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Research Article

Minimal Wave Speed in a Predator-Prey System with Distributed Time Delay

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This paper is concerned with the minimal wave speed of traveling wave solutions in a predator-prey system with distributed time delay, which does not satisfy comparison principle due to delayed intraspecific terms. By constructing upper and lower solutions, we obtain the existence of traveling wave solutions when the wave speed is the minimal wave speed. Our results complete the known conclusions and show the precisely asymptotic behavior of traveling wave solutions.

1. Introduction

Traveling wave solutions of predator-prey systems have been widely utilized to model population invasion, and the minimal wave speed of traveling wave solutions is often regarded as an important threshold to characterize the invasion feature in many examples, see Owen and Lewis [1] and Shigesada and Kawasaki [2, Chapter 8]. Moreover, Lin [3] and Pan [4] confirmed that, in a Lotka-Volterra type system, the minimal wave speed of invasion traveling wave solutions is equal to the invasion speed of the predator. Here, the invasion speed is estimated by the corresponding initial value problem when the initial value of the predator admits a nonempty compact support.

When the wave system of predator-prey system is of finite dimension, there are many important results, for example, the earlier results by Dunbar [5–7]. But when the corresponding wave system is of infinite dimension, there are some open problems on the minimal wave speed, for example, in the following system [8]:

$$\begin{aligned}\frac{\partial u_1(x,t)}{\partial t} &= d_1\Delta u_1(x,t) + r_1 u_1(x,t) F_1(u_1,u_2)(x,t),\\ \frac{\partial u_2(x,t)}{\partial t} &= d_2\Delta u_2(x,t) + r_2 u_2(x,t) F_2(u_1,u_2)(x,t),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}$, $t > 0$, $d_1 > 0$, $d_2 > 0$, $r_1 > 0$, $r_2 > 0$, $u_1 \in \mathbb{R}$, $u_2 \in \mathbb{R}$, and

$$\begin{aligned}F_1(u_1,u_2)(x,t) &= 1 - u_1(x,t)\\ &\quad - a_1\int_{-\tau}^{0} u_1(x,t+s)\,d\eta_{11}(s)\\ &\quad - b_1\int_{-\tau}^{0} u_2(x,t+s)\,d\eta_{12}(s),\\ F_2(u_1,u_2)(x,t) &= 1 - u_2(x,t)\\ &\quad - a_2\int_{-\tau}^{0} u_2(x,t+s)\,d\eta_{22}(s)\\ &\quad + b_2\int_{-\tau}^{0} u_1(x,t+s)\,d\eta_{21}(s),\end{aligned} \tag{2}$$

in which $a_1 \geq 0$, $a_2 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$, and $\tau \geq 0$ are constants such that

$$\begin{aligned}&\eta_{ij}(s) \text{ is nondecreasing on } [-\tau,0],\\ &\eta_{ij}(0) - \eta_{ij}(-\tau) = 1,\\ &i,j = 1,2.\end{aligned} \tag{3}$$

For (1), a traveling wave solution is a special solution with the form

$$u_i(x,t) = \phi_i(\xi), \quad \xi = x + ct, \ i = 1,2, \tag{4}$$

where $(\phi_1, \phi_2) \in C^2(\mathbb{R}, \mathbb{R}^2)$ is the wave profile and $c > 0$ is the wave speed. Therefore, (ϕ_1, ϕ_2) and c satisfy

$$\begin{aligned} &d_1\phi_1''(\xi) - c\phi_1'(\xi) + r_1\phi_1(\xi) F_1(\phi_1,\phi_2)(\xi) = 0, \quad \xi \in \mathbb{R}, \\ &d_2\phi_2''(\xi) - c\phi_2'(\xi) + r_2\phi_2(\xi) F_2(\phi_1,\phi_2)(\xi) = 0, \quad \xi \in \mathbb{R} \end{aligned} \tag{5}$$

with

$$\begin{aligned} F_1(\phi_1,\phi_2)(\xi) &= 1 - \phi_1(\xi) - a_1 \int_{-\tau}^{0} \phi_1(\xi + cs)\, d\eta_{11}(s) \\ &\quad - b_1 \int_{-\tau}^{0} \phi_2(\xi + cs)\, d\eta_{12}(s), \\ F_2(\phi_1,\phi_2)(\xi) &= 1 - \phi_2(\xi) - a_2 \int_{-\tau}^{0} \phi_2(\xi + cs)\, d\eta_{22}(s) \\ &\quad + b_2 \int_{-\tau}^{0} \phi_1(\xi + cs)\, d\eta_{21}(s). \end{aligned} \tag{6}$$

In Pan [8], the author defined a threshold given by

$$c^* := \max\left\{2\sqrt{d_1 r_1}, 2\sqrt{d_2 r_2}\right\} \tag{7}$$

and showed the existence (nonexistence) of desired traveling wave solutions if the wave speed $c > c^*$ ($c < c^*$). When the wave speed $c = c^*$, the author presented the existence of traveling wave solutions under special conditions. Besides [8], there are also some results on the existence of traveling wave solutions of predator-prey models similar to (1) when the wave speed is large; see Huang and Zou [9], K. Li and X. Li [10], and Lin et al. [11].

