Research Article

The Tropical Matrix Groups with Symmetric Idempotents

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In this paper we study the semigroup $M_n(\mathbb{T})$ of all $n \times n$ tropical matrices under multiplication. We give a description of the tropical matrix groups containing a diagonal block idempotent matrix in which the main diagonal blocks are real matrices and other blocks are zero matrices. We show that each nonsingular symmetric idempotent matrix is equivalent to this type of block diagonal matrix. Based upon this result, we give some decompositions of the maximal subgroups of $M_n(\mathbb{T})$ which contain symmetric idempotents.

1. Introduction

Tropical algebra (also known as max-plus algebra or max-algebra) is the algebra of the real numbers extended by adding an infinite negative element $-\infty$ when equipped with the binary operations of addition and maximum. It has applications in areas such as combinatorial optimization and scheduling, control theory, discrete event dynamic systems, and many other areas of science (see [1–9]). Many problems arising from these application areas are expressed using (tropical) linear equations, so many authors study tropical matrices, i.e., matrices over tropical algebra.

For example, consider the multi-machine interactive production process (MMIPP) [4] where products $P_1, \dots, P_m$ are prepared using n machines, every machine contributing to the completion of each product by producing a partial product. It is assumed that every machine can work for all products simultaneously and that all these actions on a machine start as soon as the machine starts to work. Let a_{ij} be the duration of the work of the j th machine needed to complete the partial product for P_i ($i = 1, \dots, m$, $j = 1, \dots, n$). If this interaction is not required for some i and j , then a_{ij} is set to $-\infty$. Denote the starting time of the j th machine by x_j . Then all partial products for P_i ($i = 1, \dots, m$) will be ready at time

$$\max \{x_1 + a_{i1}, \dots, x_n + a_{in}\}. \quad (1)$$

Hence if b_i ($i = 1, \dots, m$) are given completion times then the starting times have to satisfy the system of equations:

$$\begin{aligned} (\forall i \in \{1, \dots, m\}) \\ \max \{x_1 + a_{i1}, \dots, x_n + a_{in}\} = b_i. \end{aligned} \quad (2)$$

The problem can be converted into a related problem in tropical matrices.

From an algebraic perspective, a key object is the multiplicative semigroup of all square matrices of a given size over the tropical algebra. There are a series of papers in the literature considering this multiplicative semigroup (see [10–17]). Moreover, an important step in understanding tropical algebra is to understand the maximal subgroups of this semigroup. It is a basic fact of semigroup theory that every subgroup of a semigroup S lies in a unique maximal subgroup. Moreover, the maximal subgroups of S are precisely the $\mathcal{H}$ -classes (see Section 2 below for definitions) of S which contain idempotents element. Johnson and Kambites [16] give a classification of the maximal subgroups of the semigroup of all 2×2 tropical matrices under multiplication in 2011. Izhakian, Johnson, and Kambites [13] consider the case of matrices without $-\infty$. They prove that every subgroup of the multiplicative semigroup of $n \times n$ finite tropical matrices is isomorphic to a direct product of the form $\mathbb{R} \times \Sigma$ for some $\Sigma \leq S_n$. In the same year, Shitov [17] gives a description of the subgroups of the multiplicative semigroup of $n \times n$ tropical matrices up to isomorphism; i.e., every subgroup

of the semigroup admits a faithful representation with $n \times n$ tropical invertible matrices. In 2017, we showed that a maximal subgroup of the multiplicative semigroup of $n \times n$ tropical matrices containing a nonsingular idempotent matrix E is isomorphic to the group of all invertible matrices which commute with E as groups and proved that each maximal subgroup of the multiplicative semigroup of $n \times n$ tropical matrices with the identity of the rank r is isomorphic to some maximal subgroup of the multiplicative semigroup of $r \times r$ tropical matrices with nonsingular identity. Thus we shall turn our attention towards the invertible matrices that commute with the nonsingular idempotent. The main purpose of this paper is to study the invertible matrices that commute with a nonsingular symmetric idempotent and to give a decomposition of the maximal subgroups of $n \times n$ tropical matrices containing a nonsingular symmetric idempotent.

This paper will be divided into five sections. In Section 2 we introduce some preliminary notions and notation. The decompositions of the maximal subgroups of $n \times n$ tropical matrices containing an idempotent diagonal block matrix are established in Section 3. This result (see Theorem 11) develops the results obtained by Izhakian et al. in [13]. Finally, in the last section, we prove that each symmetric nonsingular idempotent matrix is equivalent to a block diagonal matrix and a decomposition of the maximal subgroup containing a symmetric idempotent matrix is given (see Theorem 17).

2. Preliminaries

The following notation and definitions can be found in [3, 15, 18, 19]. We write $\mathbb{T}$ for the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of maximum (denoted by $\oplus$) and addition (denoted by $\otimes$). Thus, we write

$$a \oplus b = \max\{a, b\} \quad (3)$$

and $a \otimes b = a + b$.

