left|E\Phi^2(s)+2E\Phi(s)Ex(s)\right. \\ &\left.+E^2x(s)-f_1^2(s)\right|+\frac{2}{a}\left(|Ex(s)-f_1(s)|+|Ey(s)\right. \\ &\left.-f_2(s)|\right)\left(\left|\left(E\Phi^2(s)\right)^{1/2}\left(E\Psi^2(s)\right)^{1/2}\right.\right. \\ &\left.+E\Phi(s)Ey(s)+E\Psi Ex(s)+Ex(s)Ey(s)\right| \\ &\left.+|f_1(s)f_2(s)|\right)\le\Lambda.\end{aligned} \tag{136}$$

Then, we get

$$H(t)\le\int_0^t(2rH(s)+\Lambda)\,ds. \tag{137}$$

By Gronwall's inequality, we have

$$\begin{aligned}H(t) &\le \int_0^t \Lambda ds+2r\Lambda\int_0^t se^{2r(t-s)}ds \\ &\le \Lambda\left(T+2r\left(e^T-T-1\right)\right).\end{aligned} \tag{138}$$

Then, we prove the assertion of Lemma 7. □

Theorem 8. *Suppose that $\mu^2>(16/3)aD^2$ and $(u+\sqrt{\mu^2-4aD^2})/2D>K>(2u-\sqrt{\mu^2-4aD^2})/2D$. Let $\kappa_1\ge0$ and $\kappa_2\ge0$, if the following conditions hold:*

(i) There is a real number N^* such that for all $D_1\ge N^*$, $0\le D_1\le\kappa_2D_2+\kappa_1$.

(ii) There is a probability measure ν^* on $[0,\infty)\times[0,\infty)$ and some $\tau^*>0$ such that $\mathscr{L}(x(0),y(0))=\nu^*$.

Then $\mathscr{L}(x(\tau^*),y(\tau^*))=\nu^*$ but $\mathscr{L}(x(s),y(s))\ne\nu^*$ for $0<s<\tau^*$; i.e., (4) possesses a periodic distribution.

Proof. (f_1^*,f_1^*) denote the unique periodic solution of (1), where $f_1^*(0)=\theta$, $f_2^*(0)>\vartheta$ (see [1]) and

$$\begin{aligned}\theta &= \frac{u-\sqrt{\mu^2-4aD^2}}{2D}, \\ \vartheta &= r\left(1-\frac{u-\sqrt{\mu^2-4aD^2}}{2KD}\right) \\ &\cdot\left(a+\left(\frac{u-\sqrt{\mu^2-4aD^2}}{2D}\right)^2\right).\end{aligned} \tag{139}$$

Let

$$b_2>f_2^*>b_1>\vartheta \tag{140}$$

and

$$\begin{aligned}T &= 2\max_{b_1\le b\le b_2}\min\{u>0: f_1(u)=\theta,\ f_1(u) \\ &>\vartheta,\ f_1(u)=\theta,\ f_2(u)=b,\ (f_1,f_1)\ \text{solve}\ (1)\}.\end{aligned} \tag{141}$$

$(f_1^{(i)},f_2^{(i)})$ denote the solution of system (1) starting at $f_1^{(i)}(0)=\theta$, $f_2^{(i)}(0)=b_i$ $(i=1,2)$ and let

$$\begin{aligned}\bar{t} &= \min_{b_1\le b\le b_2}\min\{u\ge0: f_1(u)=\theta,\ f_1(u)<\vartheta,\ f_1(0) \\ &=\theta,\ f_2(0)=b,\ (f_1,f_1)\ \text{solve}\ (1)\} \\ \bar{b} &:= b_2+\theta\end{aligned}$$

$$\begin{aligned}
&\overline{N} = b_2 \\
&\epsilon_1 := \min_{0\le t\le T}\left\{\left(\left(f_1^{(1)}(t)-\theta\right)^2+\left(f_2^{(1)}(t)-\vartheta\right)^2\right)^{1/2}\right\} \\
&\epsilon_2 := dist\left(\{(\theta,b);\vartheta\le b\le b_1\},\left\{\left(f_1^{(1)}(t),f_2^{(1)}(t)\right);\overline{t}\right.\right. \\
&\qquad \left.\left.\le t\le T\right\}\right) \\
&\epsilon_3 := dist\left(\{(\theta,b);b\ge b_2\},\left\{\left(f_1^{(2)}(t),f_2^{(2)}(t)\right);\overline{t}\le t\right.\right. \\
&\qquad \left.\left.\le T\right\}\right).
\end{aligned} \tag{142}$$

Since $\{(f_1^{(1)}(t), f_2^{(1)}(t)); 0 \le t \le T\}$ is a compact set but not included in (θ, ϑ), because there exists the uniqueness of the solution of system (1), it is easy to see that $\epsilon_1 > 0$ is proven. Next, we prove that $\epsilon_2 > 0$. Suppose that there exist some $\overline{t} \le t \le T$ such that $f_1^{(1)}(t) = a$ and $\vartheta \le f_2^{(1)}(t) \le b_1$. Then the line segment $\{(\theta, u), c_1 \le u \le f_1^{(1)}(t)\}$ and the curve $\{(f_1^{(1)}(s), f_2^{(1)}(s));\ 0 \le s \le t\}$ form a set Ω which is invariant, it is easy see that Ω contains the limit cycle (f_1^*, f_2^*) and $\Omega \ne \{(\theta, \vartheta)\}$, due to the globally stable limit cycle, then we prove $\epsilon_2 > 0$. By the same way, we prove that $\epsilon_3 > 0$.