The purpose of this paper is to confirm the existence of nontrivial traveling wave solutions of (1) without other conditions when the wave speed $c = c^*$. Since Pan [8, Theorem 3.5] also holds when $c = c^*$, we shall not investigate the limit behavior as $\xi \to \infty$ and focus on the existence of positive solution of (5) satisfying

$$\lim_{\xi \to -\infty} \phi_i(\xi) = 0, \quad i = 1,2. \tag{8}$$

Motivated by Lin and Ruan [12] on an abstract result of traveling wave solutions of delayed reaction-diffusion systems, we shall construct proper upper and lower solutions similar to those in Fu [13] and Lin [14] to study the existence of traveling wave solutions.

2. Main Results

When $c = c^*$, we define

$$\begin{aligned} \gamma_1 &= \frac{c - \sqrt{c^2 - 4d_1 r_1}}{2d_1}, \\ \gamma_2 &= \frac{c - \sqrt{c^2 - 4d_2 r_2}}{2d_2}. \end{aligned} \tag{9}$$

By these constants, we first present our main conclusion as follows.

Theorem 1. *Assume that $c = c^*$ holds. Then (5) admits a bounded positive solution (ϕ_1, ϕ_2) satisfying*

(1) $\lim_{\xi\to-\infty}(\phi_1(\xi)/-\xi e^{\gamma_1\xi}) = 1$, $\lim_{\xi\to-\infty}(\phi_2(\xi)/e^{\gamma_2\xi}) = 1$ *if* $c = c^* = 2\sqrt{d_1 r_1} > 2\sqrt{d_2 r_2}$;

(2) $\lim_{\xi\to-\infty}(\phi_1(\xi)/e^{\gamma_1\xi}) = 1$, $\lim_{\xi\to-\infty}(\phi_2(\xi)/-\xi e^{\gamma_2\xi}) = 1$ *if* $c = c^* = 2\sqrt{d_2 r_2} > 2\sqrt{d_1 r_1}$;

(3) $\lim_{\xi\to-\infty}(\phi_1(\xi)/-\xi e^{\gamma_1\xi}) = 1$, $\lim_{\xi\to-\infty}(\phi_2(\xi)/-\xi e^{\gamma_2\xi}) = 1$ *if* $c = c^* = 2\sqrt{d_2 r_2} = 2\sqrt{d_1 r_1}$.

We shall prove the result by three lemmas, which will study three cases $d_1 r_1 > d_2 r_2$, $d_1 r_1 < d_2 r_2$, and $d_1 r_1 = d_2 r_2$. For this purpose, we first show the following result in Lin and Ruan [12].

Lemma 2. *Suppose that $\underline{\phi}_1(\xi)$, $\overline{\phi}_1(\xi)$, $\underline{\phi}_2(\xi)$, and $\overline{\phi}_2(\xi)$ are continuous functions and*

(A1) $0 \le \underline{\phi}_1(\xi) \le \overline{\phi}_1(\xi) \le 1$, $0 \le \underline{\phi}_2(\xi) \le \overline{\phi}_2(\xi) \le 1 + b_2$, $\xi \in \mathbb{R}$;

(A2) *they are twice differentiable except a set E containing finite points of $\mathbb{R}$ and*

$$\underline{\phi}_1'(\xi), \overline{\phi}_1'(\xi), \underline{\phi}_2'(\xi), \overline{\phi}_2'(\xi), \underline{\phi}_1''(\xi), \overline{\phi}_1''(\xi), \underline{\phi}_2''(\xi), \overline{\phi}_2''(\xi) \tag{10}$$

are continuous and bounded if $\xi \in \mathbb{R} \setminus E$;

(A3) *when $x \in E$, they satisfy*

$$\begin{aligned} &\underline{\phi}_1'(\xi-) \le \underline{\phi}_1'(\xi+), \\ &\underline{\phi}_2'(\xi-) \le \underline{\phi}_2'(\xi+), \\ &\overline{\phi}_1'(\xi+) \le \overline{\phi}_1'(\xi-), \\ &\overline{\phi}_2'(\xi+) \le \overline{\phi}_2'(\xi-); \end{aligned} \tag{11}$$

