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Research Article

The Harary Index of All Unicyclic Graphs with Given Diameter

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The Harary index of G is the sum of reciprocals of distance between any two vertices in G. In this paper, we obtain the graphs with the maximum and second-maximum Harary indices among n-vertex unicyclic graphs with diameter d.

1. Introduction

In this paper, all graphs are simple, undirected, and connected. Denote by $E(G_1)$, $V(G_1)$ the edge set and vertex set of G_1 and $|E(G_1)|$, $|V(G_1)|$ the number of edges and vertices in G_1, respectively. When $|E(G_1)| = |V(G_1)|$, the graph G_1 is called a unicyclic graph. Let $d(x)$ (or $d_{G_1}(x)$), $N_{G_1}(x)$ be the degree and the neighborhood of a vertex x in G_1, respectively. When $d(x) = 1$, we call x is a pendant vertex. The distance between x_1 and y_1 $(x_1, y_1 \in V(G_1))$ is denoted by $d_{G_1}(x_1, y_1)$. The diameter d of G_1 is $d = \max\{d_{G_1}(x_1, y_1) \mid \forall x_1, y_1 \in V(G_1)\}$. The graph $G_1 - x_1y_1$ (or $G_1 + x_1y_1$) arisen by deleting the edge $x_1y_1 \in E(G_1)$ (or by adding a new edge $x_1y_1 \notin E(G_1)$). Let P_k and C_k be the path and cycle with $|V(P_k)| = |V(C_k)| = k$, respectively. Denote by $P = v_0v_1 \cdots v_k$ $(k \geq 1)$ a path of G_1, where $d_{G_1}(v_1) = d_{G_1}(v_2) = \cdots = d_{G_1}(v_{k-1}) = 2$ (if $k > 1$). When $d_{G_1}(v_0) \geq 3$, $d_{G_1}(v_k) \geq 3$, P is an internal path of G_1. When $d_{G_1}(v_0) \geq 3$, $d_{G_1}(v_k) = 1$, P is a pendant path of G_1. If $k = 1$, specially, P is a pendant edge. When the subgraph P_1 is P itself, where P_1 is induced by $V(P)$ in G_1, P is an induced path. For other terminologies and notations, we refer the readers to [1].

The Harary index $H(G_1)$, has been introduced independently in [2, 3]. Its calculation is as follows:

$$H(G_1) = \sum_{x_1, y_1 \in V(G_1)} \frac{1}{d_{G_1}(x_1, y_1)}, \tag{1}$$

where $d_{G_1}(x_1, y_1)$ is defined as above and the sum goes over all the pairs of vertices in G_1.

The set of n-vertex unicyclic graphs with diameter d is denoted by $U_{n,d}$. The graph $\triangle^1_{n,d}$ (see Figure 1) on n vertices arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 2$ pendant edges to $v_{\lfloor d/2 \rfloor}$ and adding a new vertex v_{d+1} to be adjacent to $v_{\lfloor d/2 \rfloor}$ and $v_{\lfloor d/2 \rfloor + 1}$.

The graph $\triangle^2_{n,d}$ (see Figure 1) arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 2$ pendant edges to $v_{\lfloor d/2 \rfloor + 1}$ and adding a new vertex v_{d+1} to be adjacent to $v_{\lfloor d/2 \rfloor}$ and $v_{\lfloor d/2 \rfloor + 1}$. The n-vertex graph arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 3$ pendant edges to $v_{\lfloor d/2 \rfloor}$ and adding two new adjacent vertices v_{d+1} and v_{d+2} to be adjacent to $v_{\lfloor d/2 \rfloor}$, denoted by $\nabla_{n,d}$ (see Figure 2). So, if $d = 2$, the graph $\nabla_{n,2}$ arisen from C_3 by attaching $n - 3$ pendant edges to one vertex of C_3.

Note that n-vertex graph $\triangle^3_{n,d}$ arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 3$ pendant edges to $v_{\lfloor d/2 \rfloor}$ and one pendant edge to $v_{\lfloor d/2 \rfloor + 1}$ and adding a new vertex v_{d+1} to be adjacent to $v_{\lfloor d/2 \rfloor}$ and $v_{\lfloor d/2 \rfloor + 1}$. $\triangle^4_{n,d}$ arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 2$ pendant edges to $v_{\lceil d/2 \rceil}$ and adding a new vertex v_{d+1} to be adjacent to $v_{\lceil d/2 \rceil}$ and $v_{\lceil d/2 \rceil + 1}$.