For $n = 2, 3, 4$, we take values $\overline{A}_n$ satisfying the condition of Lemma 6 with $\alpha_{n-1} = \overline{A}_{n-1}, = 2, 3, 4$ and $\alpha_1 = \overline{c}$. Let $c = \overline{A}_4$; fix Λ and β_1, β_2 satisfying $(\Lambda\beta)^{1/2} < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $rK/4 > \Lambda$. Now let $N^* \ge \overline{N}$ be large enough such that

$$\alpha = \sup_{Ez^4(0)\le b}\ \sup_{D_2\ge 0}\ \sup_{0\le D_1\le \kappa_2 D_2+\kappa_1}\ \sup_{0\le s\le T} 2E\left|\frac{\mu x(t)\,y(t)\,\Psi(t)}{a+x^2(t)}+Eg_2^2\left(y(t)\right)\right| < 2\left(D+N^*\right)\beta_1 \tag{143}$$

and

$$\begin{aligned}
\overline{\alpha} = &\sup_{Ez^4(0)\le b}\ \sup_{D_2\ge 0} \\
&\cdot \sup_{0\le D_1<r}\ \sup_{0\le s\le T} E\left|\Phi(s)\right|\left|x^2(s)+y^2(s)+Ex^2(s)+Ey^2(s)\right| \\
&+Eg_1^2\left(y(t)\right) < 2\left(r+N^*\right)\beta_2.
\end{aligned} \tag{144}$$

From (127) and (131), it is shown that $\sup_{0\le t\le T} E\Phi^2(t) \le E\Phi^2(0)$ if $E\Phi^2(0) \ge \beta_1$ and $E\Psi^2(t) \le E\Psi^2(0)$ if $E\Psi^2(0) \ge \beta_1$, $Ez^4(0) \le b$, and $D_2 > N^*$. By Lemma 6, it is easy see that $\sup_{0\le t\le T} E\overline{z}^4(t) \le b$ if $E\overline{z}^i(0) \le \overline{A}_n$ and $E\overline{z}(0) \le \overline{b}$. It implies that there are real numbers $\delta > 0$ and $\sigma > 1$ such that $\sup_{0\le t\le T} E\exp(\delta x^2(t)) \le \sigma$ whenever $E\exp(\delta x^2(0)) \le \sigma$. Let $\mathcal{U}_1([0,\infty)^2)$ denote the set on $[0,\infty)^2$ and

$$\begin{aligned}
\mathcal{U} = \Big\{&\nu\in\mathcal{U}_1\left([0,\infty)^2\right): E_\nu x(0)=\theta,\ b_1\le E_\nu x(0) \\
&\le c_2,\ E_\nu\Phi^2(0)\le\beta_1,\ E_\nu z^n(0)\le\overline{A}_n \text{ for } n \\
&= 2,3,4 \text{ and } E\exp\left(\delta x^2(0)\right)\le\sigma\Big\}
\end{aligned} \tag{145}$$

and for $\nu \in \mathcal{U}$

$$\begin{aligned}
\tau(\nu) = \inf\{&t>0: E_\nu x(t)=\theta,\ E_\nu y(t)>\vartheta,\ \exists t\ge s \\
&\ge 0: E_\nu y(t)\le\vartheta\}
\end{aligned} \tag{146}$$

and

$$\begin{aligned}
&S:\mathcal{U}\longrightarrow\mathcal{U}_1\left([0,\infty)^2\right) \\
&S(\nu)=\mathscr{L}\left(x(\tau(\nu)),y(\tau(\nu))\right).
\end{aligned} \tag{147}$$

Next, we will prove that S maps $\mathcal{U}$ into $\mathcal{U}$; hence, we show that there are real numbers $\delta > 0$ and $\sigma > 1$ satisfying $\sup_{0\le t\le T} E\exp(\delta x^2(t)) \le \sigma$ whenever $E\exp(\delta x^2(0)) \le \sigma$.

Let $C = \sup_{\delta\ge 0} g_1^2(x)$, $\delta = \overline{N}/4C$; suppose that $E\overline{z}(0) \le b$ and $\sup E\exp(\delta x^2(0)) \le \infty$. By the proof of Theorem 3, it proves that $\sup_{0\le t\le T} E\exp(\delta x^{2n}(t)) < \infty$ for all $n \ge 1$ and for $\overline{A} = c^{1/4}$

$$\begin{aligned}
dx^{2n}(t) = &\left(2nx^{2n}(t)\,r\left(1-\frac{x(t)}{K}-\frac{y(t)}{r\left(a+x^2(t)\right)}\right)\right. \\
&\left.+n(2n-1)\,x^{2n-2}g_1^2(x(t))\right)dt \\
&+x^{2n-1}g_1(x(t))\,dW(t)+D_1 2nx^{2n-1}(t) \\
&\cdot(Ex(t)-x(t))\,dt \le \left(2nx^{2n}(t)\left(r-D_1\right)\right. \\
&\left.+n(2n-1)\,x^{2n-2}B\right)dt \\
&+x^{2n-1}g_1(x(t))\,dW(t)+D_1 2nx^{2n-1}(t) \\
&\cdot Ex(t)\,dt.
\end{aligned} \tag{148}$$

Then, we have

$$\begin{aligned}
Ex^{2n}(t) \le \int^t &\left(2n\left(r-D_1\right)x^{2n}(s)\right. \\
&\left.+n(2n-1)\,Bx^{2n-2}(s)+2nD_1\overline{A}Ex^{2n-1}(t)\right)ds.
\end{aligned} \tag{149}$$

Therefore, we get

$$\begin{aligned}
&E\sum_{n=0}^{N}\frac{\delta x^{2n}(t)}{n!} \\
&\le E\exp\left(\delta x^2(0)\right) \\
&\quad+\sum_{n=1}^{N}\frac{\delta^n}{n!}\int_0^t Ex^{2n}(s)\left((2n+1)B\delta+n\left(r-D_1\right)\right)ds
\end{aligned}$$

$$+\sum_{n=1}^{N}\frac{\delta^n}{n!}\int_0^t nD_1Ex^{2n-1}(s)\left(2\overline{A}-x(s)\right)ds+\delta Bt$$
$$-\frac{\delta^{N+1}}{N!}(2N+1)B\int_0^t Ex^N(s)\,ds. \tag{150}$$

Due to $\delta=\overline{N}/4B\le D_1/4B$, we show that the first integrand is negative real number for large enough n. The second integrand is not larger than $D_1n(2\overline{A})^{2n}$ and

$$\sum_{n=1}^{N}\frac{\delta^n}{n!}D_1n\left(2\overline{A}\right)^{2n}t\le\delta D_1\left(2\overline{A}\right)^2t\exp\left(\delta\left(2\overline{A}\right)^2\right). \tag{151}$$

Hence, we get

$$B\delta\sum_{n=1}^{N}\frac{\delta^n}{n!}\int_0^t Ex^{2n}ds$$
$$+\delta\left(2B\delta-D_1\right)\sum_{n=1}^{N}\frac{\delta^{n-1}}{(n-1)!}\int_0^t Ex^{2n-2}x^2(s)\,ds \tag{152}$$

and using $2B\delta-D_1\le-(1/2)D_1$, we have

$$E\exp(\delta x(t))\le E\exp(\delta x(0))$$
$$+B\delta\int_0^t E\exp\left(\delta x^2(s)\right)ds$$
$$-\frac{1}{2}\delta D_1\int_0^t Ex^2(s)\exp\left(\delta x^2(s)\right)ds$$
$$+4\delta D_1\overline{A}^2t\exp\left(\delta\left(2\overline{A}\right)^2\right) \tag{153}$$

Clearly, $Ex^2\exp(\delta x^2)-\overline{\alpha}E\exp(\delta x^2)\longrightarrow\infty$ when $E\exp(\delta x^2)\longrightarrow\infty$ for any $-\overline{\alpha}>0$. Therefore, we choose $\sigma>1$ satisfying

$$D_1Ex^2(s)\exp\left(\delta x^2\right)-2BE\exp\left(\delta x^2\right)$$
$$>8D_1\overline{A}^2\exp\left(\delta\left(2\overline{A}\right)^2\right). \tag{154}$$

It shows that $\sup_{0\le t\le T}E\exp(\delta x^2(t))\le\sigma$ whenever $E\exp(\delta x^2(0))\ge\sigma$.