As usual, the set of all $m \times n$ tropical matrices is denoted by $M_{m \times n}(\mathbb{T})$. In particular, we shall use $M_n(\mathbb{T})$ instead of $M_{n \times n}(\mathbb{T})$. The operations $\oplus$ and $\otimes$ on $\mathbb{T}$ induce corresponding operations on tropical matrices in the obvious way. Indeed, if $A, B \in M_{m \times n}(\mathbb{T})$, $C \in M_{n \times p}(\mathbb{T})$, then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij},$$

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^n a_{ik} \otimes c_{kj}, \quad (4)$$

where x_{ij} denotes the (i, j) th entry of the matrix X . For brevity, we shall write AC in place of $A \otimes C$. It is easy to see that $(M_n(\mathbb{T}), \otimes)$ is a semigroup. Other concepts such as transpose and block matrix are defined in the usual way. Unless otherwise stated, we refer to matrix as tropical matrix in the remainder of this paper. Recall that Green's relations $\mathcal{R}$ and $\mathcal{L}$ [20] on the semigroup $M_n(\mathbb{T})$ are, respectively, given by

$$A \mathcal{R} B \iff (\exists X, Y \in M_n(\mathbb{T})) A = BX, B = AY;$$

$$A \mathcal{L} B \iff (\exists X, Y \in M_n(\mathbb{T})) A = XB, B = YA. \quad (5)$$

Green's relation $\mathcal{H}(\mathcal{D}, \text{resp.})$ is given by $\mathcal{H} = \mathcal{R} \cap \mathcal{L}(\mathcal{D} = \mathcal{R} \circ \mathcal{L}, \text{resp.})$. The $\mathcal{H}$ -class ($\mathcal{D}$ -class, resp.) containing the matrix A will be written as H_A (D_A , resp.).

We shall be interested in the space $\mathbb{T}^n$ of *affine tropical vectors*. We write x_i for the i th component of a vector $x \in \mathbb{T}^n$. We extend $\oplus$ to $\mathbb{T}^n$ componentwise so that $(x \oplus y)_i = x_i \oplus y_i$ for all i . And we define a scaling action of $\mathbb{T}$ on $\mathbb{T}^n$ by

$$\lambda \otimes (x_1, x_2, \dots, x_n) = (\lambda \otimes x_1, \lambda \otimes x_2, \dots, \lambda \otimes x_n), \quad (6)$$

for each $\lambda \in \mathbb{T}$ and each $x \in \mathbb{T}^n$. These operations give $\mathbb{T}^n$ the structure of a $\mathbb{T}$ -semimodule.

A *tropical convex set* in $\mathbb{T}^n$ is a subset closed under $\oplus$ and scaling by elements of $\mathbb{T}$, that is, a $\mathbb{T}$ -subsemimodule of $\mathbb{T}^n$. If $S \subseteq \mathbb{T}^n$, then the *tropical convex hull* of S is the smallest tropical convex set containing S , that is, the set of all vectors in $\mathbb{T}^n$ which can be written as tropical linear combinations of finitely many vectors from S .

Let X be a finitely generated tropical convex set in $\mathbb{T}^n$. A set $\{x_1, x_2, \dots, x_k\} \in X$ is called a *weak basis* of X if it is a generating set for X minimal with respect to inclusion. It is known that every finitely generated tropical convex set admits a weak basis, which is unique up to permutation and scaling (see [[19], Theorem 5]). In particular, any two weak bases have the same cardinality, in view of which we may define the *generator dimension* of a finitely generated tropical convex set X to be the cardinality of a weak basis for X , or, equivalently, the minimum cardinality of a generating set for X .

Given an $m \times n$ matrix A we define the *column space* of A , denoted by $\text{Col}(A)$, to be the tropical convex hull of the columns of A . Thus $\text{Col}(A) \in \mathbb{T}^m$. Similarly, we define the *row space* $\text{Row}(A) \in \mathbb{T}^n$ to be the tropical convex hull of the rows of A . The *column rank* of A is the generator dimension for the column space of A . The *row rank* of A is defined dually; it is well known that the row rank and column rank of a tropical matrix can differ (see [[15] Example 7.1]). The column rank (row rank, resp.) of A is denoted by $c(A)$ ($r(A)$, resp.). We denote the i -th row and the j -th column of A by $\mathbf{a}_{i*}$ and $\mathbf{a}_{*j}$, respectively. If $c(A) = s$ and $r(A) = r$, then it is easy to see that there exist s columns $\mathbf{a}_{*i_1}, \dots, \mathbf{a}_{*i_s}$ of A such that $\{\mathbf{a}_{*i_1}, \dots, \mathbf{a}_{*i_s}\}$ is a weak basis of $\text{Col}(A)$ and there exist r rows $\mathbf{a}_{j_1*}, \dots, \mathbf{a}_{j_r*}$ of A such that $\{\mathbf{a}_{j_1*}, \dots, \mathbf{a}_{j_r*}\}$ is a weak basis of $\text{Row}(A)$. The submatrix

$$\left(\begin{array}{c} [\mathbf{a}_{*i_1} \ \cdots \ \mathbf{a}_{*i_s}] \\ \left[\begin{array}{c} \mathbf{a}_{j_1*} \\ \vdots \\ \mathbf{a}_{j_r*} \end{array} \right] \end{array} \right), \left(\begin{array}{c} \left[\begin{array}{ccc} \mathbf{a}_{j_1 i_1} & \mathbf{a}_{j_1 i_2} & \cdots & \mathbf{a}_{j_1 i_s} \\ \mathbf{a}_{j_2 i_1} & \mathbf{a}_{j_2 i_2} & \cdots & \mathbf{a}_{j_2 i_s} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{j_r i_1} & \mathbf{a}_{j_r i_2} & \cdots & \mathbf{a}_{j_r i_s} \end{array} \right], \text{resp.} \end{array} \right) \quad (7)$$

of A is said to be a *column basis submatrix* of A (a *row basis submatrix* of A , a *basis submatrix* of A , resp.). If $c(A) = r(A) = r$, then r is called the *rank* of A . If $c(A) = n$ ($r(A) = m$, resp.), then A is called *column compressed* (*row compressed*, resp.)