(A4) *they satisfy the following inequalities:*

$$\begin{aligned}
&d_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi)\Big[1 - \overline{\phi}_1(\xi) \\
&\quad - a_1\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{11}(s) \\
&\quad - b_1\int_{-\tau}^{0}\underline{\phi}_2(\xi+cs)\,d\eta_{12}(s)\Big] \le 0, \\
&d_2\overline{\phi}_2''(\xi) - c\overline{\phi}_2'(\xi) + r_2\overline{\phi}_2(\xi)\Big[1 - \overline{\phi}_2(\xi) \\
&\quad - a_2\int_{-\tau}^{0}\underline{\phi}_2(\xi+cs)\,d\eta_{22}(s) \\
&\quad + b_2\int_{-\tau}^{0}\overline{\phi}_1(\xi+cs)\,d\eta_{21}(s)\Big] \le 0, \\
&d_1\underline{\phi}_1''(\xi) - c\underline{\phi}_1'(\xi) + r_1\underline{\phi}_1(\xi)\Big[1 - \underline{\phi}_1(\xi) \\
&\quad - a_1\int_{-\tau}^{0}\overline{\phi}_1(\xi+cs)\,d\eta_{11}(s) \\
&\quad - b_1\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{12}(s)\Big] \ge 0, \\
&d_2\underline{\phi}_2''(\xi) - c\underline{\phi}_2'(\xi) + r_2\underline{\phi}_2(\xi)\Big[1 - \underline{\phi}_2(\xi) \\
&\quad - a_2\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{22}(s) \\
&\quad + b_2\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{21}(s)\Big] \ge 0
\end{aligned} \tag{12}$$

for $\xi \in \mathbb{R} \setminus E$.

Then (5) has a positive solution $(\phi_1(\xi), \phi_2(\xi))$ *such that*

$$\begin{aligned}
&\underline{\phi}_1(\xi) \le \phi_1(\xi) \le \overline{\phi}_1(\xi), \\
&\underline{\phi}_2(\xi) \le \phi_2(\xi) \le \overline{\phi}_2(\xi), \\
&\qquad \xi \in \mathbb{R}.
\end{aligned} \tag{13}$$

Remark 3. In the above lemma, $(\overline{\phi}_1(\xi), \overline{\phi}_2(\xi))$ and $(\underline{\phi}_1(\xi), \underline{\phi}_2(\xi))$ are a pair of (generalized) upper and lower solutions of (5). That is, the existence of positive solutions of (5) can be obtained by the existence of (generalized) upper and lower solutions of (5).

Lemma 4. *Assume that* $d_1r_1 > d_2r_2$. *Then (1) of Theorem 1 holds.*

Proof. For simplicity, we shall denote $c^* = 2\sqrt{d_1r_1}$ by c and define

$$\gamma_3 = \frac{c + \sqrt{c^2 - 4d_2r_2}}{2d_2}. \tag{14}$$

Let $K > 0$ be a constant such that

(K1) $(-\xi + K)e^{\gamma_1\xi}$ is monotone if $\xi \le 0$;

(K2) $K > 1$ or $\sup_{\xi\le 0}(-\xi + K)e^{\gamma_1\xi} > 1$.

Moreover, select $\eta > 1$ with

$$\eta\gamma_2 < \min\left\{\gamma_2 + \frac{\gamma_1}{2}, 2\gamma_2, \gamma_3\right\} \tag{15}$$

and $M > 1 + b_2$ such that

(M1) $e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi} < 1 + b_2$ implies $(-\xi + K)e^{\gamma_1\xi} \le e^{\gamma_1\xi/2}$;

(M2) $M > -2r_2b_2/(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2) + 1 + b_2$;

(M3) $e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi} < 1 + b_2$ implies $(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2) + 2r_2b_2e^{\gamma_1\xi/2} < 0$,

and $N > 1$ such that

(N1) $e^{\gamma_2\xi} - Ne^{\eta\gamma_2\xi} > 0$ implies $e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi} < 2e^{\gamma_2\xi}$;

(N2) $N > -4r_2(1 + a_2)/(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2) + 1$.

Select $L > 1$ such that

(L1) $\xi < -L^2$ implies $(-\xi + K)e^{\gamma_1\xi} < e^{\gamma_1'\xi} < 1$, where $\gamma_1' \in (0, \gamma_1)$ such that

$$\begin{aligned}
&2\gamma_1' - \gamma_1 > 0, \\
&\gamma_1' + \gamma_2 - \gamma_1 > 0;
\end{aligned} \tag{16}$$

(L2) $\xi < -L^2$ implies $e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi} < 2e^{\gamma_2\xi}$;

(L3) $L > \sup_{\xi<-1}(4\sqrt{(-\xi)^3}[r_1(1 + a_1)e^{(2\gamma_1'-\gamma_1)\xi} + 2r_1b_1e^{(\gamma_1'+\gamma_2-\gamma_1)\xi}]/d_1)$.

The admissibility of L, M, N, and K is clear by the limit behavior of these functions as $\xi \to -\infty$. Mathematically, we first fix K, then select M, and finally define N, L. Here, N and L are independent of each other.