Next, we prove that S is weakly continuous. Hence, if we have proved that $\tau:\mathscr{U}\longrightarrow[0,T]$ is continuous, it is easy to see that S is weakly continuous on $\mathscr{U}$. Then it implies that $\tau:\mathscr{U}\longrightarrow[0,T]$ is continuous. For $\nu\in\mathscr{U}$, we have

$$\frac{d}{dt}Ex(\tau(\nu))$$
$$=E\left(r\left(x(\tau(\nu))-\frac{x(\tau(\nu))}{K}\right)-\frac{x(\tau(\nu))\,y(\tau(\nu))}{a+x^2(\tau(\nu))}\right)$$
$$=\frac{rK}{4}$$
$$-E\left(\frac{(x(\tau(\nu))-K/2)^2}{K}+\frac{x(\tau(\nu))\,y(\tau(\nu))}{a+x^2(\tau(\nu))}\right)$$
$$\ge\frac{rK}{4}-\Lambda=\varrho>0. \tag{155}$$

Let $\nu\in\mathscr{U}$ and take $t_0>0$ such that $(d/dt)E_\nu x(t)>\varrho/2$ for all $t\in[\tau(\nu)-t_0,\tau(\nu)+t_0]$. Fix $0<\epsilon\le\varrho/2$, let $\nu_n\underset{n\longrightarrow\infty}{\longrightarrow}\nu$ $(\nu_n\in\mathscr{U})$ and take n_0 such that $\sup_{0\le t\le T}|E_{\nu_n}x(t)-E_\nu x(t)|<\epsilon$ for all $n\ge n_0(\varepsilon)$. Then $|\tau(\nu)-\tau(\nu_n)|\le(2/\varrho)\epsilon$ for $n\ge n_0(\varepsilon)$; it is easy to see that τ is continuous on $\mathscr{U}$ and therefore the proof is completed. □

Corollary 9. *Suppose that* $\mu^2>(16/3)aD^2$ *and* $(u+\sqrt{\mu^2-4aD^2})/2D>K>(2u-\sqrt{\mu^2-4aD^2})/2D$, $\kappa_1>0$, $\kappa_2>0$, *and the following conditions hold:*

(i) There exists a sequence $(D_{1,n},D_{2,n})_{n\in N}$ *satisfying the conditions of Theorem 8 for every* $n\in\mathbb{N}$ *and* $D_{1,n}\underset{n\longrightarrow\infty}{\longrightarrow}\infty$.

(ii) There is a sequence ν_n^* *on* $[0,\infty)\times[0,\infty)$ *such that (4) with* $D_1=D_{1,n}$, $D_2=D_{2,n}$ *and* $\mathscr{L}(x(0),y(0))=\nu_n^*$.

Then system (4) possesses a periodic distribution and

(a) $(E_{\nu_n^*}(x(t)),E_{\nu_n^*}(y(t)))\underset{n\longrightarrow\infty}{\longrightarrow}(f_1^*(t),f_2^*(t))$;

(b) $\mathscr{L}_{\nu_n^*}(x(\cdot))\underset{n\longrightarrow\infty}{\longrightarrow}\epsilon_{f_1^*(\cdot)}$.

Proof. Let $b_1(n)$ and $b_2=b_2(n)$ converge to $f_2^*(0)$ with $D_{1,n}\ge N^*=N^*(n)$. It is easy to see that $\beta=\beta(n)$ converges to zero. Then, the first assertion of theorem is proved. Furthermore, we have

$$d\Psi(t)=\left(-\left(D+D_{2,n}\right)\Psi(t)\right.$$
$$\left.+\left(\frac{\mu x(t)\,y(t)}{a+x^2(t)}-E\frac{\mu x(t)\,y(t)}{a+x^2(t)}\right)\right)dt$$
$$+g_2\left(y(t)\right)dW(t) \tag{156}$$

and hence, solving for $\Psi(t)$, we get

$$\Psi(t)=e^{-(D+D_{2,n})t}\Psi(0)$$
$$+\int_0^t e^{-(D+D_{2,n})(t-s)}\left(\frac{\mu x(t)\,y(t)}{a+x^2(t)}-E\right.$$
$$\left.\cdot\frac{\mu x(t)\,y(t)}{a+x^2(t)}\right)dt$$
$$+\int_0^t e^{-(D+D_{2,n})(t-s)}g_2\left(y(t)\right)dW(t). \tag{157}$$

Applying Chebychev's inequality, we have

$$P\left\{\sup_{0\le t\le T}\Psi(t)\ge R\right\}\le P\{\Psi(0)\ge R\}$$
$$+P\left\{\int_0^T e^{-(D+D_{2,n})(t-s)}\left|\frac{\mu x(t)\,y(t)}{a+x^2(t)}\right.\right.$$

$$- E\frac{\mu x(t)\, y(t)}{a + x^2(t)}\Bigg| dt \geq R\Bigg\}$$
$$+ P\left\{\sup_{0\leq t\leq T}\left|\int_0^t e^{-(D+D_{2,n})(t-s)} g_2(y(t))\, dW(t)\right|\right.$$
$$\left.\geq R\right\} \leq \frac{1}{R}E\Psi(0) + \frac{1}{R}\frac{\Theta}{D+D_{2,n}} + P\left\{\sup_{0\leq t\leq T} Q_n(t)\right.$$
$$\left.\geq R\right\}, \tag{158}$$

where

$$Q_n(t) = \int_0^t e^{-(D+D_{2,n})(t-s)} g_2(y(t))\, dW(t) \tag{159}$$

and

$$\Theta = \sup_{0\leq t\leq T} E\left|\frac{\mu x(t)\, y(t)}{a + x^2(t)} - E\frac{\mu x(t)\, y(t)}{a + x^2(t)}\right|. \tag{160}$$