[21]. The matrix A is called *nonsingular* if it is both column compressed and row compressed, and *singular* otherwise.

In the sequel, the following notions and notation are needed for us.

- (i) An $n \times n$ matrix A is called a *symmetric matrix* if $A^T = A$.
- (ii) $\text{diag}(A_1, A_2, \dots, A_n)$ denotes the diagonal block matrix

$$\begin{bmatrix} A_1 & -\infty & \cdots & -\infty \\ -\infty & A_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & A_n \end{bmatrix}, \quad (8)$$

where each diagonal block A_i is a square matrix, for all $1 \leq i \leq n$. Particularly, the matrix $\text{diag}(a_1, a_2, \dots, a_n)$ will be called *diagonal* if all of $a_1, a_2, \dots, a_n$ are real numbers.

- (iii) I_n denotes the *identity matrix*, i.e., the $n \times n$ matrix $\text{diag}(1, 1, \dots, 1)$.
- (iv) An $n \times n$ matrix A is called *invertible* if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called an inverse of A and is denoted by A^{-1} .
- (v) An $n \times n$ matrix is called a *monomial matrix* if it has exactly one entry in each row and column which is not equal to $-\infty$.
- (vi) An $n \times n$ matrix is called a *permutation matrix* if it is formed from the identity matrix by reordering its columns and/or rows.
- (vii) $-\infty$ denotes the *zero matrix*, i.e., the matrix whose entries are all $-\infty$.

It is well known that an $n \times n$ matrix A is invertible if and only if A is monomial [22]. Also, the inverse of a permutation matrix is its transpose. Denote the set of all $n \times n$ monomial matrices (permutation matrices, resp.) by $GL_n(\mathbb{T})$ ($P_n(\mathbb{T})$, resp.). Then $GL_n(\mathbb{T})$ and $P_n(\mathbb{T})$ are group under the matrix multiplication.

There are two types of elementary matrices corresponding to the two types of elementary operations.

Type 1. An elementary matrix of Type 1 is a matrix obtained by interchanging two rows (columns, resp.) of I_n . We write $E_{i,j}$ as the matrix obtained by trading places of rows (or columns) i and j of I_n .

Type 2. An elementary matrix of Type 2 is a matrix obtained by multiplying a row (column, resp.) of I_n by a constant $a \neq -\infty$. We write $E_i(a)$ as the matrix obtained by multiplying row (or column) i of the identity matrix by $a \neq -\infty$.

Recall that if A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single elementary row (column, resp.) operation, then there is an elementary matrix of size m (n , resp.) that will convert A to B via matrix multiplication on the left (right, resp.). Thus it is easy to see that a matrix is monomial if and only if it may be decomposed

into the product of a finite number of elementary matrices. Also, it is worth mentioning that an elementary column (row, resp.) operation on a matrix does not change the linear relationship among the row (column, resp.) vectors. That is to say, if $A, B \in M_{m \times n}(\mathbb{T})$ and $A = BM$ for some $n \times n$ monomial matrix M , then

$$\begin{aligned} \mathbf{a}_{k*} &= \lambda_1 \otimes \mathbf{a}_{i_1*} \oplus \cdots \oplus \lambda_s \mathbf{a}_{i_s*} \iff \\ \mathbf{b}_{k*} &= \lambda_1 \otimes \mathbf{b}_{i_1*} \oplus \cdots \oplus \lambda_s \mathbf{b}_{i_s*}, \end{aligned} \quad (9)$$

where $\mathbf{a}_{k*}, \mathbf{a}_{i_1*}, \dots, \mathbf{a}_{i_s*}$ are some rows of A , $\mathbf{b}_{k*}, \mathbf{b}_{i_1*}, \dots, \mathbf{b}_{i_s*}$ are the corresponding rows of B , and $\lambda_1, \dots, \lambda_s \in \mathbb{T}$.

We say that matrices A and B are *equivalent* [23] (notation $A \equiv B$) if $B = PNP^T$ for some permutation matrix P , that is, B can be obtained by a simultaneous permutation of the rows and columns of A .

3. Tropical Matrix Groups Containing a Diagonal Block Idempotent

In this section, we study the tropical matrix groups containing a diagonal block idempotent. First, we will need the following notation and results in [13]. Let E be an $n \times n$ nonsingular idempotent matrix. We denote the set of all monomial matrices commuting with E by G_E . That is to say,

$$G_E = \{M \mid M \in GL_n(\mathbb{T}), ME = EM\}. \quad (10)$$

The $\mathcal{H}$ -classes containing an $n \times n$ idempotent matrix are the maximal subgroups of the semigroup $M_n(\mathbb{T})$. By Theorems 4.3 and 5.3 in [13], we have the following.

Lemma 1. *Let E be an idempotent of rank r . Then H_E is isomorphic to $G_{\bar{E}}$ as groups, where $\bar{E}$ is a basis submatrix of E .*

Since each basis submatrix of an idempotent is a nonsingular idempotent matrix, we need only to study the group G_E , in which E is a nonsingular idempotent matrix. Indeed it is easy to see the following.

