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Research Article

Lebesgue-p Norm Convergence Analysis of PD$^{\alpha}$-Type Iterative Learning Control for Fractional-Order Nonlinear Systems

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The first-order and second-order PD$^{\alpha}$-type iterative learning control (ILC) schemes are considered for a class of Caputo-type fractional-order nonlinear systems. Due to the imperfection of the λ-norm, the Lebesgue-p (L^p) norm is adopted to overcome the disadvantage. First, a generalization of the Gronwall integral inequality with singularity is established. Next, according to the reached generalized Gronwall integral inequality and the generalized Young inequality, the monotonic convergence of the first-order PD$^{\alpha}$-type ILC is investigated, while the convergence of the second-order PD$^{\alpha}$-type ILC is analyzed. The resultant condition shows that both the learning gains and the system dynamics affect the convergence. Finally, numerical simulations are exploited to verify the results.

1. Introduction

Iterative learning control (ILC) is an effective control developed for target trajectory tracking [1]. The key feature of the ILC is to improve the quality of control iteratively by using proportional, integral, and/or derivative tracking errors obtained from previous operation and finally to generate the control input that causes the desired output trajectory. Due to its satisfactory tracking performance by using less a prior knowledge, ILC has been widely applied to repetitive operations including robot manipulations and batch processes [2–4].

Fractional calculus is a mathematical topic with more than three-hundred-year-old history, but its application to physics and engineering has attracted a lot of attention in the latest decades [5, 6]. They have been verified to be a powerful technique to model the memory and hereditary properties of many materials and processes [7–9]. Further, it has been acknowledged that a fractional-order controller performs well compared to an integer-order controller for a fractional-order system. This pushes the development of the fractional-order controllers [10, 11].

Among different fractional-order controllers, fractional-order iterative learning control (FOILC) is becoming one of active research areas. In the 2010s, for an αth-order linear system, the authors investigated the αth-order derivative-type (D^{α}-type) ILC in time domain [12]. This investigation showed that the optimal ILC for an αth-order linear system is the ILC order being α. In the following years, many FOILC problems are presented for various fractional-order systems. Up to now, the FOILC area has attracted much attention, of which the convergence analysis is one of key issues. For more details, readers can refer to the works [13–19] and the references therein. Despite the nice results of existing investigations, there still remain some undesirable problems between the theoretical development and its practical application.

The first one is that many nice results are derived under a questionable assumption that the desired control input exists [20–22]. However, from the engineering application perspective, the desired output trajectory should be predetermined by the target to be tracked, rather than constructed under the assumption that the desired control input existed. That is, the convergence analysis process relies on the information that seems to be known but is actually unknown to the desired control input.

The second one is that the existing FOILC investigations analyzed the convergence by using λ-norm-based analytical methods, and the convergence is guaranteed with the sufficiently large λ. As commented in [23, 24], the larger parameter λ may greatly inhibit the actual tracking error and ignore the influence of the state matrix and proportional learning gains to the condition. This conveys that the results of both the ILCs and FOILCs are mathematically good, but practically, it may result in the normal tracking error exceeding the practical tolerance even though the λ-norm tracking error is satisfactory. Thus, both the existing ILCs and FOILCs need to be refined. For this aspect, reference [25] has adopted L^p-norm to evaluate the tracking error and has derived the monotonic convergence of the conventional first-order PD-type ILC for a class of integer-order linear systems. The derivation tells that the convergence is not only dominated by the system input and output matrices and the derivative learning gain, but also by the system state matrix and the proportional learning gain. The result reflects the relationship between the system dynamics and the learning mechanism to the convergence. To be specific, as L^2-norm measures the tracking error in the concept of energy, the convergence result in the sense of L^p-norm may boost the theoretical development near to practical execution. However, the result of the FOILC is a hanging issue. It is necessary to state that the integer derivative is a local operator, while the fractional derivative is a nonlocal operator which has many different properties; thus many theoretical approaches based on the integer-order control systems cannot have been directly applied to the fractional ones. Therefore, it is worthwhile to apply the L^p-norm for convergence analysis of existing FOILCs.

In addition, as discussed in FOILCs [12, 15–17], the higher-order learning algorithms, which employ preceding control information of more than one iteration, have utility to lead a better performance in terms of both convergence rate and robustness, which is taken advantage of. As a matter of fact, with different choice of learning gains, the higher-order classical ILC algorithm can be perform slower and faster than or equivalent to the lower-order ones in terms of convergence rate [25]. However, such affirmation has not been seen valid for fractional-order iterative learning control systems.