Since

$$E\Psi(0) \leq \left(E\Psi^2(0)\right)^2 \leq \beta^{1/2}(n) \underset{n\longrightarrow\infty}{\longrightarrow} 0 \tag{161}$$

then we prove the second assertion of theorem that is

$$P\left\{\sup_{0\leq t\leq T}\left|Q_n(t)\right| \geq R\right\} \underset{n\longrightarrow\infty}{\longrightarrow} 0 \quad \text{for every } R > 0. \tag{162}$$

By [31, pp 142], we get

$$Q_n(t) = -(D + D_{2,n})\, Q_n(t)\, dt + g_2(y(t))\, dW(t), \qquad Q_n(0) = 0. \tag{163}$$

Therefore, for

$$f(y) = \begin{cases} y^4, & |y| \leq R^4 \\ \text{bounded}, & \mathscr{C}^2(\mathfrak{R}), \end{cases} \tag{164}$$

then

$$f(Q_n(t)) + (D + D_{2,n})\int_0^t f'(Q_n(s))\, Q_n(s)\, ds - \frac{1}{2}\int_0^t f''(Q_n(s))\, g_2(y(s))\, ds \tag{165}$$

is a martingale. Denoting the stopping time,

$$\tau := \inf\{t \geq 0 : |Q_n(s)| = R\} \wedge T. \tag{166}$$

By Chebychev's inequality, we get

$$P\left\{\sup_{0\leq t\leq T}\left|Q_n(t)\right| \geq R\right\} = P\{f(Q_n(\tau)) \geq R\} \leq \frac{1}{R^4} \cdot Ef(Q_n(\tau)) = \frac{1}{R^4}E\int_0^\tau \left(6Q_n^2(s)\, g_2^2(y(s)) - 4(D + D_{2,n})\, Q_n^4(s)\right) ds. \tag{167}$$

For $-R^4 \leq y \leq R^4$,

$$6y^2 g_2^2(y(s)) - 4(D + D_{2,n})\, y^4 = -4(D + D_{2,n})\left(x^2 - \frac{3g_2^2(y(s))}{4(D + D_{2,n})}\right)^2 + \frac{9g_2^4(y(s))}{4(D + D_{2,n})} \tag{168}$$

and hence, we have

$$6y^2 g_2^2(y(s)) - 4(D + D_{2,n})\, y^4 \leq \begin{cases} \dfrac{9g_2^4(y(s))}{4(D + D_{2,n})} & \text{if } g_2^2(y(s)) \leq R^8(D + D_{2,n}) \\ 6R^8 g_2^2(y(s)) - 4(D + D_{2,n})\, R^{16} & \text{if } g_2^2(y(s)) > R^8(D + D_{2,n}); \end{cases} \tag{169}$$

since $\sup_n E\int_0^T g_2^4(y(s))ds < \infty$, it implies

$$P\left\{\sup_{0\leq t\leq T}\left|Q_n(t)\right| \geq R\right\} \leq \frac{1}{R^4} \cdot E\int_0^T \left(1_{g_2^2(y(s))\leq R^8(D+D_{2,n})}\frac{9g_2^4(y(s))}{4(D + D_{2,n})} + 1_{g_2^2(y(s))>R^8(D+D_{2,n})}\frac{9g_2^4(y(s))}{4(D + D_{2,n})} \cdot \frac{6g_2^4(y(s))}{D + D_{2,n}}\right) ds \underset{n\longrightarrow\infty}{\longrightarrow} 0. \tag{170}$$

Therefore, the proof is completed. □

5. Fluctuations

In the section, under some suitable assumptions, we study the fluctuations of the periodic in distribution processes for mean-field stochastic predator-prey systems when the white noise converges to zero.

Theorem 10. *Suppose that condition of Theorem 8 is satisfied. The following conditions hold:*

(i) *There exist real numbers N_1 and N_2 satisfying $D_1 \geq N_1$ and $D_2 \geq N_2$.*

(ii) *There is a probability measure μ_ε on $[0,\infty)^2$ such that*

$dx(t)$

$$
\begin{aligned}
&= x(t)\, r\left(1-\frac{x(t)}{K}-\frac{y(t)}{r\left(a+x^2(t)\right)}\right)dt \\
&\quad + \varepsilon g_1(x(t))\, dW(t) + D_1(Ex(t)-x(t))\, dt, \\
dy(t) &= y(t)\left(\frac{\mu x(t)}{a+x^2(t)}-D\right)dt + \varepsilon g_2(y(t))\, dW(t) \\
&\quad + D_2(Ey(t)-y(t))\, dt, \\
\mathfrak{L}(x(0), y(0)) &= \mu_\varepsilon.
\end{aligned} \tag{171}
$$

Then system (171) exists a periodic distribution for all $1 \geq \varepsilon \geq 0$.

Furthermore, there exists a periodic solution of (171) with $(x_\varepsilon, y_\varepsilon) \underset{\varepsilon\longrightarrow 0}{\longrightarrow} (f_1^*, f_2^*)$ weakly on $\mathscr{C}([0,\infty), \mathfrak{R}^2)$.

Proof. By the proof of Theorem 8, we prove only $\sup_{t\geq 0} Ez^4 < \infty$ but, using Lemma 6, it is easy to show that there is a periodic solution such that $\sup_{t\geq 0} Ez^8 < \infty$. Then the first assertion of the theorem is proven.

Next, let $\varepsilon_n \downarrow 0$. Due to $Ex_{\varepsilon_n}(0) = \theta$ and $Ey_{\varepsilon_n}(0) \leq b_2$, $\{\nu_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is family of compact. By the same proof of Theorem 8, it is easy to see that there is a weakly convergent subsequence of $(x_{\varepsilon_n}, x_{\varepsilon_n})$ approaching to a solution (x, y) of

$$
\begin{aligned}
\dot{x}(t) &= x(t)\, r\left(1-\frac{x(t)}{K}-\frac{y(t)}{r\left(a+x^2(t)\right)}\right) \\
&\quad + D_1(Ex(t)-x(t)), \quad Ex(0)=\theta \\
\dot{y}(t) &= y(t)\left(\frac{\mu x(t)}{a+x^2(t)}-D\right) + D_2(Ey(t)-y(t)), \\
&\qquad b_1 \leq Ey(0) \leq b_2.
\end{aligned} \tag{172}
$$

Let τ_ε denote the period of $(x_\varepsilon, y_\varepsilon)$; we choose a subsequence ε_{n_k} $(k = 1, 2, 3, \ldots)$ satisfying $\mathscr{L}(x_{\varepsilon_{n_k}}, y_{\varepsilon_{n_k}})$ approaches to a solution of (172), and $\tau = \lim_{k\longrightarrow\infty} \tau_{\varepsilon_{n_k}}$ exists. Based on Theorem 11.1.4 in [18, pp 264], it is easy to see that that the map $(\nu, \varepsilon, t) \longmapsto \mathscr{L}_{\nu,\varepsilon}(x(t), y(t))$ from $\mathscr{U} \times [0,1] \times [0,T]$ to $\mathscr{U}_1(C([0,T], \mathfrak{R}_+^2))$ is continuous, where $\mathscr{L}_{\nu,\varepsilon}(x(t), y(t))$ is the solution of (171) with $\mathscr{L}(x(0), y(0)) = \nu$ at t.