Let $P_d = v_0v_1 \cdots v_d$, $U_0(v_i)$ be the graph with $d + 2$ vertices arisen from P_d by adding a new vertex u_0 to be adjacent to one vertex v_i $(v_i \in V(P_d), 1 \leq i \leq d - 1)$. $P_d(C_3; v_i)$ arisen from $P_d = v_0v_1 \cdots v_d$ by attaching $n - d - 3$ pendant vertices to v_i $(1 \leq i \leq d - 1)$ and identifying v_i with one vertex of C_3, denote $U^1_{n,d} = \{P_d(C_3; v_i) \mid 1 \leq i \leq d-1\}$. Let U_{d+2} be a $(d+2)$-vertex unicyclic graph arisen from $P_d = v_0v_1 \cdots v_d$ by adding a new vertex v_{d+1} to be adjacent to v_k and v_{k+1} (see Figure 2), $U_{d+2}(n - d - 2; v_j)$ arisen from U_{d+2} by attaching $n - d - 2$

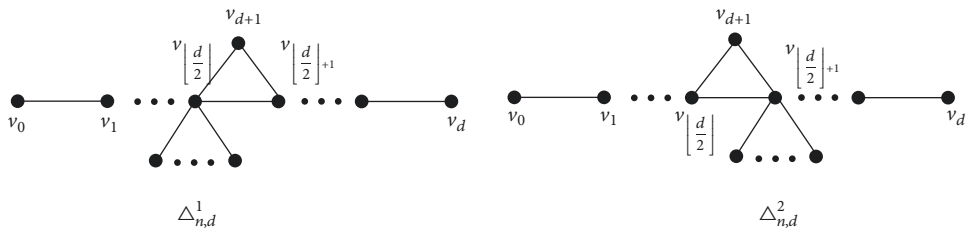

Figure 1: The graphs $\triangle^1_{n,d}$ and $\triangle^2_{n,d}$.

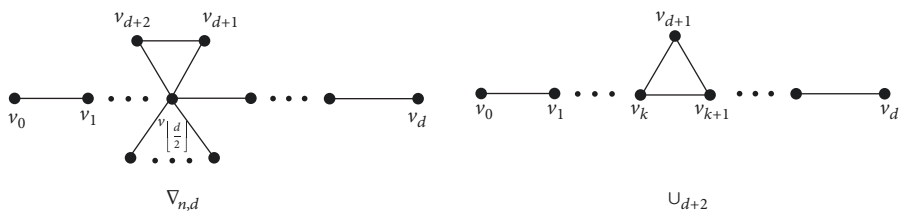

Figure 2: The graphs $\nabla_{n,d}$ and U_{d+2}.

pendant vertices to v_j ($1 \le j \le d-1$, *or* $j = d+1$), and denote $U^2_{n,d} = \{U_{d+2}(n-d-2; v_j) \mid 1 \le j \le d-1,$ *or* $j = d+1\}$.

Lately, the investigations on the spectral radius or eigenvalue or topological index of a special class of graphs and the topological properties of a certain class of networks remain a popular topic to researchers. For example, J.Q. Fei and J.H. Tu [4] characterized the complete characterization of the degree Kirchhoff index of bicyclic graphs. J.B. Liu et al. [5] discussed the topological properties of certain neural networks. B. Ning and B. Li [6] discussed the spectral radius of connected claw-free graphs. Y.Y. Wu and Y.J. Chen [7] analyzed the eccentric connectivity index of graphs. C. Wang et al. [8] obtained the least eigenvalue of the graphs whose complements are connected and have pendant paths. Many results can be seen in [2–16] and other papers. In particular, A. Ilic et al. [13] discussed the Harary index among the trees with fixed diameter. In this paper, we determine the graphs which have the maximum and second-maximum Harary indices among all the n-vertex unicyclic graphs with diameter d.