Motivated by the aforementioned hanging issue regarding the fractional-order systems and FOILC schemes, this paper develops the first-order and the second-order proportional-Caputo-fractional-order-derivative-type (PD$^\alpha$-type) ILCs for a class of Caputo-type fractional-order nonlinear dynamic systems and then applies the L^p-norm to investigate their convergence in an objective manner. The main contributions of this paper are that we establish a theoretical analysis framework on the monotonic convergence of the first-order PD$^\alpha$-type ILC for a Caputo-type fractional-order nonlinear system in the sense of L^p-norm. In the theoretical analysis, there is no need for the questionable assumption that the desired input exists and a novel Growall integral inequality with singularity is established for the strict convergence analysis. And then, the convergence is derived for the case when the second-order PD$^\alpha$-type is implemented on the systems and the convergent speed comparison of the second-order law with the first-order one is generalized to FOILCs.

The rest of the paper is organized as follows. In Section 2, the basic concepts, properties, and lemmas are described. In Section 3, the monotonic convergence of the first-order PD$^\alpha$-type ILC scheme and the convergence of the second-order PD$^\alpha$-type ILC are given. In Section 4, examples are presented to validate the theoretical results. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we briefly give some basic definitions and properties related to fractional calculus [5, 6].

Definition 1. For an arbitrary integrable function $f(t)$, the definition of the fractional integrals of order $\alpha > 0$ is defined as

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad t \in [0,\infty), \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

Definition 2. For a given number $\alpha > 0$, the α-order Caputo-type derivative of the function $f(t)$ is defined as

$$ \begin{gathered} {}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \\ n-1 < \alpha < n, \; t \in [0,\infty), \end{gathered} \tag{2}$$

where n is an integer and $f^{(n)}(t) = (d^n/dt^n) f(t)$.

Property 3. If $0 < \alpha < 1$, then ${}_0D_t^{-\alpha}({}_0^CD_t^{\alpha} f(t)) = f(t) - f(0)$.

Definition 4. The Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \; \beta > 0, \; z \in C^{n\times n}, \tag{3}$$

where $E_\alpha(z) = E_{\alpha,1}(z)$ and $E_{1,1}(z) = e^z$.

Definition 5 (see [26]). For a given scalar function $f : [0,T] \to R$, its L^p-norm is defined as

$$ \|f(\cdot)\|_p = \left[\int_0^T |f(t)|^p dt\right]^{1/p}, \quad 1 \le p \le \infty. \tag{4}$$

For a time-varying vector function $f : [0,T] \to R^m$, $f(t) = [f^1(t), \ldots, f^m(t)]^T$, its L^p-norm is defined as

$$ \|f(\cdot)\|_p = \left[\int_0^T \left(\max_{1\le i\le m} |f^i(t)|\right)^p dt\right]^{1/p}, \quad 1 \le p \le \infty. \tag{5}$$

Lemma 6 (see [27]). *If the function $f(t) \in C^n[0,T]$, then the initial value problem,*

$$ \begin{gathered} {}_0^CD_t^{\alpha} x(t) = f(x(t),t), \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{gathered} \tag{6}$$

is equivalent to the following nonlinear Volterra integral equation:

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x(\tau),\tau)\,d\tau, \tag{7}$$

and its solutions are continuous.

For brevity, we set ${}_0D_t^{\alpha} = {}_0^CD_t^{\alpha}$ in the following section.

Lemma 7 (generalized Young inequality of convolution integral [26]). *For Lebesgue integrable scalar functions g, h : $[0,T] \in R$, the generalized Young inequality of their convolution integral is*

$$\|g * h(\cdot)\|_r \leq \|g(\cdot)\|_q \|h(\cdot)\|_p, \tag{8}$$

where $1 \leq p,\ q, r \leq \infty$ satisfy $1/r = 1/p + 1/q - 1$. Particularly, when $r = p$ and thus $q = 1$, then the inequality of convolution integral is

$$\|g * h(\cdot)\|_p \leq \|g(\cdot)\|_1 \|h(\cdot)\|_p. \tag{9}$$

Lemma 8 (see [2]). *Let $\{a_n\}$ be a positive real sequence defined as*

$$a_n \leq \rho_1 a_{n-1} + \rho_2 a_{n-2}. \tag{10}$$

If $\rho_1,\ \rho_2$ are nonnegative numbers satisfying

$$\rho = \rho_1 + \rho_2 < 1, \tag{11}$$

then the following holds:

$$\lim_{n\to\infty} a_n = 0. \tag{12}$$

Let us establish an extended Gronwall integral inequality with singularity, which is important to the convergence analysis in the next section. The proof is based on an iteration argument.