We now define

$$\begin{aligned}
\overline{\phi}_1(\xi) &= \begin{cases} (-\xi + K)\,e^{\gamma_1\xi}, & \xi \le \xi_1, \\ 1, & \xi \ge \xi_1, \end{cases} \\
\underline{\phi}_1(\xi) &= \begin{cases} \left(-\xi - L\sqrt{-\xi}\right)e^{\gamma_1\xi}, & \xi \le -L^2, \\ 0, & \xi > -L^2, \end{cases}
\end{aligned} \tag{17}$$

where $\xi_1 < 0$ such that $\overline{\phi}_1(\xi)$ is continuous by (K1)-(K2) and

$$\begin{aligned}
\overline{\phi}_2(\xi) &= \min\left\{e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi}, 1 + b_2\right\}, \\
\underline{\phi}_2(\xi) &= \max\left\{0, e^{\gamma_2\xi} - Ne^{\eta\gamma_2\xi}\right\}.
\end{aligned} \tag{18}$$

If these functions satisfy (12), then our result holds by Lemma 2. Now, we are in a position of verifying these

inequalities. For $\overline{\phi}_1(\xi)$, we shall prove the first inequality of (12) when $\xi \neq \xi_1$. If $\xi > \xi_1$ and $\overline{\phi}_1(\xi) = 1$, then

$$\begin{aligned} 1 - \overline{\phi}_1(\xi) - a_1 \int_{-\tau}^{0} \underline{\phi}_1(\xi + cs)\, d\eta_{11}(s) \\ - b_1 \int_{-\tau}^{0} \underline{\phi}_2(\xi + cs)\, d\eta_{12}(s) \leq 0 \end{aligned} \tag{19}$$

and the first inequality of (12) is clear. When $\overline{\phi}_1(\xi) = (-\xi + K)e^{\gamma_1\xi} < 1$, then

$$\begin{aligned} d_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi) \\ = (-\xi + K)\left(d_1\gamma_1^2 - c\gamma_1 + r_1\right)e^{\gamma_1\xi} \\ - \left(2d_1\gamma_1 - c\right)e^{\gamma_1\xi} = 0 \end{aligned} \tag{20}$$

and the verification on the first inequality of (12) is finished.

When the second inequality on $\overline{\phi}_2(\xi)$ is concerned, it is also clear if $\overline{\phi}_2(\xi) = 1 + b_2 < e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi}$. When $\overline{\phi}_2(\xi) = e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi} < 1 + b_2$, then (M1) leads to

$$\begin{aligned} r_2\overline{\phi}_2(\xi)\Big[1 - \overline{\phi}_2(\xi) - a_2 \int_{-\tau}^{0} \underline{\phi}_2(\xi + cs)\, d\eta_{22}(s) \\ + b_2 \int_{-\tau}^{0} \overline{\phi}_1(\xi + cs)\, d\eta_{21}(s)\Big] \leq r_2\overline{\phi}_2(\xi)\big[1 \\ + b_2\overline{\phi}_1(\xi)\big] \leq r_2\left(e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi}\right)\left(1 + b_2e^{\gamma_1\xi/2}\right) \\ = r_2\left(e^{\gamma_2\xi} + Me^{\eta\gamma_2\xi}\right) + r_2b_2\left(e^{(\gamma_2+\gamma_1/2)\xi} \\ + Me^{(\eta\gamma_2+\gamma_1/2)\xi}\right). \end{aligned} \tag{21}$$

Note that

$$\begin{aligned} d_2\overline{\phi}_2''(\xi) - c\overline{\phi}_2'(\xi) = \left(d_2\gamma_2^2 - c\gamma_2\right)e^{\gamma_2\xi} \\ + M\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2\right)e^{\eta\gamma_2\xi} \end{aligned} \tag{22}$$

then the definition of γ_2 implies that the desired inequality is true if

$$\begin{aligned} M\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right)e^{\eta\gamma_2\xi} \\ + r_2b_2\left(e^{(\gamma_2+\gamma_1/2)\xi} + Me^{(\eta\gamma_2+\gamma_1/2)\xi}\right) \leq 0 \end{aligned} \tag{23}$$

or

$$\begin{aligned} 2M\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right) \\ + 2r_2b_2\left(e^{(\gamma_2+\gamma_1/2-\eta\gamma_2)\xi} + Me^{\gamma_1\xi/2}\right) \leq 0. \end{aligned} \tag{24}$$

On the one hand, (M2) leads to

$$M\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right) + 2r_2b_2 < 0. \tag{25}$$

At the same time, we have

$$\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right) + 2r_2b_2e^{\gamma_1\xi/2} < 0 \tag{26}$$

by (M3). Therefore, (23) is true, as is the case for the second inequality of (12).

On the third inequality of (12), it is clear if $\underline{\phi}_1(\xi) = 0$. Otherwise,

$$\begin{aligned} \underline{\phi}_1'(\xi) &= \gamma_1\underline{\phi}_1(\xi) - e^{\gamma_1\xi} + \frac{L}{2\sqrt{-\xi}}e^{\gamma_1\xi}, \\ \underline{\phi}_1''(\xi) &= \gamma_1^2\underline{\phi}_1(\xi) - 2\gamma_1e^{\gamma_1\xi} + \frac{L\gamma_1}{\sqrt{-\xi}}e^{\gamma_1\xi} \\ &\quad + \frac{L}{4\sqrt{(-\xi)^3}}e^{\gamma_1\xi}. \end{aligned} \tag{27}$$