Lemma 2. *$E = \text{diag}(E_1, E_2, \dots, E_k)$ is a nonsingular idempotent matrix if and only if $E_1, E_2, \dots, E_k$ are nonsingular idempotent matrices.*

We can say immediately that $G_{I_n} = GL_n(\mathbb{T})$, which is isomorphic to $\mathbb{R} \wr S_n$ as groups. More generally, we have the following.

Lemma 3. *If*

$$E = \begin{bmatrix} F & -\infty & \cdots & -\infty \\ -\infty & F & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & F \end{bmatrix} \quad (11)$$

is an $n \times n$ nonsingular idempotent matrix, where the diagonal blocks are k real square matrices, then $G_E =$

$$\left\{ M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{bmatrix} \in GL_n(\mathbb{T}) \mid (\exists \sigma \in S_k) (\forall i, j \in [k]) \begin{matrix} M_{ij} \in G_F & j = \sigma(i) \\ M_{ij} = -\infty & j \neq \sigma(i) \end{matrix} \right\}. \quad (12)$$

Proof. Suppose that $E = \text{diag}(F, F, \dots, F)$ is an $n \times n$ nonsingular idempotent matrix and that F is a real matrix. Then by Lemma 2 we can find that F is an $(n/k) \times (n/k)$ real nonsingular idempotent matrix. If $M \in G_E$, then partition M in the same manner of E , i.e.,

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{bmatrix}, \quad (13)$$

where M_{ij} are all $(n/k) \times (n/k)$ matrices, and we have

$$EM = ME. \quad (14)$$

Thus we can see that

$$FM_{ij} = M_{ij}F, \quad (15)$$

for any $i, j \in [k]$.

Now we claim that

$$\begin{aligned} & \text{if } M_{ij} \notin GL_{n/k}(\mathbb{T}), \\ & \text{then } M_{ij} = -\infty. \end{aligned} \quad (16)$$

If $M_{ij} \notin GL_{n/k}(\mathbb{T})$, then M_{ij} has some row where entries are all $-\infty$ or M_{ij} has some column where entries are all $-\infty$, since M_{ij} is a submatrix of the monomial matrix M . Without loss of generality, we assume that M_{ij} has one row where entries are all $-\infty$; thus $M_{ij}F = FM_{ij}$ has one row where entries are all $-\infty$. Since F is real matrix, it follows that $M_{ij} = -\infty$, for otherwise FM_{ij} does not have one row where entries are all $-\infty$.

If, on the other hand, $M_{ij} \in GL_{n/k}(\mathbb{T})$ such that (15), then $M_{ij} \in G_F$. This completes our proof. $\square$

For any matrix F , we denote the matrix $\text{diag}(F, F, \dots, F)$ by $\tilde{F}$.

As a consequence, we have the following.

Corollary 4. $G_{\tilde{F}}$ is isomorphic to $G_F \wr S_{n/k}$ as groups, in which the matrix $\tilde{F}$ has the form given in Lemma 3.

Next, we shall want to consider the type of matrices in Lemma 9. And we need some lemmas at first. By [21, Theorem 102], we immediately have the following.

Lemma 5. Let E be an $n \times n$ nonsingular idempotent matrix. Then

$$D_E = \{MEN \mid M, N \in GL_n(\mathbb{T})\}. \quad (17)$$

Lemma 6 (see [24] Proposition 4.5). Let E be a nonsingular idempotent matrix. If there exists a monomial matrix M , such that $EME = E$, then $M = I_n$.

Lemma 7. Let E, F be nonsingular idempotent matrices. Then $E \mathcal{D} F$ if and only if there exists a monomial matrix M such that $E = MFM^{-1}$, i.e., such that $EM = MF$.

Proof. Suppose that E, F are nonsingular idempotent matrices.

If $E \mathcal{D} F$, then by Lemma 5 we can see that $E = MFN$, for some monomial matrices M and N . It follows that $MFN = E = E^2 = MFNMFN$. This implies that $F = FNMF$. Now by Lemma 6 we have that $NM = I_n$. Hence $E = MFM^{-1}$, and so $EM = MF$.

To prove the converse half, if there exists a monomial matrix M such that $EM = MF$, then we let $C = EM = MF$, and we can see that $E \mathcal{R} C$ and $C \mathcal{R} F$. Hence $E \mathcal{D} F$ as required. $\square$

If $M = (m_{ij})_{n \times n}$ is a monomial matrix, then there exists a unique $\sigma \in S_n$, such that $m_{i\sigma(i)} \in \mathbb{R}$ and $m_{ij} = -\infty$ for all $j \neq \sigma(i)$. Thus from the definition of matrix multiplication it is easy to show that the map

$$\varphi : GL_n(\mathbb{T}) \longrightarrow S_n, \quad M \longrightarrow \sigma \quad (18)$$

is a homomorphism of groups. Now we can show that

Proposition 8. Let $E = (e_{ij})_{n \times n}$ and $F = (f_{ij})_{n \times n}$ be real nonsingular idempotent matrices. Then $E \mathcal{D} F$ if and only if there exists $\sigma \in S_n$, such that, for all $i, j \in [n]$,

$$e_{i1} - e_{j1} - e_{ij} = f_{\sigma(i)\sigma(1)} - f_{\sigma(j)\sigma(1)} - f_{\sigma(i)\sigma(j)}, \quad (19)$$

Proof. Suppose that $E = (e_{ij})_{n \times n}$ and $F = (f_{ij})_{n \times n}$ are real nonsingular idempotent matrices.