Let $\nu := \lim_{k\infty} \nu_{\varepsilon_{n_k}}$; it will show that the periodic distribution solution of (172) is (f_1^*, f_2^*). Denoting $\overline{z} = g(y(t)) + x(t)$ $(0 \leq t \leq T)$, where g is defined as in Lemma 5 with $\overline{b} = b_2$, then we have

$$
\begin{aligned}
\frac{d\overline{z}(t)}{dt} &= x(t)\, r\left(1-\frac{x(t)}{K}\right) \\
&\quad - \frac{x(t)\, y(t)}{\left(a+x^2(t)\right)}\left(1-\mu g'(y(t))\right) \\
&\quad + D_1(Ex(t)-x(t)) - Dg'(y(t))\, y(t) \\
&\quad + D_2 g'(y(t))(Ey(t)-y(t)) \\
&\leq \frac{rK}{4} - r\left(x-\frac{K}{2}\right)^2 - Dg'(y(t))\, y(t) \\
&\quad + D_1(Ex(t)-x(t)),
\end{aligned} \tag{173}
$$

for all $y \leq \sup_{0\leq t\leq T} y(t) \leq b$, then $g'(y) = 0$. The right hand side of (173) is negative constant if either x or y is large enough; i.e., there has some $\delta > 0$ satisfying $d\overline{z}(t)/dt < 0$ whenever $\overline{z}(t) \geq \delta$; it easily shows that the support of $\mathscr{L}(x(t), y(t))$ which is periodic is included in

$$
\{(x, y) \mid x \geq 0,\ y \geq 0,\ x + y \leq \delta\} \tag{174}
$$

for every $t \geq 0$.

Let $\Phi = x - Ex$ and $\Psi = y - Ey$; we get

$$
\frac{x(t)\, y(t)}{a+x^2(t)} \leq \frac{1}{2\sqrt{a}} y(t) = \frac{1}{2\sqrt{a}}\left(\Psi + Ey(t)\right). \tag{175}
$$

Hence, (w.p.1) $0 \leq x + y \leq \delta$, and therefore $0 \leq Ex, Ey \leq \delta$ and $-\delta \leq \Phi, \Psi \leq \delta$

$$
\begin{aligned}
E\Phi\frac{x(t)\, y(t)}{a+x^2(t)} &\leq \frac{1}{2\sqrt{a}} E|\Phi|\left|\Psi + Ey(t)\right| \\
&\leq \frac{1}{4\sqrt{a}}\left(E\Phi^2 + E\Psi^2\right) \\
E\Psi\frac{x(t)\, y(t)}{a+x^2(t)} &\leq \frac{\delta}{a} E|\Psi|\,|\Phi + Ex(t)| \\
&\leq \frac{\delta}{2a}\left(E\Phi^2 + E\Psi^2\right) \\
E\Phi x^2 &\leq \delta E|\Phi(\Phi + Ex)| \leq \delta E\Phi^2
\end{aligned} \tag{176}
$$

and

$$
\begin{aligned}
&\frac{d}{dt} E\left(\Phi^2 + \Psi^2\right) \\
&\quad \leq -\left(D_1 - r - \frac{\delta r}{K} - \frac{1}{4\sqrt{a}} - \frac{\mu\delta}{2a}\right) E\Phi^2 \\
&\qquad - \left(D + D_2 - \frac{1}{4\sqrt{a}} - \frac{\mu\delta}{2a}\right) E\Psi^2.
\end{aligned} \tag{177}
$$

Then, we have

$$
\frac{d}{dt} E\left(\Phi^2 + \Psi^2\right) < 0 \tag{178}
$$

whenever $E(\Phi^2 + \Psi^2) > 0$ if D_1 and D_2 are sufficiently large. Because $E\Phi^2$ and $E\Psi^2$ are periodic, it easily shows that $E\Phi^2 \equiv E\Psi^2 \equiv 0$; i.e., $(x(t), y(t))$ is deterministic. There exists only one periodic solution of system (1) with $x(0) = \theta$ and $y(0) \geq b_1$. Therefore, the proof is completed. □

Theorem 11. *Suppose that* $\mu^2 > (16/3)aD^2$ *and* $(u + \sqrt{\mu^2 - 4aD^2})/2D > K > (2u - \sqrt{\mu^2 - 4aD^2})/2D$. *Let*

$\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are nonnegative constants. For $D_1 \le \kappa_1 D_2 + \kappa_2$ and $D_2 \le \kappa_3 D_1 + \kappa_4$, D_1 and D_2 are large enough,

$$\sup_{0<\varepsilon\le 1} \frac{\delta_\varepsilon}{\varepsilon} < \infty, \tag{179}$$

where

$$\begin{aligned} \delta_\varepsilon &= \varepsilon \sup_{t\ge 0} \left\{ \left(E\Phi_\varepsilon^4(t)\right)^{1/4}, \left(E\Psi_\varepsilon^4(t)\right)^{1/4} \right\}, \\ \Phi_\varepsilon(t) &= \varepsilon^{-1}\left(x_\varepsilon - Ex_\varepsilon\right), \\ \Psi_\varepsilon(t) &= \varepsilon^{-1}\left(y_\varepsilon - Ey_\varepsilon\right). \end{aligned} \tag{180}$$

Then there exists a periodic solution $(x_\varepsilon, y_\varepsilon)$ of system (4).

Proof. According to $\sup_{0<\varepsilon\le 1}\delta_\varepsilon < \infty$, $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon = 0$ and Theorem 10. It will prove $\delta_\varepsilon = O(\varepsilon)$ for $\varepsilon \downarrow 0$.