With these results, we obtain

$$\begin{aligned} d_1\underline{\phi}_1''(\xi) - c\underline{\phi}_1'(\xi) + r_1\underline{\phi}_1(\xi) = \left(d_1\gamma_1^2 - c\gamma_1 + r_1\right) \\ \cdot \underline{\phi}_1(\xi) - \left(2d_1\gamma_1 - c\right)e^{\gamma_1\xi} + Le^{\gamma_1\xi}\Bigg[\frac{d_1\gamma_1}{\sqrt{-\xi}} - \frac{c}{2\sqrt{-\xi}} \\ + \frac{d_1}{4\sqrt{(-\xi)^3}}\Bigg] = \frac{Ld_1e^{\gamma_1\xi}}{4\sqrt{(-\xi)^3}}, \\ r_1\underline{\phi}_1(\xi)\Big[-\underline{\phi}_1(\xi) - a_1\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\, d\eta_{11}(s) \\ - b_1\int_{-\tau}^{0}\overline{\phi}_2(\xi + cs)\, d\eta_{12}(s)\Big] \geq -r_1\underline{\phi}_1^2(\xi) \\ - r_1a_1\overline{\phi}_1(\xi)\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\, d\eta_{11}(s) - r_1b_1\overline{\phi}_1(\xi) \\ \cdot \int_{-\tau}^{0}\overline{\phi}_2(\xi + cs)\, d\eta_{12}(s) \geq -r_1\left(1 + a_1\right) \\ \cdot \overline{\phi}_1^2(\xi) - 2r_1b_1\overline{\phi}_1(\xi)\overline{\phi}_2(\xi) \geq -r_1\left(1 + a_1\right)e^{2\gamma_1'\xi} \\ - 2r_1b_1e^{(\gamma_1'+\gamma_2)\xi} \end{aligned} \tag{28}$$

by (L1)-(L2). Therefore, it suffices to prove that

$$\frac{Ld_1e^{\gamma_1\xi}}{4\sqrt{(-\xi)^3}} \geq r_1\left(1 + a_1\right)e^{2\gamma_1'\xi} + 2r_1b_1e^{(\gamma_1'+\gamma_2)\xi} \tag{29}$$

or

$$\frac{Ld_1}{4\sqrt{(-\xi)^3}} \geq r_1\left(1 + a_1\right)e^{(2\gamma_1'-\gamma_1)\xi} + 2r_1b_1e^{(\gamma_1'+\gamma_2-\gamma_1)\xi}, \tag{30}$$

which is true by (L3).

We now consider $\underline{\phi}_2(\xi)$, that is, the forth inequality of (12). When $\underline{\phi}_2(\xi) > 0$, the definition implies

$$\begin{aligned} & r_2\left[-\underline{\phi}_2^2(\xi) - a_2\underline{\phi}_2(\xi)\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{22}(s)\right. \\ & \left. + b_2\underline{\phi}_2(\xi)\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{21}(s)\right] \\ & \geq r_2\left[-\underline{\phi}_2^2(\xi) - a_2\underline{\phi}_2(\xi)\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{22}(s)\right] \\ & \geq r_2\left[-\overline{\phi}_2^2(\xi) - a_2\overline{\phi}_2(\xi)\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{22}(s)\right] \\ & \geq -4r_2(1+a_2)e^{2\gamma_2\xi} \end{aligned} \tag{31}$$

by (N1) as well as

$$\begin{aligned} & d_2\underline{\phi}_2''(\xi) - c\underline{\phi}_2'(\xi) + r_2\underline{\phi}_2(\xi) \\ & = e^{\gamma_2\xi}\left(d_2\gamma_2^2 - c\gamma_2 + r_2\right) \\ & \quad - Ne^{\eta\gamma_2\xi}\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right) \\ & = -Ne^{\eta\gamma_2\xi}\left(d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2\right). \end{aligned} \tag{32}$$

Thus, the desired inequality is true if

$$N > \frac{-4r_2(1+a_2)}{d_2\eta^2\gamma_2^2 - c\eta\gamma_2 + r_2} + 1 > 1 \tag{33}$$

since $\xi < 0$ such that $e^{2\gamma_2\xi} < e^{\eta\gamma_2\xi}$, which holds by (N2). The proof is complete. □

Lemma 5. *Assume that $d_1r_1 < d_2r_2$. Then (2) of Theorem 1 is true.*

Proof. Similar to the proof of the previous lemma, it suffices to construct proper upper and lower solutions. When $c = 2\sqrt{d_2r_2}$, let

$$\gamma_3 = \frac{c + \sqrt{c^2 - 4d_1r_1}}{2d_1}. \tag{34}$$

Fix $\eta > 1$ such that

$$\eta \in \left(1, \min\left\{2, \frac{\gamma_3}{\gamma_1}, \frac{\gamma_1 + \gamma_2/2}{\gamma_1}\right\}\right). \tag{35}$$

Select $N_1 > 1$ such that

$$\sup_{\xi<-1}\left\{\left(-\xi + N\sqrt{-\xi}\right)e^{\gamma_2\xi}\right\} \geq 1 + b_2, \quad N > N_1. \tag{36}$$

Let $\xi_2 < -1$ be the smaller root of $(-\xi + N\sqrt{-\xi})e^{\gamma_2\xi} = 1 + b_2$. Clearly, if $N \to \infty$, then $\xi_2 \to -\infty$.