If $E \mathcal{D} F$, then by Lemma 7 we have that there exists a matrix $M = (m_{ij})_{n \times n} \in G_E$ such that $EM = MF$. It follows that

$$EM = (e_{ij})(m_{ij}) = MF = (m_{ij})(f_{ij}). \quad (20)$$

This implies that, for any $i, j \in [n]$,

$$e_{ij} \otimes m_{j\sigma(j)} = m_{i\sigma(i)} \otimes f_{\sigma(i)\sigma(j)}. \quad (21)$$

Since for all $i, j \in [n]$, $e_{ij}, m_{j\sigma(j)}, m_{i\sigma(i)}$ and $f_{\sigma(i)\sigma(j)}$ are real numbers, then we have

$$e_{ij} + m_{j\sigma(j)} = m_{i\sigma(i)} + f_{\sigma(i)\sigma(j)}. \quad (22)$$

Thus we can see that, for any $i, j \in [n]$,

$$\begin{aligned} m_{i\sigma(i)} - m_{j\sigma(j)} &= e_{ij} - f_{\sigma(i)\sigma(j)} \\ &= m_{i\sigma(i)} - m_{1\sigma(1)} - (m_{j\sigma(j)} - m_{1\sigma(1)}) \\ &= e_{i1} - f_{\sigma(i)\sigma(1)} - (e_{j1} - f_{\sigma(j)\sigma(1)}). \end{aligned} \quad (23)$$

Hence for any $i, j \in [n]$ we have

$$e_{i1} - e_{j1} - e_{ij} = f_{\sigma(i)\sigma(1)} - f_{\sigma(j)\sigma(1)} - f_{\sigma(i)\sigma(j)}. \quad (24)$$

Conversely, if there exists $\sigma \in S_n$ such that, for any $i, j \in [n]$,

$$e_{i1} - e_{j1} - e_{ij} = f_{\sigma(i)\sigma(1)} - f_{\sigma(j)\sigma(1)} - f_{\sigma(i)\sigma(j)}, \quad (25)$$

then the system

$$\begin{aligned} m_{2\sigma(2)} - m_{1\sigma(1)} &= e_{21} - f_{\sigma(2)\sigma(1)} \\ m_{3\sigma(3)} - m_{1\sigma(1)} &= e_{31} - f_{\sigma(3)\sigma(1)} \\ &\dots \\ m_{n\sigma(n)} - m_{1\sigma(1)} &= e_{n1} - f_{\sigma(n)\sigma(1)} \\ m_{1\sigma(1)} - m_{2\sigma(2)} &= e_{12} - f_{\sigma(1)\sigma(2)} \\ m_{3\sigma(3)} - m_{2\sigma(2)} &= e_{32} - f_{\sigma(3)\sigma(2)} \\ &\dots \\ m_{n\sigma(n)} - m_{2\sigma(2)} &= e_{n2} - f_{\sigma(n)\sigma(2)} \\ &\dots \\ m_{n-1,\sigma(n-1)} - m_{n\sigma(n)} &= e_{n-1,n} - f_{\sigma(n-1)\sigma(n)} \end{aligned} \quad (26)$$

has the solutions

$$(m_{i\sigma(i)}) = \lambda \otimes (e_{i1} - f_{\sigma(i)\sigma(1)}), \quad (27)$$

where $\lambda \in \mathbb{R}$. This means that if σ satisfies (24), then there exists a monomial matrix M , whose $(i, \sigma(i))$ th entry is the real number $m_{i\sigma(i)}$ and the other entries are $-\infty$, such that $EM = MF$, and so $E \mathcal{D} F$. $\square$

Lemma 9. Let

$$E = \begin{bmatrix} E_1 & -\infty & \cdots & -\infty \\ -\infty & E_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & E_k \end{bmatrix} \quad (28)$$

be an $n \times n$ nonsingular idempotent matrix, where the matrix E_i is a real matrix of order n_i , $i \in [k]$, and for any $i, j \in [k]$, $(E_i, E_j) \notin \mathcal{D}$ ($i \neq j$). Then

$$\begin{aligned} G_E &= \{M = \text{diag}(M_{11}, M_{22}, \dots, M_{kk}) \\ &\in GL_n(\mathbb{T}) \mid (\forall i \in [k]) \ M_{ii} \in G_{E_i}\}. \end{aligned} \quad (29)$$

Proof. Let $E = \text{diag}(E_1, E_2, \dots, E_k)$ be an $n \times n$ nonsingular idempotent matrix. Then by Lemma 2 we can see that E_i is an $(n_i) \times (n_i)$ real nonsingular idempotent matrix. Suppose that $M \in G_E$. Then partition M into k^2 blocks

$$\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & & \vdots \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{bmatrix}, \quad (30)$$

where M_{ij} is an $n_i \times n_j$ matrix. It follows $EM = ME$ that

$$E_i M_{ij} = M_{ij} E_j, \quad (31)$$

for any $i, j \in [k]$. Since M_{ij} is a submatrix of the monomial matrix M , it has at most one entry in each row and column which is not equal to $-\infty$. We now distinguish two cases:

$$\begin{aligned} (i) \quad &i \neq j, \\ (ii) \quad &i = j. \end{aligned} \quad (32)$$

In case (i), suppose that M_{ij} is a monomial matrix such that (31). Then by Lemma 7 we have that $E_i \mathcal{D} E_j$. This contradiction implies that M_{ij} is not a monomial matrix. It follows by a closely similar proof of the claim (16) that $M_{ij} = -\infty$.