Define Lyapunov functions

$$\begin{aligned} V_\varepsilon(\Phi,\Psi) &= \lambda_\varepsilon(\Phi+\Psi)\varphi(\Phi,\Psi) \\ &\quad + \left(1-\lambda_\varepsilon(\Phi+\Psi)\right)\psi(\Phi,\Psi), \\ &\qquad \Phi,\Psi \ge \frac{M}{\varepsilon} \end{aligned} \tag{181}$$

where

$$\begin{aligned} M &= \sup_{1\ge\varepsilon>0, t\ge 0}\left(Ex_\varepsilon + Ey_\varepsilon\right), \\ \varphi(\Phi,\Psi) &= \Phi^4 + \Psi^4, \\ \psi(\Phi,\Psi) &= 17(\Phi+\Psi)^4, \\ &\lambda \in \mathscr{C}^2([0,\infty),\mathfrak{R}) \text{ satisfies } \frac{d\lambda}{dv} \le 0 \\ \text{and } \lambda(v) &= \begin{cases} 1 & v \le 1 \\ 0 & v \ge 2 \end{cases} \end{aligned} \tag{182}$$

and

$$\begin{aligned} \lambda_\varepsilon(v) &= \lambda\left(\alpha^{-1}\varepsilon v\right), \\ \text{and } \alpha &= M \cdot \max\{\kappa_1, \kappa_3, 1\}. \end{aligned} \tag{183}$$

Next, we prove that

$$\sup_{t\ge 0}\sup_{1\ge\varepsilon>0}\sup_{D_1,D_2} EV\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) < \infty \tag{184}$$

Let $L_{t,\varepsilon}$ denote the generator of the diffusion $(\Phi_\varepsilon(t), \Psi_\varepsilon(t))$. Then, we have

$$\begin{aligned} &L_{t,\varepsilon}V\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) \\ &\quad = \lambda_\varepsilon L_{t,\varepsilon}\varphi + \left(1+\lambda_\varepsilon\right)L_{t,\varepsilon}\psi \\ &\qquad + \alpha^{-1}\varepsilon\lambda_\varepsilon'(\varphi-\psi)\left(\Phi'+\Psi'\right) \\ &\qquad + \frac{1}{2}\alpha^{-2}\varepsilon^2\lambda_\varepsilon''(\varphi-\psi)\left(g_1^2+g_2^2\right) \\ &\qquad + \alpha^{-1}\varepsilon\lambda_\varepsilon'\left((\varphi_\Phi-\psi_\Phi)g_1^2 + (\psi_\Psi-\psi_\Psi)g_2^2\right). \end{aligned} \tag{185}$$

Firstly, we consider the case $\Phi + \Psi \ge 2\alpha/\varepsilon$ which shows $\lambda_\varepsilon = 0$ and hence

$$L_{t,\varepsilon}V\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) = L_{t,\varepsilon}\psi. \tag{186}$$

In case $\Phi + \Psi \ge \alpha/\varepsilon$

$$\begin{aligned} &\frac{1}{17}L_{t,\varepsilon}\psi\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) \\ &\quad \le (\Phi+\Psi)^3\left((r-D_1+x+Ex)\Phi - (D+D_2)\Psi\right) \\ &\qquad + 6(\Phi+\Psi)^2\left(g_1^2(\varepsilon\Phi+Ex) + g_2^2(\varepsilon\Psi+Ey)\right) \\ &\quad \le (\Phi+\Psi)^3\left(-(D_1-r-2M)\Phi - (D+D_2)\Psi\right) \\ &\qquad + 6(\Phi+\Psi)^2K^2\left((\varepsilon\Phi+M)^2 + (\varepsilon\Psi+M)^2\right). \end{aligned} \tag{187}$$

Now, for $\Phi, \Psi \ge -M/\varepsilon$, $\Phi + \Psi \ge \alpha/\varepsilon$, and $D_1 \ge r + 2M + 1$, we get

$$\begin{aligned} &(D_1-r-2M-1)\Phi + (D+D_2-1)\Psi \\ &\quad \ge \min\left\{(D_1-r-2M-1)\left(-\frac{M}{\varepsilon}\right)\right. \\ &\qquad + (D+D_2-1)\left(\Phi+\Psi+\frac{M}{\varepsilon}\right)(D_1-r-2M-1) \\ &\qquad \left.\cdot\left(\Phi+\Psi+\frac{M}{\varepsilon}\right) + (D+D_2-1)\left(-\frac{M}{\varepsilon}\right)\right\} \ge \varepsilon^{-1} \\ &\qquad \cdot \min\{(r+2M-1-\kappa_1D_2-\kappa_2)M \\ &\qquad + (D+D_2+1)(\alpha+M)(D_1-r-2M-1) \\ &\qquad \cdot(\alpha+M) - (D+1+\kappa_3D_1+\kappa_4)M\} = \varepsilon^{-1} \\ &\qquad \cdot \min\{D_2(\alpha+M-\kappa_1M) + (r+2M-\kappa_2-1)M \\ &\qquad + (D+1)(\alpha+M)D_1(\alpha+M-\kappa_3M) \\ &\qquad - (1+r+2M)(\alpha+M) - (D+\kappa_4-1)M\} \ge 0, \end{aligned} \tag{188}$$

if D_1 and D_2 are large enough. Then, we have

$$-(D_1-r-2M)\Phi - (D+D_2)\Psi \le -x-y. \tag{189}$$

Furthermore, since $\epsilon\Phi + M > 0$ and $\varepsilon\Psi + M > 0$, we have

$$\begin{aligned} &6(\Phi+\Psi)^2K^2\left((\varepsilon\Phi+M)^2 + (\varepsilon\Psi+M)^2\right) \\ &\quad \le 6(\Phi+\Psi)^2K^2(\varepsilon\Phi+M+\varepsilon\Psi+M)^2 \\ &\quad \le 6(\Phi+\Psi)^2K^2(\varepsilon(\Phi+\Psi)+2\alpha)^2 \\ &\quad \le 6(\Phi+\Psi)^4K^29\varepsilon^2 \le (\Phi+\Psi)^4 \end{aligned} \tag{190}$$

for all $\Phi + \Psi \ge \alpha/\varepsilon$ and $\varepsilon \le (54K^2)^{-1/2}$. Then

$$\begin{aligned} \frac{1}{17}L_{t,\varepsilon}\psi\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) &\le -3(\Phi+\Psi)^4 \\ &= -\frac{3}{17}\psi\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right) \end{aligned} \tag{191}$$

if ε is small enough.