By these constants, we define

$$\begin{aligned} \overline{\phi}_1(\xi) &= \min\left\{1, e^{\gamma_1\xi}\right\}, \\ \underline{\phi}_1(\xi) &= \max\left\{0, e^{\gamma_1\xi} - Qe^{\eta\gamma_1\xi}\right\}, \\ \overline{\phi}_2(\xi) &= \begin{cases} 1 + b_2, & \xi \geq \xi_2, \\ \left(-\xi + N\sqrt{-\xi}\right)e^{\gamma_2\xi}, & \xi < \xi_2, \end{cases} \\ \underline{\phi}_2(\xi) &= \begin{cases} 0, & \xi \geq -R^2, \\ \left(-\xi - R\sqrt{-\xi}\right)e^{\gamma_2\xi}, & \xi < -R^2, \end{cases} \end{aligned} \tag{37}$$

where N, Q, and R are positive constants satisfying that

(N1) $N > N_1$ is large such that $-d_2/4 + r_2b_2(-\xi)^{5/2}e^{\gamma_1\xi}/N + r_2b_2(-\xi)^2e^{\gamma_1\xi} < 0$, $\xi < \xi_2$;

(Q1) $Q \geq Q_1 > 1$ such that $e^{\gamma_1\xi} - Q_1e^{\eta\gamma_1\xi} > 0$ implies $(-\xi + N\sqrt{-\xi})e^{\gamma_2\xi} < e^{\gamma_2\xi/2}$;

(Q2) $Q \geq (r_1(1+a_1) + r_1b_1)/-(d_1\eta^2\gamma_1^2 - c\eta\gamma_1 + r_1) + Q_1$,

and

(R0) $R_0 > 1$ is a constant such that $\xi < -R_0^2 < -1$ implies $\underline{\phi}_2(\xi) \leq \overline{\phi}_2(\xi)$,

(R1) $R \geq R_1 > R_0$ such that $\xi < -R_1^2$ implies $\overline{\phi}_2(\xi) < e^{\gamma_2'\xi}$, where γ_2' satisfies

$$\begin{aligned} 2\gamma_2' - \gamma_2 &> 0, \\ \gamma_2' + \gamma_1 - \gamma_2 &> 0, \\ \gamma_2' &\in (0, \gamma_2), \end{aligned} \tag{38}$$

(R2) $R > R_2 > R_1$ such that $d_2/4\sqrt{(-\xi)^3} - r_2(1 + a_2)e^{(2\gamma_2'-\gamma_2)\xi}/R_2 > 0$, $\xi \leq -R_2^2$.

On the first inequality of (12), if $\overline{\phi}_1(\xi) = 1 < e^{\gamma_1\xi}$, then

$$\begin{aligned} & 1 - \overline{\phi}_1(\xi) - a_1\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{11}(s) \\ & \quad - b_1\int_{-\tau}^{0}\underline{\phi}_2(\xi+cs)\,d\eta_{12}(s) \leq 0 \end{aligned} \tag{39}$$

and the result is clear. If $\overline{\phi}_1(\xi) = e^{\gamma_1\xi} < 1$, then

$$\begin{aligned} & 1 - \overline{\phi}_1(\xi) - a_1\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{11}(s) \\ & \quad - b_1\int_{-\tau}^{0}\underline{\phi}_2(\xi+cs)\,d\eta_{12}(s) \leq 1, \\ & d_1\overline{\phi}_1''(\xi) - c\overline{\phi}_1'(\xi) + r_1\overline{\phi}_1(\xi) = \left(d_1\gamma_1^2 - c\gamma_1 + r_1\right)e^{\gamma_1\xi} \\ & \quad = 0, \end{aligned} \tag{40}$$

which completes the verification on $\overline{\phi}_1(\xi)$. On the second inequality, it is evident if $\overline{\phi}_2(\xi) = 1 + b_2$. When $\overline{\phi}_2(\xi) = (-\xi + N\sqrt{-\xi})e^{\gamma_2\xi} < 1 + b_2$, we have