In case (ii), M_{ii} is a monomial matrix such that (31), since $M_{ij} = -\infty$ ($i \neq j$) and M is a monomial matrix. This implies that $M_{ii} \in GL_{n_i}(\mathbb{T})$ such that $E_i M_{ii} = M_{ii} E_i$, and so $M_{ii} \in G_{E_i}$. This completes our proof. $\square$

We now immediately deduce the following.

Corollary 10. If the matrix E has the form in Lemma 9, then G_E is isomorphic to $G_{E_1} \times G_{E_2} \times \cdots \times G_{E_k}$ as groups.

By the connection between the elementary operations and elementary matrices, it follows by Lemma 7 that if $E = \text{diag}(E_1, E_2, \dots, E_k)$ is a nonsingular idempotent matrix, then there exists a monomial matrix N , such that

$$NEN^{-1} = \text{diag}(\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_s), \quad (33)$$

where $E_{i_1}, E_{i_2}, \dots, E_{i_s}$ are diagonal blocks of E and for any $h, g \in \{1, \dots, s\}$, $(E_{i_h}, E_{i_g}) \notin \mathcal{D}$ ($h \neq g$). It is easy to see that the mapping $\varphi : G_E \rightarrow G_{NEN^{-1}}$ defined by

$$\varphi(M) = NMN^{-1} \quad (M \in G_E) \quad (34)$$

is a group isomorphism. Thus we obtain that G_E is isomorphic to $G_{NEN^{-1}}$ as groups. Hence we have the following theorem.

Theorem 11. Let

$$E = \begin{bmatrix} E_1 & -\infty & \cdots & -\infty \\ -\infty & E_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & E_k \end{bmatrix} \quad (35)$$

be an $n \times n$ nonsingular idempotent matrix, where $E_1, E_2, \dots, E_k$ are real square matrices. Then there exists a monomial matrix N , such that

$$NEN^{-1} = \begin{bmatrix} \tilde{E}_{i_1} & -\infty & \cdots & -\infty \\ -\infty & \tilde{E}_{i_2} & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & \tilde{E}_{i_s} \end{bmatrix}, \quad (36)$$

where $E_{i_1}, E_{i_2}, \dots, E_{i_s}$ are diagonal blocks of E and for any $h, g \in \{1, \dots, s\}$, $(E_{i_h}, E_{i_g}) \notin \mathcal{D}(h \neq g)$. Furthermore, G_E is isomorphic to

$$(G_{E_{i_1}} \wr S_{k_1}) \times (G_{E_{i_2}} \wr S_{k_2}) \times \dots \times (G_{E_{i_s}} \wr S_{k_s}) \quad (37)$$

as groups, where n_h is the order of the matrix $\tilde{E}_{i_h}$ and k_h is the number of the diagonal blocks of $\tilde{E}_{i_h}$, $h \in \{1, \dots, s\}$.

It follows by Lemma 1 and Theorem 11 that each tropical matrix group containing an idempotent of the form in Theorem 11 is isomorphic to some direct products of some wreath products. This result develops the decomposition of maximal subgroups of the semigroup of $n \times n$ real matrices under multiplication as direct products of $\mathbb{R}$ with finite groups in [13].

4. Tropical Matrix Groups Containing a Symmetric Nonsingular Idempotent Matrix

In this section we shall prove that each symmetric nonsingular idempotent matrix is similar to a diagonal block matrix. On this basis, we give a decomposition of the maximal subgroups containing an idempotent of this kind. For this aim, the following lemmas are needed.

Lemma 12 (see [24] Corollary 4.4). *All main diagonal entries of a nonsingular idempotent matrix are 0.*

It is easy to verify the following lemma.

Lemma 13. *Let E be an $n \times n$ matrix. Then the following are true.*

(i) *If $E = (e_{ij})_{n \times n}$ is a nonsingular idempotent matrix, then*

$$e_{ik} \otimes e_{kj} \leq e_{ij}, \quad (38)$$

for all $i, j, k \in [n]$;

(ii) *If E is a nonsingular symmetric idempotent matrix, then so is PEP^T for any $P \in P_n(\mathbb{T})$.*

We can now prove the following proposition.

Proposition 14. *Let E be a nonsingular symmetric idempotent matrix. Then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} E_1 & -\infty & \cdots & -\infty \\ -\infty & E_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & E_k \end{bmatrix}, \quad (39)$$

where $E_1, E_2, \dots, E_k$ are real nonsingular symmetric idempotent matrices.

Proof. Suppose that $E = E^{(1)} = (e_{ij}^{(1)})$ is an $n \times n$ nonsingular symmetric idempotent matrix. Then we shall show that E can be reduced to a diagonal block form using some simultaneous elementary rows and columns operations.

Step 1. Since E is a nonsingular idempotent matrix, it follows by Lemma 12 that all main diagonal entries of E are 0. If the i -th row of E has the most $-\infty$ entries, then we can interchange 1-row and i -row of E and interchange 1-column and i -column of E . By Lemma 13 (ii), a new nonsingular symmetric idempotent matrix obtained will be

$$E^{(2)} = P_1 E^{(1)} P_1^T = (e_{ij}^{(2)})_{n \times n}, \quad (40)$$

where $P_1 = E_{1,i}$ is an elementary matrix.