2. Lemmas

Lemma 1 (see [14]). *Let A, H_1, and H_2 be three connected, pairwise disjoint graphs; note that $u_1, u_2 \in V(A)$, $v_1 \in V(H_1)$, $w_1 \in V(H_2)$. The graph G is obtained by identifying the vertices u_1, v_1 and u_2, w_1 between A, H_1, H_2, respectively. G' is obtained by identifying the vertices u_1, v_1, w_1 from A, H_1, H_2 and G'' is obtained by identifying the vertices u_2, v_1, w_1 from A, H_1, H_2; then we can get*

$$H(G) < H(G') \text{ or } H(G) < H(G''). \tag{2}$$

Lemma 2 (see [9]). *Let G_1 be an n-vertex unicyclic graph and $\nabla_{n,2}$ be the graph defined as above. Then*

$$H(G_1) \le H(\nabla_{n,2}), \tag{3}$$

if and only if $G_1 \cong \nabla_{n,2}$, and the equality holds. When $d = 1$, then $G_1 \cong C_3$, and when $d = 2$, $n \le 5$, then $G_1 \cong C_4$.

Lemma 3. *Let $U_0(v_i)$ be the graph defined as above; then*

$$H(U_0(v_{\lfloor d/2 \rfloor})) \ge H(U_0(v_i)); \tag{4}$$

if and only if $i = \lfloor d/2 \rfloor$, the equality holds.

Proof. By the calculation of Harary index, we can get the following.

(1) If $1 \le i < \lfloor d/2 \rfloor$,

$$H(U_0(v_{i+1})) - H(U_0(v_i)) = \left(\frac{1}{i+2} - \frac{1}{d-i+1}\right) > 0. \tag{5}$$

(2) If $\lfloor d/2 \rfloor < i \le d-1$,

$$H(U_0(v_{i-1})) - H(U_0(v_i)) = \left(\frac{1}{d-i+2} - \frac{1}{i+1}\right) > 0. \tag{6}$$

When d is odd, we can get $H(U_0(v_{\lfloor d/2 \rfloor})) = H(U_0(v_{\lfloor d/2 \rfloor+1}))$.

Thus the result holds. □

Lemma 4. *Let $U^2_{n,d} = \{U_{d+2}(n-d-2; v_j) \mid 1 \le j \le d-1,$ *or* $j = d+1\}$ and $\triangle^1_{n,d}$ and $\triangle^2_{n,d}$ be the set and two graphs defined as above, respectively. $G_1 \in U^2_{n,d}$; then*

$$H(G_1) \le H(\triangle^1_{n,d}); \tag{7}$$

if and only if $G_1 \cong \triangle^1_{n,d}$, the equality holds.

Proof. Choose a graph $G_1 \in U^2_{n,d}$ $(3 \le d \le n-2)$, such that $H(G_1)$ is the maximum. The following claims play a crucial role.

Claim 1. $j \in \{k, k+1, \lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1\}$.

Proof. First, we prove $j \ne d+1$. Otherwise, $j = d+1$; denote $u_i \in N_{G_1}(v_{d+1}) \setminus \{v_k, v_{k+1}\}$, $1 \le i \le n-d-2$; let

$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_{d+1}u_i + \sum_{i=1}^{n-d-2} v_k u_i \text{ or}$$
$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_{d+1}u_i + \sum_{i=1}^{n-d-2} v_{k+1} u_i; \tag{8}$$

using Lemma 1, we get $H(G_1) < H(G_1^*)$, a contraction.

So, $1 \le j \le d-1$, let $w_i \in N_{G_1}(v_j) \setminus \{v_{j-1}, v_{j+1}\}$, $1 \le i \le n-d-2$, $N_{P_d}(v_j) = \{v_{j-1}, v_{j+1}\}$.

(1) If $1 \le j < k \le \lfloor d/2 \rfloor$, let

$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_j w_i + \sum_{i=1}^{n-d-2} v_k w_i; \tag{9}$$

if $\lfloor d/2 \rfloor \le k < k+1 < j \le d-1$, let

$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_j w_i + \sum_{i=1}^{n-d-2} v_{k+1} w_i. \tag{10}$$

(2) If $1 \le k+1 < j \le \lfloor d/2 \rfloor$ or $\lfloor d/2 \rfloor \le j < k < k+1 \le d-1$, let

$$G_1^* = G_1 - v_{d+1}v_k - v_{d+1}v_{k+1} + v_{d+1}v_j + v_{d+1}v_{j+1}. \tag{11}$$