$$\begin{aligned}
&r_2\overline{\phi}_2(\xi)\left[1 - \overline{\phi}_2(\xi) - a_2\int_{-\tau}^{0}\underline{\phi}_2(\xi + cs)\,d\eta_{22}(s)\right.\\
&\quad\left. + b_2\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\,d\eta_{21}(s)\right] \le r_2\overline{\phi}_2(\xi)\left[1\right.\\
&\quad\left. + b_2\overline{\phi}_1(\xi)\right] = r_2\overline{\phi}_2(\xi) + r_2b_2\overline{\phi}_2(\xi)e^{\gamma_1\xi},\\
&\overline{\phi}_2'(\xi) = \gamma_2\overline{\phi}_2(\xi) - e^{\gamma_2\xi} - \frac{Ne^{\gamma_2\xi}}{2\sqrt{-\xi}}\\
&\overline{\phi}_2''(\xi) = \gamma_2^2\overline{\phi}_2(\xi) - 2\gamma_2e^{\gamma_2\xi} - \frac{N\gamma_2e^{\gamma_2\xi}}{\sqrt{-\xi}} - \frac{Ne^{\gamma_2\xi}}{4(-\xi)^{3/2}}.
\end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}
&d_2\overline{\phi}_2''(\xi) - c\overline{\phi}_2'(\xi) + r_2\overline{\phi}_2(\xi)\times\left[1 - \overline{\phi}_2(\xi)\right.\\
&\quad - a_2\int_{-\tau}^{0}\underline{\phi}_2(\xi + cs)\,d\eta_{22}(s)\\
&\quad\left. + b_2\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\,d\eta_{21}(s)\right] \le d_2\left[\gamma_2^2\overline{\phi}_2(\xi)\right.\\
&\quad\left. - 2\gamma_2e^{\gamma_2\xi} - \frac{N\gamma_2e^{\gamma_2\xi}}{\sqrt{-\xi}} - \frac{Ne^{\gamma_2\xi}}{4(-\xi)^{3/2}}\right] - c\left[\gamma_2\overline{\phi}_2(\xi)\right.\\
&\quad\left. - e^{\gamma_2\xi} - \frac{Ne^{\gamma_2\xi}}{2\sqrt{-\xi}}\right] + r_2\overline{\phi}_2(\xi) + r_2b_2\overline{\phi}_2(\xi)e^{\gamma_1\xi}\\
&= -\frac{d_2Ne^{\gamma_2\xi}}{4(-\xi)^{3/2}} + r_2b_2\left(-\xi + N\sqrt{-\xi}\right)e^{\gamma_2\xi}e^{\gamma_1\xi}\\
&= e^{\gamma_2\xi}\left[-\frac{d_2N}{4(-\xi)^{3/2}} + r_2b_2\left(-\xi + N\sqrt{-\xi}\right)e^{\gamma_1\xi}\right]\\
&= \frac{e^{\gamma_2\xi}}{(-\xi)^{3/2}}\left[-\frac{d_2N}{4}\right.\\
&\quad\left. + r_2b_2\left((-\xi)^{5/2} + N(-\xi)^2\right)e^{\gamma_1\xi}\right] = \frac{Ne^{\gamma_2\xi}}{(-\xi)^{3/2}}\left[-\frac{d_2}{4}\right.\\
&\quad\left. + \frac{r_2b_2(-\xi)^{5/2}e^{\gamma_1\xi}}{N} + r_2b_2(-\xi)^2e^{\gamma_1\xi}\right] \le 0
\end{aligned} \tag{42}$$

by (N1).

On the third inequality, it is clear if $\underline{\phi}_1(\xi) = 0$. When $e^{\gamma_1\xi} - Qe^{\eta\gamma_1\xi} > 0$, (Q1) implies

$$\begin{aligned}
&r_1\underline{\phi}_1(\xi)\left[-\underline{\phi}_1(\xi) - a_1\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\,d\eta_{11}(s)\right.\\
&\quad\left. - b_1\int_{-\tau}^{0}\overline{\phi}_2(\xi + cs)\,d\eta_{12}(s)\right] \ge -r_1\underline{\phi}_1^2(\xi)\\
&\quad - r_1a_1\overline{\phi}_1(\xi)\int_{-\tau}^{0}\overline{\phi}_1(\xi + cs)\,d\eta_{11}(s) - r_1b_1\overline{\phi}_1(\xi)\\
&\quad\cdot\int_{-\tau}^{0}\overline{\phi}_2(\xi + cs)\,d\eta_{12}(s) \ge -r_1(1 + a_1)\\
&\quad\cdot\overline{\phi}_1^2(\xi) - r_1b_1\overline{\phi}_1(\xi)\overline{\phi}_2(\xi) \ge -r_1(1 + a_1)e^{2\gamma_1\xi}\\
&\quad - r_1b_1e^{(\gamma_1+\gamma_2/2)\xi}.
\end{aligned} \tag{43}$$

Since

$$\begin{aligned}
&d_1\underline{\phi}_1''(\xi) - c\underline{\phi}_1'(\xi) + r_1\underline{\phi}_1(\xi)\\
&\quad = e^{\gamma_1\xi}\left(d_1\gamma_1^2 - c\gamma_1 + r_1\right)\\
&\qquad - Qe^{\eta\gamma_1\xi}\left(d_1\eta^2\gamma_1^2 - c\eta\gamma_1 + r_1\right)\\
&\quad = -Qe^{\eta\gamma_1\xi}\left(d_1\eta^2\gamma_1^2 - c\eta\gamma_1 + r_1\right)
\end{aligned} \tag{44}$$

then the third is true by (Q2).