Step 2. By some synchronous permutations of the rows and columns of $E^{(2)}$, we can move the all $-\infty$ entries of the first row to the end of this row. This means that we can take a suitable permutation matrix P_2 and obtain another new matrix

$$E^{(3)} = P_2 E^{(2)} P_2^T = \begin{bmatrix} 0 & e_{12}^{(3)} & \cdots & e_{1k}^{(3)} & -\infty & \cdots & -\infty \\ e_{21}^{(3)} & 0 & \cdots & e_{2k}^{(3)} & e_{2,k+1}^{(3)} & \cdots & e_{2n}^{(3)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{k1}^{(3)} & e_{k2}^{(3)} & \cdots & 0 & e_{k,k+1}^{(3)} & \cdots & e_{kn}^{(3)} \\ e_{k+1,1}^{(3)} & e_{k+1,2}^{(3)} & \cdots & e_{k+1,k}^{(3)} & 0 & \cdots & e_{k+1,n}^{(3)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{n1}^{(3)} & e_{n2}^{(3)} & \cdots & e_{nk}^{(3)} & e_{n,k+1}^{(3)} & \cdots & 0 \end{bmatrix}, \quad (41)$$

where the first row has the most $-\infty$ entries and $e_{1j}^{(3)} = -\infty$ iff $j > k$. By Lemma 13 (ii) we have that $E^{(3)}$ is a nonsingular symmetric idempotent matrix. It follows by Lemma 13 (i) that $e_{1t}^{(3)} \otimes e_{tj}^{(3)} \leq e_{1j}^{(3)}$, for all $t, j \in [n]$. When $t \leq k$, $j > k$, we can see that $e_{1t}^{(3)} \in \mathbb{R}$ and $e_{1j}^{(3)} = -\infty$, and so $e_{tj}^{(3)} = -\infty$. Thus we have

$$E^{(3)} = \left[\begin{array}{cccc|ccc} 0 & e_{12}^{(3)} & \cdots & e_{1k}^{(3)} & -\infty & \cdots & -\infty \\ e_{21}^{(3)} & 0 & \cdots & e_{2k}^{(3)} & -\infty & \cdots & -\infty \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{k1}^{(3)} & e_{k2}^{(3)} & \cdots & 0 & -\infty & \cdots & -\infty \\ \hline e_{k+1,1}^{(3)} & e_{k+1,2}^{(3)} & \cdots & e_{k+1,k}^{(3)} & 0 & \cdots & e_{k+1,n}^{(3)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{n1}^{(3)} & e_{n2}^{(3)} & \cdots & e_{nk}^{(3)} & e_{n,k+1}^{(3)} & \cdots & 0 \end{array} \right] \quad (42)$$

$$= \begin{bmatrix} E_{11}^{(3)} & -\infty \\ E_{21}^{(3)} & E_{22}^{(3)} \end{bmatrix}.$$

On the other hand, since $E^{(3)}$ is symmetric, it now follows that $E_{21}^{(3)} = -\infty$. Hence

$$E^{(3)} = \begin{bmatrix} E_{11}^{(3)} & -\infty \\ -\infty & E_{22}^{(3)} \end{bmatrix}. \quad (43)$$

We can find that $E_{11}^{(3)}$ is a real matrix, since the first row of $E^{(3)}$ has the most $-\infty$ entries. Now the matrices $E_{11}^{(3)}$ and $E_{22}^{(3)}$ are nonsingular symmetric idempotent matrices. It follows that we can use the same method to reduce $E_{22}^{(3)}$.

After finite steps we will end up with a diagonal block matrix

$$E^{(m)} = \begin{bmatrix} E_1 & -\infty & \cdots & -\infty \\ -\infty & E_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & E_k \end{bmatrix} = P_{m-1} E^{(m-1)} P_{m-1}^T \quad (44)$$

$$= P_{m-1} P_{m-2} E^{(m-2)} P_{m-2}^T P_{m-1}^T$$

$$= P_{m-1} P_{m-2} \cdots P_1 E^{(1)} P_1^T \cdots P_{m-2}^T P_{m-1}^T$$

$$= P E^{(1)} P^T = P E P^T,$$

where $P = P_{m-1} P_{m-2} \cdots P_1$ is a permutation matrix and $E_1, E_2, \dots, E_k$ are real nonsingular symmetric idempotent matrices. This completes our proof. $\square$

This proposition shows that each nonsingular symmetric idempotent matrix E is equivalent to a diagonal block matrix $\text{diag}(E_1, E_2, \dots, E_k)$, which is a Frobenius normal form [23] of E , where $E_1, E_2, \dots, E_k$ are real matrices. In the following, we will study G_E , where E is a diagonal block idempotent whose diagonal blocks are all real matrices.

By a similar argument in Proposition 8, we have the following.

Lemma 15. *Let E be an $n \times n$ real nonsingular idempotent matrix. Then*

$$\varphi(G_E) = \left\{ \sigma \in S_n \mid (\forall i, j \in \{1, 2, \dots, n\}) e_{i1} - e_{j1} - e_{ij} \right. \\ \left. = e_{\sigma(i)\sigma(1)} - e_{\sigma(j)\sigma(1)} - e_{\sigma(i)\sigma(j)} \right\} \quad (45)$$

and G_E is isomorphic to the group $\mathbb{R} \times \varphi(G_E)$.