(3) If $1 \le j < \lfloor d/2 \rfloor < k, k+1 \le d-1$, let

$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_j w_i - v_{d+1}v_k - v_{d+1}v_{k+1} + \sum_{i=1}^{n-d-2} v_{\lfloor d/2 \rfloor} w_i + v_{d+1}v_{\lfloor d/2 \rfloor} + v_{d+1}v_{\lfloor d/2 \rfloor + 1}; \tag{12}$$

i.e., $G_1^* \cong \triangle^1_{n,d}$;

if $1 \le k < k+1 < \lfloor d/2 \rfloor + 1 < j \le d-1$, let

$$G_1^* = G_1 - \sum_{i=1}^{n-d-2} v_j w_i - v_{d+1}v_k - v_{d+1}v_{k+1} + \sum_{i=1}^{n-d-2} v_{\lfloor d/2 \rfloor + 1} w_i + v_{d+1}v_{\lfloor d/2 \rfloor} + v_{d+1}v_{\lfloor d/2 \rfloor + 1}; \tag{13}$$

i.e., $G_1^* \cong \triangle^2_{n,d}$.

From Lemma 3 and the calculation of $H(G_1)$, we can calculate that $H(G_1) < H(G_1^*)$, a contraction.

Claim 2. $k = \lfloor d/2 \rfloor$.

Proof. Otherwise, if $k = j < \lfloor d/2 \rfloor$, let $G_1^* = G_1 - v_{d-1}v_d + v_0 v_d$; if $k = j > \lfloor d/2 \rfloor$, let $G_1^* = G_1 - v_0 v_1 + v_0 v_d$. From Lemma 3, we can get $H(G_1) < H(G_1^*)$, a contraction.

By Claims 1-2, we have the graph $G_1 \cong \triangle^1_{n,d}$ or $G_1 \cong \triangle^2_{n,d}$.

Claim 3. $H(\triangle^2_{n,d}) \le H(\triangle^1_{n,d})$.

Proof. If $d = 2t$ is even, we have

$$\begin{aligned} H\left(\triangle^2_{n,d}\right) - H\left(\triangle^1_{n,d}\right) &= H\left(\triangle^2_{n,2t}\right) - H\left(\triangle^1_{n,2t}\right) = (n \\ &- 2t - 2)\left\{\left(1 + \frac{1}{t+1} + \frac{1}{t+2} + 2\sum_{i=2}^{t}\frac{1}{i}\right)\right. \\ &\left. - \left(1 + 2\sum_{i=2}^{t+1}\frac{1}{i}\right)\right\} = (n-2t-2)\left(\frac{1}{t+2} - \frac{1}{t+1}\right) \\ &< 0. \end{aligned} \tag{14}$$

If $d = 2t+1$ is odd, $\triangle^1_{n,2t+1} \cong \triangle^2_{n,2t+1}$, $H(\triangle^1_{n,d}) - H(\triangle^2_{n,d}) = 0$.

By Claims 1-3, we have $G_1 \cong \triangle^1_{n,d}$.

Thus the result follows. □

3. Main Results

In this section, we will list our main results.

Theorem 5. *Let $U_{n,d}$ and $\triangle^1_{n,d}$ be defined in Section 1, the n-vertex graph $G_1 \in U_{n,d}$ $(3 \le d \le n-2)$; then*

$$H\left(\triangle^1_{n,d}\right) \ge H\left(G_1\right); \tag{15}$$

if and only if $G_1 \cong \triangle^1_{n,d}$, the equality holds.

Proof. Let $G_1 \in U_{n,d}$; using Lemma 2, the result holds for $d = 1, 2$. If $d = n-1$, then $G_1 \cong C_n$. So, we discuss that $3 \le d \le n-2$.

Choose a graph $G_1 \in U_{n,d}$ with $H(G_1)$ being the maximum. Note that C_q is the only cycle and $P_d = v_0 v_1 \cdots v_d$ is the induced path in G_1; assume that $d(v_0) = 1$; we consider the following claims.

Claim 1. $|V(C_q) \bigcap V(P_d)| > 0$.