We now consider the fourth inequality, which is clear if $\underline{\phi}_2(\xi) = 0$. When $\underline{\phi}_2(\xi) > 0$, we have

$$\begin{aligned}
&\underline{\phi}_2'(\xi) = \gamma_2\underline{\phi}_2(\xi) - e^{\gamma_2\xi} + \frac{R}{2\sqrt{-\xi}}e^{\gamma_2\xi},\\
&\underline{\phi}_2''(\xi) = \gamma_2^2\underline{\phi}_2(\xi) - 2\gamma_2e^{\gamma_2\xi} + \frac{R\gamma_2}{\sqrt{-\xi}}e^{\gamma_2\xi}\\
&\qquad + \frac{R}{4\sqrt{(-\xi)^3}}e^{\gamma_2\xi},
\end{aligned} \tag{45}$$

which implies

$$\begin{aligned}
&d_2\underline{\phi}_2''(\xi) - c\underline{\phi}_2'(\xi) + r_2\underline{\phi}_2(\xi)\\
&\quad = \left(d_2\gamma_2^2 - c\gamma_2 + r_2\right)\underline{\phi}_2(\xi) - \left(2d_2\gamma_2 - c\right)e^{\gamma_2\xi}\\
&\qquad + Re^{\gamma_2\xi}\left[\frac{d_2\gamma_2}{\sqrt{-\xi}} - \frac{c}{2\sqrt{-\xi}} + \frac{d_2}{4\sqrt{(-\xi)^3}}\right]\\
&\quad = \frac{Rd_2e^{\gamma_2\xi}}{4\sqrt{(-\xi)^3}}.
\end{aligned} \tag{46}$$

Moreover, (R0) and (R1) imply that

$$\begin{aligned} r_2\underline{\phi}_2(\xi)\Big[-\underline{\phi}_2(\xi) - a_2\int_{-\tau}^{0}\overline{\phi}_2(\xi+cs)\,d\eta_{22}(s) \\ + b_2\int_{-\tau}^{0}\underline{\phi}_1(\xi+cs)\,d\eta_{21}(s)\Big] \geq -r_2(1+a_2)\overline{\phi}_2^2(\xi) \\ \geq -r_2(1+a_2)e^{2\gamma_2'\xi}, \end{aligned} \tag{47}$$

and so

$$\begin{aligned} &\frac{Rd_2e^{\gamma_2\xi}}{4\sqrt{(-\xi)^3}} - r_2(1+a_2)e^{2\gamma_2'\xi} \\ &\quad = Re^{\gamma_2\xi}\left[\frac{d_2}{4\sqrt{(-\xi)^3}} - \frac{r_2(1+a_2)e^{(2\gamma_2'-\gamma_2)\xi}}{R}\right] > 0 \end{aligned} \tag{48}$$

by (R2), which completes the verification and proof. □

Lemma 6. *Assume that $d_1r_1 = d_2r_2$. Then (3) of Theorem 1 is true.*

Proof. Utilizing the parameters similar to those in Lemmas 4 and 5, we define

$$\begin{aligned} \overline{\phi}_1(\xi) &= \begin{cases} (-\xi+K)e^{\gamma_1\xi}, & \xi \leq \xi_1, \\ 1, & \xi \geq \xi_1, \end{cases} \\ \underline{\phi}_1(\xi) &= \begin{cases} \left(-\xi - L\sqrt{-\xi}\right)e^{\gamma_1\xi}, & \xi \leq -L^2, \\ 0, & \xi > -L^2, \end{cases} \\ \overline{\phi}_2(\xi) &= \begin{cases} \left(-\xi + N\sqrt{-\xi}\right)e^{\gamma_2\xi}, & \xi \leq \xi_2, \\ 1+b_2, & \xi \geq \xi_2, \end{cases} \\ \underline{\phi}_2(\xi) &= \begin{cases} \left(-\xi - R\sqrt{-\xi}\right)e^{\gamma_2\xi}, & \xi \leq -R^2, \\ 0, & \xi > -R^2, \end{cases} \end{aligned} \tag{49}$$

where $\xi_1 < 0$ and $\xi_2 < 0$ such that $\overline{\phi}_1(\xi)$ and $\overline{\phi}_2(\xi)$ are continuous. Similar to the proof of Lemmas 4 and 5, we can complete the proof. □

Before ending this paper, we make the following remarks on the minimal wave speed.

Remark 7. In Lin [15] and Pan [16], the authors studied the asymptotic spreading of (1) if $\tau = 0$, in which one species spreads in the minimal wave speed of traveling wave solutions. However, (1) does not satisfy the comparison principle of classical predator-prey systems in [15, 16]; there are also some technique problems in estimating the asymptotic spreading of (1), which will be further investigated in our future research.

Remark 8. From Pan [8], we see that a traveling wave solution with large wave speed decays exponentially as $\xi \to -\infty$. However, when the minimal wave speed is concerned, it does not decay exponentially as $\xi \to -\infty$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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