In [13], Izhakian, Johnson, and Kambites give a result that $G_E \cong \mathbb{R} \times \Sigma$ for some $\Sigma \in S_n$. We use a different method to prove this result in the above lemma and give a necessary and sufficient condition for some permutation σ in Σ . And we can easily verify that

$$\sigma \in \Sigma \iff$$

$$(\forall i, j \in \{1, 2, \dots, n\}) e_{i1} - e_{j1} - e_{ij}$$

$$= e_{\sigma(i)\sigma(1)} - e_{\sigma(j)\sigma(1)} - e_{\sigma(i)\sigma(j)} \iff \quad (46)$$

$$(\forall i, j, k \in \{1, 2, \dots, n\}) e_{ik} - e_{jk} - e_{ij}$$

$$= e_{\sigma(i)\sigma(k)} - e_{\sigma(j)\sigma(k)} - e_{\sigma(i)\sigma(j)}.$$

Especially if E is an $n \times n$ symmetric real nonsingular idempotent matrix, then we have the following.

Proposition 16. *Let E be an $n \times n$ real symmetric nonsingular idempotent matrix. Then*

$$\varphi(G_E)$$

$$= \left\{ \sigma \in S_n \mid (\forall i, j \in \{1, 2, \dots, n\}) e_{ij} = e_{\sigma(i)\sigma(j)} \right\} \quad (47)$$

and

$$G_E = \{ \lambda P \mid \lambda \in \mathbb{R}, P \in G_E \cap P_n(\mathbb{T}) \}, \quad (48)$$

which is isomorphic to the group $\mathbb{R} \times \varphi(G_E)$.

Proof. Following the proof of Proposition 8, we have that for all $i, j \in \{1, 2, \dots, n\}$

$$e_{ij} = e_{ji} \quad (49)$$

$$\text{and } e_{\sigma(i)\sigma(j)} = e_{\sigma(j)\sigma(i)}$$

Thus (26) reduce to

$$m_{i\sigma(i)} - m_{j\sigma(j)} = e_{ij} - e_{\sigma(i)\sigma(j)} = e_{ji} - e_{\sigma(j)\sigma(i)} \quad (50)$$

$$= m_{j\sigma(j)} - m_{i\sigma(i)}.$$

Then we know that the set of solutions to (50) is not empty if and only if

$$e_{ij} = e_{\sigma(i)\sigma(j)} \quad (\forall i, j \in \{1, 2, \dots, n\}) \quad (51)$$

and the solutions are

$$(m_{i\sigma(i)}) = (\lambda) \quad (\forall i \in \{1, 2, \dots, n\}), \quad (52)$$

in which $\lambda \in \mathbb{R}$. Hence there exist a real number λ and a permutation matrix $P \in G_E$, such that

$$M = \lambda P. \quad (53)$$

Thus $G_E = \{ \lambda P \mid \lambda \in \mathbb{R}, P \in G_E \cap P_n(\mathbb{T}) \}$, and so G_E is isomorphic to $\mathbb{R} \times \varphi(G_E)$. $\square$

Proposition 16 enables us to compile the following algorithm.

If the idempotent matrix E is real nonsingular, we have discussed G_E . In the following, we will study the symmetric nonsingular idempotent matrix, which is only a real matrix. In summation, from Theorem 11 and Proposition 16, we have the following.

Theorem 17. *Let E be an $n \times n$ symmetric nonsingular idempotent matrix. Then there exists a monomial matrix N , such that*

$$NEN^{-1} = \begin{bmatrix} \tilde{E}_1 & -\infty & \cdots & -\infty \\ -\infty & \tilde{E}_2 & \cdots & -\infty \\ \vdots & \vdots & & \vdots \\ -\infty & -\infty & \cdots & \tilde{E}_s \end{bmatrix}, \quad (54)$$

where $E_1, E_2, \dots, E_s$ are symmetric real nonsingular idempotent matrices and for any $i, j \in \{1, \dots, s\}$, $(E_i, E_j) \notin \mathcal{D}$ ($i \neq j$). Moreover, G_E is isomorphic to

$$\begin{aligned} &((\mathbb{R} \times \Sigma_1) \wr S_{k_1}) \times ((\mathbb{R} \times \Sigma_2) \wr S_{k_2}) \times \cdots \\ &\times ((\mathbb{R} \times \Sigma_s) \wr S_{k_s}) \end{aligned} \quad (55)$$

as groups, where $\Sigma_i \leq S_{n_i/k_i}$, n_i is the order of the matrix $\tilde{E}_i$, and k_i is the number of the diagonal blocks of $\tilde{E}_i$, $i \in \{1, 2, \dots, s\}$.

Since each basis submatrix of a symmetric idempotent matrix is a symmetric nonsingular idempotent matrix, it follows by Lemma 1 and Theorem 17 that each tropical matrix group containing a symmetric idempotent matrix is isomorphic to some direct products of some wreath products.

Our next aim is to provide an algorithm for G_E of any nonsingular idempotent $E \in M_n(\mathbb{T})$.

Data Availability

Previously reported data were used to support this study and are available at [https://doi.org/10.1155/2018/4797638]. These prior studies (and datasets) are cited at relevant places within the text as references [1–24].

Conflicts of Interest

The author declares that they have no conflicts of interest.

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