Lemma 9. *Suppose $a, b \geq 0$ (constant) and $\alpha > 0$, $c(t)$, $x(t)$ and $y(t)$ are nonnegative and locally integrable on $[0, T_0)$ ($T_0 \leq +\infty$). If*

$$x(t) \leq c(t) + \int_0^t (t-s)^{\alpha-1} \left[ay(s) + bx(s)\right] ds, \quad t \in [0, T_0). \tag{13}$$

Then

$$x(t) \leq c(t) + \int_0^t \left[\Gamma(\alpha) \cdot \Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)) \cdot (bc(s) + ay(s))\right] ds, \tag{14}$$

where $\Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)) = (t-s)^{\alpha-1} E_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)^{\alpha})$.

Proof. See Appendix. □

3. PD$^\alpha$-Type ILCs and Convergence Analysis for Fractional-Order Nonlinear Systems

Consider the following nonlinear α-order ($0 < \alpha < 1$) systems:

$$\begin{aligned} {}_0^C D_t^\alpha x_k(t) &= f(x_k(t), t) + Bu_k(t), \\ y_k(t) &= Cx_k(t), \quad t \in [0, T] \\ x_k(0) &= 0, \end{aligned} \tag{15}$$

where k refers to the operation number and $[0, T]$ is an operation time interval while $\alpha \in (0, 1)$. $x_k(t) \in R^n$, $u_k(t) \in R$ and $y_k(t) \in R$ denote n-dimensional state vector, scalar control input, and output, respectively. B and C are matrices with appropriate dimensions. The function $f(x_k(t), t)$ satisfies the global Lipschitz condition:

$$\begin{aligned} &\left|C\left(f(x_{k+1}(t), t) - f(x_k(t), t)\right)\right| \\ &\quad \leq L_0 \left|C(x_{k+1}(t) - x_k(t))\right| = L_0 \left|y_{k+1}(t) - y_k(t)\right|, \end{aligned} \tag{16}$$

where L_0 is positive Lipschitz constant.

In this section, the sufficient conditions are derived for convergence of the first-order and second-order PD$^\alpha$-type ILC algorithms for fractional-order nonlinear systems. Now, we give our main results.

4. Monotonic Convergence Analysis for First-Order PD$^\alpha$-Type ILC

To control the systems stated in (15), the first-order PD$^\alpha$-type ILC is given as follows:

(I_1)

$$\begin{aligned} &u_1(t) \text{ is given arbitrarily}, \\ &u_{k+1}(t) = u_k(t) + L_{p_1} e_k(t) + L_{d_1\, 0}D_t^\alpha e_k(t), \\ &\qquad t \in [0, T], \quad k = 2, 3, 4, \ldots. \end{aligned} \tag{17}$$

Here, L_{p_1} and L_{d_1} are the first-order proportional and fractional-order derivative learning gains, respectively. The expression $e_k(t) = y_d(t) - y_k(t)$ denotes the tracking error between the desired trajectory $y_d(t)$ and the system output $y_k(t)$ of the system (15) driven by $u_k(t)$ at the kth iteration.

Theorem 10. *For the first-order PD$^\alpha$-type iterative learning control rue (I_1) is applied to system (15), if the system matrices B, C, the order α and the Lipchitz constant L_0 together with learning gains L_{p_1}, L_{d_1} satisfy the following condition:*

$$\begin{aligned} \rho_1 &= \left|1 - CBL_{d_1}\right| + \left(\left|CBL_{p_1}\right| + L_0 + L_0\left|1 - CBL_{d_1}\right|\right) \\ &\quad \cdot \left\|\Phi_{\alpha,\alpha}(L_0 \cdot (\cdot))\right\|_1 < 1. \end{aligned} \tag{18}$$

Here, $|\cdot|$ stands for the absolute value and $\|\cdot\|_1$ stands for the L^1-norm of the function defined on the operation time interval $[0, T]$.

Then, the output error is strictly monotonic convergence in L^p-norm; that is,

(1) $\|e_{k+1}(\cdot)\|_p < \|e_k(\cdot)\|_p$;

(2) $\lim_{k\to\infty} \|e_{k+1}(\cdot)\|_p = 0$.