If $\Phi + \Psi \le 2\alpha/\varepsilon$, then we have

$$\begin{aligned} L_{t,\varepsilon}\varphi(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) &\le 4\Phi^3\Big(-(D_1 - r)\Phi \\ &+ \frac{r}{\varepsilon K}\left(x^2 - Ex^2\right) + h(\epsilon, x, y, t)\Big) \\ &+ 4\Psi^3\left(-(D + D_2)\Psi + \mu h(\epsilon, x, y, t)\right) \\ &+ 6\Phi^2 g_1^2(\varepsilon\Phi + Ex) + 6\Psi^2 g_2^2(\varepsilon\Psi + Ey) \le -4(D_1 \\ &- r)\Phi^4 - 4(D + D_2)\Psi^4 + \frac{4r}{K}(2\alpha + M)\Phi^4 \\ &+ 4\left(\left|\Phi^3\right| + \left|\Psi^3\right|\right)\left|h(\epsilon, x, y, t)\right| + 6\left(\Phi^2 + \Psi^2\right) \\ &\cdot K^2(2\alpha + 2M)^2, \end{aligned} \tag{192}$$

where

$$\begin{aligned} \left|h(\epsilon, x, y, t)\right| &= \varepsilon^{-1}\left|E\frac{x(t)y(t)}{a + x^2(t)} - \frac{x(t)y(t)}{a + x^2(t)}\right| \\ &\le \frac{\varepsilon^{-1}}{a}\left|Ex(t)y(t)\right| + \frac{\varepsilon^{-1}}{a}\left|x(t)y(t)\right| \\ &\le \frac{5\varepsilon^{-1}}{2a}\left(Ex^2 + Ey^2\right) \\ &\quad + \frac{1}{2a}\left(\varepsilon\Phi^2 + 2M\Phi + \varepsilon\Psi^2 + 2M\Psi\right) \\ &\le \frac{5\varepsilon^{-1}}{2a}\left(E(\varepsilon\Phi + Ex)^2 + E(\epsilon\Psi + Ey)^2\right) \\ &\quad + \frac{1}{2a}\left(\varepsilon\Phi^2 + 2M\Phi + \varepsilon\Psi^2 + 2M\Psi\right). \end{aligned} \tag{193}$$

Hence, there is a constant b_3 satisfying

$$\begin{aligned} &\left(\left|\Phi^3\right| + \left|\Psi^3\right|\right)\left(\varepsilon\Phi^2 + 2M\Phi + \varepsilon\Psi^2 + 2M\Psi\right) \\ &\quad \le b_3\left(\Phi^4 + \Psi^4\right). \end{aligned} \tag{194}$$

Furthermore, let

$$\delta_\varepsilon(t) := \varepsilon \max\left\{E\left(\Phi_\varepsilon^4(t)\right)^{1/4}, E\left(\Psi_\varepsilon^4(t)\right)^{1/4}\right\}, \tag{195}$$

then

$$\begin{aligned} \varepsilon E\Phi_\varepsilon^2(t) &\le \frac{\delta_\varepsilon^2(t)}{\varepsilon}, \\ \varepsilon E\Psi_\varepsilon^2(t) &\le \frac{\delta_\varepsilon^2(t)}{\varepsilon}, \end{aligned} \tag{196}$$

and it shows

$$\begin{aligned} &\left(\left|\Phi^3\right| + \left|\Psi^3\right|\right)\frac{5\varepsilon^{-1}}{2a}\left(E(\varepsilon\Phi + Ex)^2 + E(\epsilon\Psi + Ey)^2\right) \\ &\quad \le \Phi^4 + \Psi^4 + 2\left(\frac{20\delta_\varepsilon^2(t)}{\varepsilon a}\right)^4 \end{aligned} \tag{197}$$

provided ε is small enough such that $\delta_\varepsilon \le 1$.

Furthermore, we have

$$\begin{aligned} &-4\left(D_1 - r - \frac{4r}{K}(2\alpha + M)\right)\Phi^4 - 4(D + D_2)\Psi^4 \\ &\quad + 6\left(\Phi^2 + \Psi^2\right)K^2(2\alpha + 2M)^2 \\ &\quad \le -\min\left\{D_1 - r - \frac{4r}{K}(2\alpha + M), 4(D + D_2)\right\} \\ &\quad \cdot\left(\Phi^4 + \Psi^4\right) + b_4 \end{aligned} \tag{198}$$

as D_1 and D_2 are large enough and b_4 is a suitable constant.

By (197) and (198), it has proven that there are constants b_4 and b_5 satisfying

$$\begin{aligned} L_{t,\varepsilon}\varphi(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) &\le -2\left(\Phi^4 + \Psi^4\right) + b_5\delta_\varepsilon^8(t)\varepsilon^{-4} \\ &\quad + b_4 \end{aligned} \tag{199}$$

if D_1 and D_2 are large enough and ε is small enough.

Next, our aim is to consider the remaining terms in $L_{t,\varepsilon}V(\Phi_\varepsilon(t), \Psi_\varepsilon(t))$; it contains derivatives of λ_ε.

Since $\lambda_\varepsilon' \le 0$,

$$-(D_1 - r - 2M)\Phi - (D + D_2)\Psi \le 0, \tag{200}$$

where $\lambda_\varepsilon' \ne 0$ and

$$\begin{aligned} \varphi(\Phi, \Psi) = \Phi^4 + \Psi^4 &\le \left(-\frac{M}{\varepsilon}\right)^4 + \left(\Phi + \Psi + \frac{M}{\varepsilon}\right)^4 \\ &\le (\Phi + \Psi)^4 + (2(\Phi + \Psi))^4 = 17(\Phi + \Psi)^4 \\ &= \psi(\Phi, \Psi) \quad \text{for } \frac{\alpha}{\varepsilon} \le (\Phi + \Psi) \le \frac{2\alpha}{\varepsilon}. \end{aligned} \tag{201}$$

Then, we have $\alpha^{-1}\varepsilon\lambda_\varepsilon'(\varphi - \psi)(-(D_1 - r - 2M)\Phi - (D + D_2)) \le 0$.