Proof. Otherwise, suppose that there exists a path $P_k = v_h v_{g+1} \cdots v_{g+k-1} v_l$ connecting the cycle C_q and the path P_d with $v_h \in V(P_d)$, $v_l \in V(C_q)$; denote $u_s \in N_{G_1}(v_h) \setminus \{v_{g+1}\}$, $1 \le s \le d(v_h) - 1$ and $w_j \in N_{G_1}(v_l) \setminus \{v_{g+k-1}\}$, $1 \le j \le d(v_l) - 1$. Let

$$G_1^* = G_1 - \sum_{s=1}^{d(v_h)-1} v_h u_s + \sum_{s=1}^{d(v_h)-1} v_l u_s \text{ or}$$
$$G_1^* = G_1 - \sum_{j=1}^{d(v_l)-1} v_l w_j + \sum_{j=1}^{d(v_l)-1} v_h w_j. \tag{16}$$

Then, applying Lemma 1, $H(G_1) < H(G_1^*)$, a contraction.

Using Claim 1, we note that $V(C_q)\cap V(P_d) = \{v_t, v_{t+1}, \cdots, v_{t+k}\}$, $V(C_q) \setminus V(P_d) = \{v_{d+1}, v_{d+2}, \cdots, v_s\}$.

Claim 2. For any $v \in V(G_1) \setminus (V(P_d) \bigcup V(C_q))$, $d(v) = 1$.

Proof. Otherwise, assume that there exists a vertex $u \in V(G_1)\setminus (V(P_d) \bigcup V(C_q))$ with uv_b $(1 \le b \le d(u) - 1)$, $uv_f \in E(G_1)$, $v_b \in V(G_1) \setminus (V(P_d) \bigcup V(C_q))$, $v_f \in V(P_d) \bigcup V(C_q)$. Suppose that $u'_s \in N_{G_1}(v_f) \setminus \{u\}$, $1 \le s \le d(v_f) - 1$. Let

$$G_1^* = G_1 - \sum_{s=1}^{d(v_f)-1} v_f u'_s + \sum_{s=1}^{d(v_f)-1} uu'_s \text{ or}$$
$$G_1^* = G_1 - \sum_{b=1}^{d(u)-1} v_b u + \sum_{b=1}^{d(u)-1} v_b v_f. \tag{17}$$

Then, using Lemma 1, $H(G_1) < H(G_1^*)$, a contraction.

By Claim 2, we have any vertex $v \in V(G_1) \setminus (V(P_d) \bigcup V(C_q))$ is pendant vertex.

Claim 3. $|V(C_q) \cap V(P_d)| \le 2$.

Proof. Otherwise, suppose that $V(C_q)\cap V(P_d) = \{v_t, v_{t+1}, \cdots, v_{t+k}\}$, $k \ge 2$.

(1) If $t \le \lfloor d/2 \rfloor - 1$, denote $u_i \in N_{G_1}(v_t) \setminus \{v_s, v_{t-1}, v_{t+1}\}$, $1 \le i \le d(v_t) - 3$ (if exists), $N_{P_d}(v_t) = \{v_{t-1}, v_{t+1}\}$. Let

$$G_1^* = G_1 - v_s v_t + v_s v_{t+1} - \sum_{i=1}^{d(v_t)-3} v_t u_i + \sum_{i=1}^{d(v_t)-3} v_{t+1} u_i. \tag{18}$$

(2) If $t + k \ge \lfloor d/2 \rfloor + 2$, denote $u_s \in N_{G_1}(v_{t+k}) \setminus \{v_{d+1}, v_{t+k-1}, v_{t+k+1}\}$, $1 \le s \le d(v_{t+k}) - 3$ (if exists), $N_{P_d}(v_{t+k}) = \{v_{t+k-1}, v_{t+k+1}\}$. Let

$$G_1^* = G_1 - v_{d+1} v_{t+k} + v_{d+1} v_{t+k-1} - \sum_{s=1}^{d(v_{t+k})-3} v_{t+k} u_s + \sum_{s=1}^{d(v_{t+k})-3} v_{t+k-1} u_s. \tag{19}$$

Then, applying Lemma 3 and the calculation of $H(G_1)$, $H(G_1) < H(G_1^*)$, a contraction.

Claim 4. (1) *if* $|V(C_q)\cap V(P_d)| = 2$, $s = d+1$; (2) *if* $|V(C_q)\cap V(P_d)| = 1$, $s = d + 2$.