Proof. For the dynamic system (15) and the PD$^\alpha$-type ILC scheme (I_1), from Lemma 6, we have

$$\begin{aligned} x_{k+1}(t) &= x_{k+1}(0) + \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-\tau)^{\alpha-1} \left(f(x_{k+1}(\tau), \tau) + Bu_{k+1}(\tau)\right) d\tau, \end{aligned} \tag{19}$$

and, then, we get

$$\begin{aligned} e_{k+1}(t) &= y_d(t) - y_{k+1}(t) = y_d(t) - y_k(t) - \left[y_{k+1}(t) - y_k(t)\right] = e_k(t) - C\left[x_{k+1}(t) - x_k(t)\right] = e_k(t) \\ &\quad - C\left[x_{k+1}(0) + \frac{1}{\Gamma(\alpha)} \cdot \int_0^t (t-\tau)^{\alpha-1}\left(f\left(x_{k+1}(\tau),\tau\right) + Bu_{k+1}(\tau)\right) d\tau\right] \\ &\quad + C\left[x_k(0) + \frac{1}{\Gamma(\alpha)} \cdot \int_0^t (t-\tau)^{\alpha-1}\left(f\left(x_k(\tau),\tau\right) + Bu_k(\tau)\right) d\tau\right] \\ &= e_k(t) - C \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \cdot \left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right) d\tau - CB \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\left(u_{k+1}(\tau) - u_k(\tau)\right) d\tau = e_k(t) \\ &\quad - C \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \cdot \left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right) d\tau - CBL_{p_1} \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_k(\tau)\, d\tau - CBL_{d_1} \cdot \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-\tau)^{\alpha-1} {}_0D_\tau^\alpha e_k(\tau)\, d\tau. \end{aligned} \tag{20}$$

Applying Property 3 to the last term on the right side of (20), we have

$$\begin{aligned} e_{k+1}(t) &= \left(1 - CBL_{d_1}\right) e_k(t) - C \cdot \frac{1}{\Gamma(\alpha)} \cdot \int_0^t (t-\tau)^{\alpha-1}\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right) d\tau \\ &\quad - CBL_{p_1} \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_k(\tau)\, d\tau. \end{aligned} \tag{21}$$

Taking absolution on both sides of (21) yields

$$\begin{aligned} \left|e_{k+1}(t)\right| &\le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \cdot \left|C\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)\right| d\tau \\ &\quad + \left|CBL_{p_1}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau. \end{aligned} \tag{22}$$

Applying Lipschitz condition to the second term on the right side of (22), we get

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \cdot \left|C\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)\right| d\tau \le L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_{k+1}(\tau) - y_d(\tau)\right| d\tau + L_0 \\ &\quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_d(\tau) - y_k(\tau)\right| d\tau = L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k+1}(\tau)\right| d\tau + L_0 \cdot \frac{1}{\Gamma(\alpha)} \\ &\quad \cdot \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau. \end{aligned} \tag{23}$$

Taking (23) into (22) obtains

$$\begin{aligned} \left|e_{k+1}(t)\right| &\le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k+1}(\tau)\right| d\tau \\ &\quad + \left(\left|CBL_{p_1}\right| + L_0\right) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau. \end{aligned} \tag{24}$$

Using Lemma 9 to inequality (24), we have

$$\begin{aligned} \left|e_{k+1}(t)\right| &\le \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + \int_0^t \left[\left(\left|CBL_{p_1}\right| + L_0 + L_0 \left|1 - CBL_{d_1}\right|\right) \cdot \Phi_{\alpha,\alpha}\left(L_0 (t-\tau)\right)\right] \left|e_k(\tau)\right| d\tau. \end{aligned} \tag{25}$$

Taking the L^p-norm on both sides of (25) and adopting the generalized Young inequality of convolution integral, we get

$$\left\|e_{k+1}(\cdot)\right\|_p \le \rho_1 \left\|e_k(\cdot)\right\|_p. \tag{26}$$

This completes the proof of Theorem 10. □

Remark 11. We can see, from the above derivation, that the convergence condition is quantified directly from the L^p-norm, not by using the sufficiently large λ, and analyzed in terms of the tracking error rather than the control input error. Besides, the monotonic property of convergence can ensure the first-order PD$^\alpha$-type ILC rule to be practically implementable. Further, from condition (18), we can observe that the convergence is affected not only by the derivative learning gain and the system dynamics, but also by the proportional learning gain. That is, the result reflects the features of system dynamics and the mechanism of the algorithm to the convergence. Actually, the impact of the state dynamics and proportional learning gain, which are neglected in the existing

FOILC investigations, exists but is significantly suppressed by the sufficiently large parameter λ. It should be noted that, as mentioned in [23], a large value of λ may have a huge tracking error, which is not allowable in practice.