Furthermore, if ε is small enough. Then we have

$$\begin{aligned} &L_{t,\varepsilon}V(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) \\ &\quad \le -2\lambda_\varepsilon(\Phi + \Psi)\varphi(\Phi, \Psi) \\ &\qquad - 3\left(1 - \lambda_\varepsilon(\Phi + \Psi)\right)\psi(\Phi, \Psi) \\ &\qquad + \min\left(\varphi(\Phi, \Psi), \psi(\Phi, \Psi)\right) + b_7 + b_5\delta_\varepsilon^8(t)\varepsilon^{-4} \\ &\quad \le -V(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) + b_7 + b_5\delta_\varepsilon^8(t)\varepsilon^{-4} \end{aligned} \tag{202}$$

for $b_7 = b_4 + b_6$, $\Phi, \Psi \ge -M/\varepsilon$. Hence, we have

$$\begin{aligned} \frac{d}{dt}EV(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) &\le EL_{t,\varepsilon}V(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) \\ &\le -EV(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) + b_7 \\ &\quad + b_5\delta_\varepsilon^4\max\left\{E\Phi_\varepsilon^4(t), E\Psi_\varepsilon^4(t)\right\} \\ &\le -EV(\Phi_\varepsilon(t), \Psi_\varepsilon(t)) + b_7 \\ &\quad + b_5\delta_\varepsilon^4 EV(\Phi_\varepsilon(t), \Psi_\varepsilon(t)). \end{aligned} \tag{203}$$

Because $EV(\Phi_\varepsilon(t), \Psi_\varepsilon(t))$ are periodic and $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon^4 = 0$, hence it shows that

$$\sup_{t\geq 0}\sup_{1\geq\varepsilon>0}\sup_{D_1,D_2} EV\left(\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right)<\infty \tag{204}$$

if D_1 and D_2 satisfy the condition above and are large enough. Therefore, the proof is completed. □

Theorem 12. *Suppose that condition of Theorem 11 is satisfied. If D_1 and D_2 are large enough, as $\varepsilon\downarrow 0$, then $(\Phi_\varepsilon,\Psi_\varepsilon)$ converge weakly to (Φ_0,Ψ_0) which possesses the unique time-invariant distribution solution of*

$$\begin{pmatrix} d\Phi_0(t)\\ d\Psi_0(t)\end{pmatrix}=\begin{pmatrix} r-D_1-\dfrac{2rf_1^*(t)}{K}-\dfrac{af_2^*(t)-\left(f_1^*(t)\right)^2f_2^*(t)}{\left(a+\left(f_1^*(t)\right)^2\right)^2} & -\dfrac{f_1^*(t)}{a+\left(f_1^*(t)\right)^2}\\ -\mu\dfrac{af_2^*(t)-\left(f_1^*(t)\right)^2f_2^*(t)}{\left(a+\left(f_1^*(t)\right)^2\right)^2} & -\dfrac{\mu f_1^*(t)}{a+\left(f_1^*(t)\right)^2}-D_2\end{pmatrix}\begin{pmatrix}\Phi_0(t)\\ \Psi_0(t)\end{pmatrix}dt + \begin{pmatrix} g_1\left(f_1^*(t)\right)dW(t)\\ g_2\left(f_2^*(t)\right)dW(t)\end{pmatrix}. \tag{205}$$

Proof. By Theorem 11, it can easily see that the sequence $\mathscr{L}(\Phi_{\varepsilon_n}(0),\Psi_{\varepsilon_n}(0))$ is tight for any sequence $\varepsilon_n\downarrow 0$. Furthermore

$$\begin{aligned} d\Phi_\varepsilon(t)&=\Big(\left(r-D_1\right)\Phi_\varepsilon(t)-\frac{r}{K}\left(x^2-Ex^2\right)\\ &\quad-h\left(\varepsilon,\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right)\Big)dt+g_1\left(\varepsilon\Phi_\varepsilon(t)\right.\\ &\quad\left.+Ex_\varepsilon\right)dW(t)\\ d\Psi_\varepsilon(t)&=-\left(D+D_2\right)\Psi_\varepsilon(t)\\ &\quad+\mu h\left(\varepsilon,\Phi_\varepsilon(t),\Psi_\varepsilon(t)\right)dt+g_2\left(\varepsilon\Psi_\varepsilon(t)\right.\\ &\quad\left.+Ey_\varepsilon\right)dW(t),\end{aligned} \tag{206}$$

where h defined in (194).

By Theorem 11.1.4 in [29], if we can prove that (205) is no larger than one solution such that $\mathscr{L}(\Phi_0(\overline{\tau}),\Psi_0(\overline{\tau}))=\mathscr{L}(\Phi_0(0),\Psi_0(0))$ for some $\overline{\tau}>0$, $(\Phi_\varepsilon(t),\Psi_\varepsilon(t))$ converge weakly to the solution of (205). For any solution (Φ_0,Ψ_0) of (205), it can be described by

$$\begin{aligned}&\begin{bmatrix}\Phi_0(t)\\ \Psi_0(t)\end{bmatrix}\\ &\quad=Q(t)\left[\begin{bmatrix}\Phi_0(0)\\ \Psi_0(0)\end{bmatrix}+\int_0^t Q^{-1}(s)\Gamma(s)\begin{pmatrix}dW(t)\\ dW(t)\end{pmatrix}\right]\end{aligned} \tag{207}$$

where

$$Q(t)=\begin{pmatrix} g_1\left(f_1^*(s)\right) & 0\\ 0 & g_2\left(f_1^*(s)\right)\end{pmatrix}. \tag{208}$$

Let $\Phi_0^{(n)}(t)=\Phi_0^{(n)}(t+n\tau)$, $\Psi_0^{(n)}(t)=\Psi_0^{(n)}(t+n\tau)$, $t\geq 0$, then $(\Phi_0^{(n)},\Psi_0^{(n)})$ converge in $\mathscr{U}_1(C[0,\infty),\mathfrak{R}^2)$ to the Gaussian process

$$\begin{aligned}K(t)&=Q(t)\int_{-\infty}^t Q^{-1}(s)\Gamma(s)\,d\widehat{W}(t)\\ &=Q(t+n\tau)\int_{-\infty}^{t+n\tau}Q^{-1}(s)\Gamma(s)\,d\widehat{W}(s-n\tau)\end{aligned} \tag{209}$$

where $K(t)$ is a periodic. For all $n\in\mathbb{N}$, because the law of $(\Phi_0(0),\Psi_0(0))$ is periodic, it shows that $\mathscr{L}(\Phi_0(\tau),\Psi_0(\tau))=\mathscr{L}(\Phi_0(0),\Psi_0(0))$; therefore the laws of $(\Phi_0^{(n)},\Psi_0^{(n)})$ are identical with (Φ_0,Ψ_0); it is easy to see that the laws of (Φ_0,Ψ_0) are identical with $K(t)$ if system (205) shows the uniqueness periodic solution. Therefore, the proof is completed. □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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