Proof. (1) Otherwise, we assume that $s \ge d + 2$. The graph G_1^* arisen by deleting all edges in $E(C_q)$ of G_1 and incidence with C_q except the edges $v_t v_{t+1}, v_{t+1} v_{d+1}$ of G_1, adding the edge $v_t v_{d+1}$, connecting all isolated vertices to the vertex v_t (if $t \ge \lfloor d/2 \rfloor$) or to the vertex v_{t+1} (if $t + 1 \le \lfloor d/2 \rfloor$) of G_1. Together with the definition of the Harary index and Lemma 3, $H(G_1) < H(G_1^*)$, a contraction.

(2) Otherwise, we assume that $s \ge d + 3$. The graph G_1^* is obtained from G_1 by deleting all edges in $E(C_q)$ and incidence with C_q except the edges $v_t v_{d+1}, v_{d+1} v_{d+2}$, adding the edge $v_t v_{d+2}$, connecting all isolated vertices to the vertex v_t. We can calculate that $H(G_1) < H(G_1^*)$, a contraction.

By Claim 4 and Lemmas 1 and 3, we get $G_1 \in U_{n,d}^1 \bigcup U_{n,d}^2$. If $G_1 \in U_{n,d}^1$, applying Lemma 3, $G_1 \cong \nabla_{n,d}$ (see Figure 2); if $G_1 \in U_{n,d}^2$, applying Lemma 4, $G_1 \cong \triangle_{n,d}^1$.

Claim 5. $H(\triangle_{n,d}^1) > H(\nabla_{n,d})$.

Proof. If $d = 2t$ is even, we have

$$\begin{aligned} H\left(\triangle_{n,d}^1\right) - H\left(\nabla_{n,d}\right) &= H\left(\triangle_{n,2t}^1\right) - H\left(\nabla_{n,2t}\right) \\ &= \left\{\left(\frac{1}{t+1} + 2\sum_{i=1}^{t}\frac{1}{i} + \frac{1}{2}\right) - \left(1 + 2\sum_{i=2}^{t+1}\frac{1}{i} + 1\right)\right\} \\ &= \frac{1}{2} - \frac{1}{t+1} > 0. \end{aligned} \tag{20}$$

If $d = 2t + 1$ is odd, we have

$$\begin{aligned} H\left(\triangle_{n,d}^1\right) - H\left(\nabla_{n,d}\right) &= H\left(\triangle_{n,2t+1}^1\right) - H\left(\nabla_{n,2t+1}\right) \\ &= \left\{\left(2\sum_{i=1}^{t+1}\frac{1}{i} + \frac{1}{2}\right) - \left(1 + 2\sum_{i=2}^{t+1}\frac{1}{i} + \frac{1}{t+2} + 1\right)\right\} \\ &= \frac{1}{2} - \frac{1}{t+2} > 0. \end{aligned} \tag{21}$$

By Claims 1-5, we have $G_1 \cong \triangle_{n,d}^1$.

Thus the result follows. □

Theorem 6. *Let $U_{n,d}, \triangle_{n,d}^1$ be defined in Section 1; the graph $G_1 \in U_{n,d} \setminus \{\triangle_{n,d}^1\}$ $(3 \le d \le n - 2)$; then*

(1) If $d = 2t$ is even, we obtain

$$H\left(\nabla_{n,d}\right) \ge H\left(G_1\right), \quad \text{when } 2n \ge t^2 + 5t + 2; \tag{22}$$

if and only if $G_1 \cong \nabla_{n,d}$, the equality holds.

$$H\left(\triangle_{n,d}^2\right) \ge H\left(G_1\right), \quad \text{when } 4t + 8 \le 2n \le t^2 + 5t + 2; \tag{23}$$

if and only if $G_1 \cong \triangle_{n,d}^2$, the equality holds.

(2) If $d = 2t + 1$ is odd, we have

$$H\left(\triangle_{n,d}^4\right) \ge H\left(G_1\right); \tag{24}$$

if and only if $G_1 \cong \triangle_{n,d}^4$, the equality holds.

Proof. Choose a graph $G_1 \in U_{n,d} \setminus \{\triangle_{n,d}^1\}$, such that $H(G_1)$ is as large as possible; together with Lemmas 1, 3, and 4 and Theorem 5, we have the following.