Remark 12. Note that the convergence analysis for fractional-order linear time-invariant system has been investigated in my early work [28], in which the derivation is made by means of the state transition matrix in an equality form, and the convergence condition is

$$\rho = \left|1 - CBL_{d_1}\right| + \left\|C\widetilde{\Phi}_{\alpha,\alpha}(\cdot)\left(BL_{p_1} + ABL_{d_1}\right)\right\|_1, \quad (27)$$

where A is the system matrix and $\widetilde{\Phi}_{\alpha,\alpha}(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})$ is the state transition matrix. However, in this paper, the investigated fractional-order system is nonlinear with its nonlinearity unknown; the proof of convergence is derived by means of inequality Lemma 9. Thus, the convergence condition ρ_1 cannot degenerate to the condition ρ when the fractional-order nonlinear system is reduced to the corresponding linear case.

4.1. Convergence Analysis for Second-Order PD$^{\alpha}$-Type ILC. In this section, we go on considering the second-order PD$^{\alpha}$-type ILC algorithm, which is constructed by employing the control inputs and their output errors of the latest previously adjacent operations in a weighting average form as follows:

$$(I_2)$$

$$\begin{aligned} &u_1(t) \text{ is given arbitrarily,} \\ &u_2(t) = u_1(t) + L_{p_1}e_1(t) + L_{d_1}\,{}_0D_t^{\alpha}e_1(t), \\ &u_{k+1}(t) = r_1\left[u_k(t) + L_{p_1}e_k(t) + L_{d_1}\,{}_0D_t^{\alpha}e_k(t)\right] \\ &\quad + r_2\left[u_{k-1}(t) + L_{p_2}e_{k-1}(t) + L_{d_2}\,{}_0D_t^{\alpha}e_{k-1}(t)\right], \\ &\qquad k = 2, 3, 4, \ldots. \end{aligned} \quad (28)$$

Here, L_{p_2} and L_{d_2} denote the second-order proportional and fractional-order derivative learning gains, respectively. The weighting coefficients r_1 and r_2 satisfy $0 \le r_1 < 1$, $0 \le r_2 \le 1$, and $r_1 + r_2 = 1$.

Theorem 13. *For the second-order PD$^{\alpha}$-type iterative learning control rule (I_2) is applied to system (15), if the system matrices B, C, the order α, and the Lipchitz constant L_0 together with the learning gains L_{p_1}, L_{p_2}, L_{d_1}, L_{d_2} satisfy the following conditions:*

(1) $\rho_1 = |1 - CBL_{d_1}| + (|CBL_{p_1}| + L_0 + L_0|1 - CBL_{d_1}|)\|\Phi_{\alpha,\alpha}(L_0\cdot(\cdot))\|_1 < 1$.

(2) $\rho_2 = |1 - CBL_{d_2}| + (|CBL_{p_2}| + L_0 + L_0|1 - CBL_{d_2}|)\|\Phi_{\alpha,\alpha}(L_0\cdot(\cdot))\|_1 < 1$.

Then, the learning scheme (I_2) is convergent, that is, $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p = 0$.

Proof. From the dynamic system (15) and the PD$^{\alpha}$-type ILC scheme (I_2), we have

$$\begin{aligned} e_{k+1}(t) &= y_d(t) - y_{k+1}(t) = r_1\left[y_d(t) - y_k(t)\right] \\ &+ r_2\left[y_d(t) - y_{k-1}(t)\right] - \left[y_{k+1}(t) - r_1y_k(t)\right. \\ &\left.- r_2y_{k-1}(t)\right] = r_1e_k(t) + r_2e_{k-1}(t) - C\left[x_{k+1}(t)\right. \\ &\left.- r_1x_k(t) - r_2x_{k-1}(t)\right] = r_1e_k(t) + r_2e_{k-1}(t) \\ &- C\left[x_{k+1}(0) + \frac{1}{\Gamma(\alpha)}\right. \\ &\left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_{k+1}(\tau),\tau\right) + Bu_{k+1}(\tau)\right)d\tau\right] \\ &+ r_1C\left[x_k(0) + \frac{1}{\Gamma(\alpha)}\right. \\ &\left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_k(\tau),\tau\right) + Bu_k(\tau)\right)d\tau\right] \\ &+ r_2C\left[x_{k-1}(0) + \frac{1}{\Gamma(\alpha)}\right. \\ &\left.\cdot\int_0^t(t-\tau)^{\alpha-1}\left(f\left(x_{k-1}(\tau),\tau\right) + Bu_{k-1}(\tau)\right)d\tau\right] \\ &= r_1e_k(t) + r_2e_{k-1}(t) - r_1C\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)d\tau - r_2C \\ &\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_{k-1}(\tau),\tau\right)\right)d\tau - CB \\ &\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(u_{k+1}(\tau) - r_1u_k(\tau) - r_2u_{k-1}(\tau)\right)d\tau = r_1e_k(t) \\ &+ r_2e_{k-1}(t) - r_1C\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_k(\tau),\tau\right)\right)d\tau - r_2C \\ &\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(f\left(x_{k+1}(\tau),\tau\right) - f\left(x_{k-1}(\tau),\tau\right)\right)d\tau - CB \\ &\cdot\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(r_1L_{p_1}e_k(\tau) + r_2L_{p_2}e_{k-1}(\tau)\right)d\tau - CB\cdot\frac{1}{\Gamma(\alpha)} \\ &\cdot\int_0^t(t-\tau)^{\alpha-1} \\ &\cdot\left(r_1L_{d_1}\,{}_0D_{\tau}^{\alpha}e_k(\tau) + r_2L_{d_2}\,{}_0D_{\tau}^{\alpha}e_{k-1}(\tau)\right)d\tau. \end{aligned} \quad (29)$$

Applying Property 3 to the last term on the right side of (29) yields

$$\begin{aligned} & CB \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ & \quad \cdot \left(r_1 L_{d_1} {}_0D_\tau^\alpha e_k(\tau) + r_2 L_{d_2} {}_0D_\tau^\alpha e_{k-1}(\tau)\right) d\tau \\ & \quad = r_1 CBL_{d_1} e_k(t) + r_2 CBL_{d_2} e_{k-1}(t). \end{aligned} \tag{30}$$

Submitting (30) into (29) and taking absolute values on both sides of (29) obtain

$$\begin{aligned} & \left|e_{k+1}(t)\right| \le r_1 \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + r_2 \left|1 - CBL_{d_2}\right| \\ & \quad \cdot \left|e_{k-1}(t)\right| + r_1 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ & \quad \cdot \left|C\left(f\left(x_{k+1}(\tau), \tau\right) - f\left(x_k(\tau), \tau\right)\right)\right| d\tau + r_2 \\ & \quad \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ & \quad \cdot \left|C\left(f\left(x_{k+1}(\tau), \tau\right) - f\left(x_{k-1}(\tau), \tau\right)\right)\right| d\tau \\ & \quad + r_1 \left|CBL_{p_1}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau \\ & \quad + r_2 \left|CBL_{p_2}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k-1}(\tau)\right| d\tau \\ & \quad \le r_1 \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + r_2 \left|1 - CBL_{d_2}\right| \left|e_{k-1}(t)\right| \\ & \quad + r_1 L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_{k+1}(\tau) - y_d(\tau)\right| d\tau \\ & \quad + r_1 L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_d(\tau) - y_k(\tau)\right| d\tau \\ & \quad + r_2 L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_{k+1}(\tau) - y_d(\tau)\right| d\tau \\ & \quad + r_2 L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|y_d(\tau) - y_{k-1}(\tau)\right| d\tau \\ & \quad + r_1 \left|CBL_{p_1}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau \\ & \quad + r_2 \left|CBL_{p_2}\right| \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k-1}(\tau)\right| d\tau \\ & \quad = r_1 \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + r_2 \left|1 - CBL_{d_2}\right| \left|e_{k-1}(t)\right| \\ & \quad + L_0 \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k+1}(\tau)\right| d\tau + r_1 \left(L_0\right. \\ & \quad \left. + \left|CBL_{p_1}\right|\right) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_k(\tau)\right| d\tau + r_2 \left(L_0\right. \\ & \quad \left. + \left|CBL_{p_2}\right|\right) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left|e_{k-1}(\tau)\right| d\tau. \end{aligned} \tag{31}$$

Using Lemma 9 to equality (31) derives that

$$\begin{aligned} & \left|e_{k+1}(t)\right| \le r_1 \left|1 - CBL_{d_1}\right| \left|e_k(t)\right| + r_2 \left|1 - CBL_{d_2}\right| \\ & \quad \cdot \left|e_{k-1}(t)\right| \\ & \quad + r_1 \int_0^t \left[\left(\left|CBL_{p_1}\right| + L_0 + L_0 \left|1 - CBL_{d_1}\right|\right)\right. \\ & \quad \left. \cdot \Phi_{\alpha,\alpha}\left(L_0 (t-\tau)\right)\right] \left|e_k(\tau)\right| d\tau \\ & \quad + r_2 \int_0^t \left[\left(\left|CBL_{p_2}\right| + L_0 + L_0 \left|1 - CBL_{d_2}\right|\right)\right. \\ & \quad \left. \cdot \Phi_{\alpha,\alpha}\left(L_0 (t-\tau)\right)\right] \left|e_{k-1}(\tau)\right| d\tau. \end{aligned} \tag{32}$$