(1) If $d = 2t$ is even, $G_1 \cong \triangle_{n,d}^2$ or $G_1 \cong \triangle_{n,d}^3$ or $G_1 \cong \nabla_{n,d}$.

In fact,

$$\begin{aligned}
&H\left(\triangle_{n,d}^{3}\right)-H\left(\triangle_{n,d}^{2}\right)\\
&\quad=(n-d-3)\left(\frac{11}{6}+2\sum_{i=2}^{t+1}\frac{1}{i}\right)\\
&\qquad-(n-d-3)\left(2+2\sum_{i=2}^{t}\frac{1}{i}+\frac{1}{t+1}+\frac{1}{t+2}\right)\\
&\quad=(n-d-3)\left(\frac{1}{t+1}-\frac{1}{t+2}-\frac{1}{6}\right)<0,\\
&H\left(\triangle_{n,d}^{3}\right)-H\left(\nabla_{n,d}\right)\\
&\quad=\left(\frac{3}{2}+2\sum_{i=2}^{t}\frac{1}{i}+\frac{1}{t+1}+\frac{1}{t+2}+\frac{n-d-3}{3}\right)\\
&\qquad-\left(2+2\sum_{i=2}^{t+1}\frac{1}{i}+\frac{n-d-3}{2}\right)\\
&\quad=-\frac{1}{(t+1)(t+2)}-\frac{1}{2}-\frac{n-d-3}{6})<0,\\
&H\left(\nabla_{n,d}\right)-H\left(\triangle_{n,d}^{2}\right)\\
&\quad=(n-d-3)\left(2+2\sum_{i=2}^{t+1}\frac{1}{i}\right)\\
&\qquad-(n-d-3)\left(2+2\sum_{i=2}^{t}\frac{1}{i}+\frac{1}{t+1}+\frac{1}{t+2}\right)\\
&\qquad+\left(3+4\sum_{i=2}^{t+1}\frac{1}{i}\right)\\
&\qquad-\left(\frac{7}{2}+4\sum_{i=2}^{t}\frac{1}{i}+\frac{2}{t+1}+\frac{1}{t+2}\right)\\
&\quad=(n-d-2)\left(\frac{1}{t+1}-\frac{1}{t+2}\right)+\left(\frac{1}{t+1}-\frac{1}{2}\right).
\end{aligned}\tag{25}$$

(1.1) When $2n \geq t^2+5t+2$, $H(\nabla_{n,d})-H(\triangle_{n,d}^2)\geq 0$.

(1.2) When $4t+8\leq 2n\leq t^2+5t+2$, $H(\nabla_{n,d})-H(\triangle_{n,d}^2)\leq 0$.

(2) If $d=2t+1$ is odd, $H(\triangle_{n,d}^1)=H(\triangle_{n,d}^2)$, so $G_1\cong\triangle_{n,d}^3$ or $G_1\cong\triangle_{n,d}^4$ or $G_1\cong\nabla_{n,d}$.

$$\begin{aligned}
&H\left(\triangle_{n,d}^{3}\right)-H\left(\triangle_{n,d}^{4}\right)\\
&\quad=\frac{1}{3}(n-d-3)-\frac{1}{2}(n-d-3)\\
&\qquad+\left(3+\frac{1}{t+2}+4\sum_{i=2}^{t+1}\frac{1}{i}\right)\\
&\qquad-\left(3+\frac{2}{t+2}+3\sum_{i=2}^{t+1}\frac{1}{i}+\sum_{i=2}^{t}\frac{1}{i}\right)\\
&\quad=(n-d-3)\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{t+1}-\frac{1}{t+2}\right)<0,\\
&H\left(\nabla_{n,d}\right)-H\left(\triangle_{n,d}^{4}\right)\\
&\quad=\left(3+\frac{2}{t+2}+4\sum_{i=2}^{t+1}\frac{1}{i}\right)\\
&\qquad-\left(\frac{7}{2}+\frac{2}{t+2}+3\sum_{i=2}^{t+1}\frac{1}{i}+\sum_{i=2}^{t}\frac{1}{i}\right)\\
&\quad=\left(\frac{1}{t+1}-\frac{1}{2}\right)<0.
\end{aligned}\tag{26}$$

Thus the result follows. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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