Adopting the L^p-norm on both sides of (32) implies that

$$\left\|e_{k+1}(\cdot)\right\|_p \le r_1 \rho_1 \left\|e_k(\cdot)\right\|_p + r_2 \rho_2 \left\|e_{k-1}(\cdot)\right\|_p, \tag{33}$$

and, then, according to Lemma 8 and assumptions (1) and (2), it is therefore finally evident that $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p = 0$. □

Remark 14. It can be observed that when $r_2 = 0$, the second-order PD$^\alpha$-type ILC updating law (I_2) degenerates to the first-order PD$^\alpha$-type ILC updating law (I_1). Then the convergence becomes

$$\begin{aligned} \rho_1 = & \left|1 - CBL_{d_1}\right| + \left(\left|CBL_{p_1}\right| + L_0 + L_0 \left|1 - CBL_{d_1}\right|\right) \\ & \cdot \left\|\Phi_{\alpha,\alpha}\left(L_0 \cdot (\cdot)\right)\right\|_1 < 1. \end{aligned} \tag{34}$$

Corollary 15. *Assume that* $\lim_{k\to\infty}\|e_{k+1}(\cdot)\|_p/\|e_k(\cdot)\|_p$ *exists. Analogous to the discussion regarding the convergent speed in [25], we can make assertions as follows:*

(1) *If* $\rho_2 < \rho_1^2 < 1$, *then the second-order algorithm* (I_2) *is* Q_p*-faster than the first-order* (I_1).

(2) *If* $\rho_1^2 = \rho_2 < 1$, *then the scheme* (I_2) *is* Q_p*-equivalent to the scheme* (I_1).

(3) *If* $\rho_1^2 < \rho_2 < 1$, *then the second-order strategy* (I_2) *is* Q_p*-slower than the first-order* (I_1).

Remark 16. From the view point of speed, comparing (1) and (3) of Corollary 15, we can conclude that if we choose a suitable learning gains, the second-order updating law (I_2) may not be a preferred candidate for the systems. However, if we pursue more freedom in choosing the learning gains and better robustness to noise, the second-order updating law (I_2) is a useful alternative.

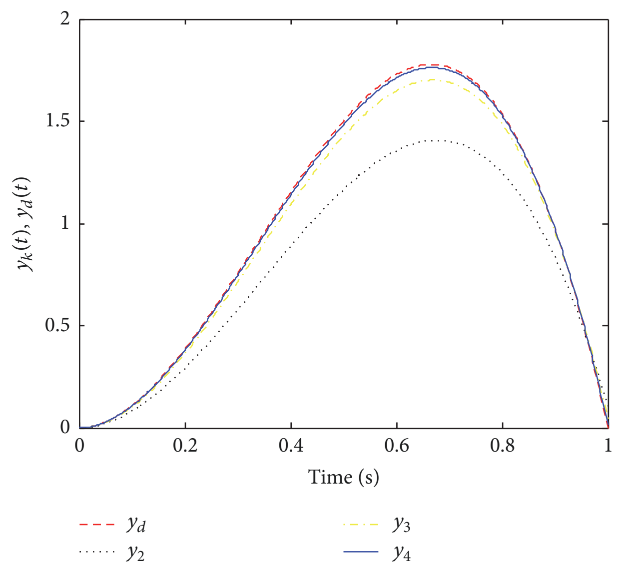

Figure 1: System outputs of (I_1).

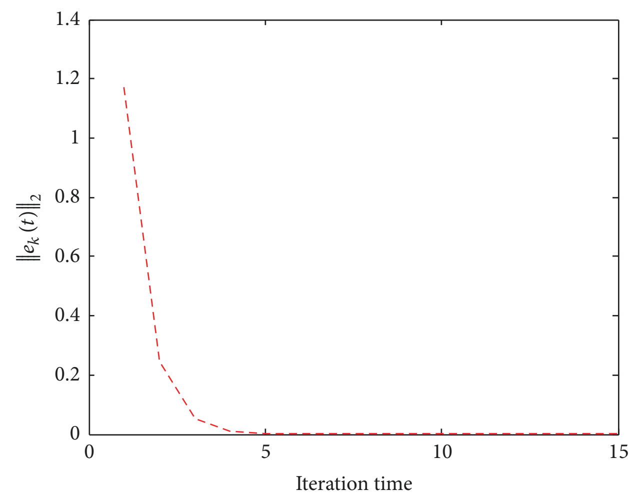

Figure 2: Tracking errors in the sense of L^2-norm.

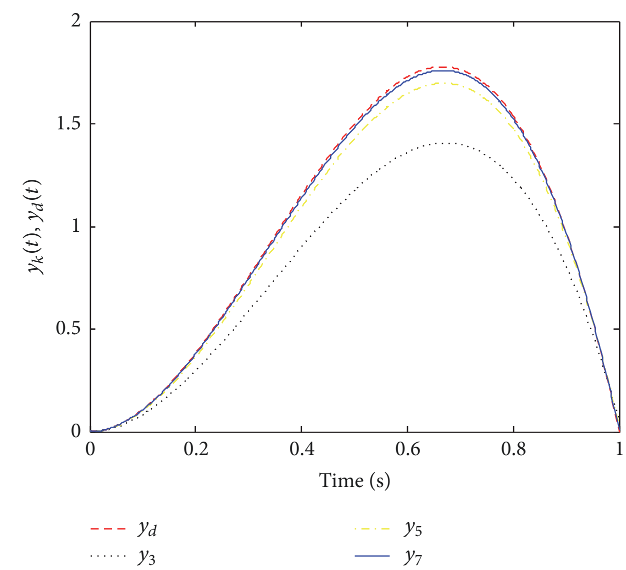

Figure 3: System outputs of (I_2).

5. Simulation Illustrations

In this simulation, we consider the following fractional-order nonlinear system:

$$\begin{gathered}\begin{bmatrix} {}_0D_t^{0.5} x_1(t) \\ {}_0D_t^{0.5} x_2(t) \end{bmatrix} = \begin{bmatrix} 0.2x_1(t) \\ 0.1\sin x_1(t) + 0.2x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{gathered} \tag{35}$$

The operation time period is $[0, 1]$, the desired trajectory is $y_d(t) = 12t^2(1-t)$, and the initial control is $u_1(t) = 0$.

For the first-order PD$^{0.5}$-type ILC scheme (I_1), the first-order proportional and fractional-order derivative learning gains are chosen as $L_{p_1} = 0.2$ and $L_{d_1} = 1.5$, respectively. It can be seen $L_0 = 0.2$ and $\|\Phi_{\alpha,\alpha}(L_0 \cdot (\cdot))\|_1 = 1.3630$; then it is easy to verify that $\rho_1 = 0.7271 < 1$, which means that the monotonic convergence condition (18) is satisfied. The outputs by the scheme (I_1) at the 2nd, 3rd, and 4th operations are shown in Figure 1, respectively. The monotonic tracking error in the sense of L^2-norm is shown in Figure 2.

For the second-order PD$^{0.5}$-type ILC (I_2), we consider two cases as follows.

Case 1. In schemes (I_1) and (I_2), the first-order learning gains are set as $L_{p_1} = 0.2$ and $L_{d_1} = 1.5$, respectively. In scheme (I_2), the weighting coefficients are assigned as $r_1 = 0.5$ and $r_2 = 0.5$, and the second-order learning gains are selected as $L_{p_2} = 0.1$ and $L_{d_2} = 1.2$, respectively. It is computed that $\rho_1 = 0.7271 < 1$ and $\rho_2 = 0.8498 < 1$, which is included in the case that $\rho_1^2 < \rho_2$. It shows that the output error of (I_1) convergence is faster than that of (I_2). The corresponding outputs by scheme (I_2) at the 3rd, 5th, and 7th operations are displayed in Figure 3. It shows that the output follows the desired trajectory well as the iteration increases. The comparison of the tracking error in the sense of L^2-norm made by the updating laws (I_1) and (I_2) is shown in Figure 4. It shows that the tracking errors of both the first-order and second-order laws are convergent and the first-order law (I_1) is convergence faster than second-order law (I_2).

Case 2. In schemes (I_1) and (I_2), the first-order learning gains are chosen as $L_{p_1} = 0.1$ and $L_{d_1} = 1$, respectively. In (I_2), the weighting coefficients are assigned as $r_1 = 0.3$ and $r_2 = 0.7$, and the second-order learning gains are set as $L_{p_2} = 0.1$ and $L_{d_2} = 1.7$, respectively. It is computed that $\rho_1 = 0.9771 < 1$

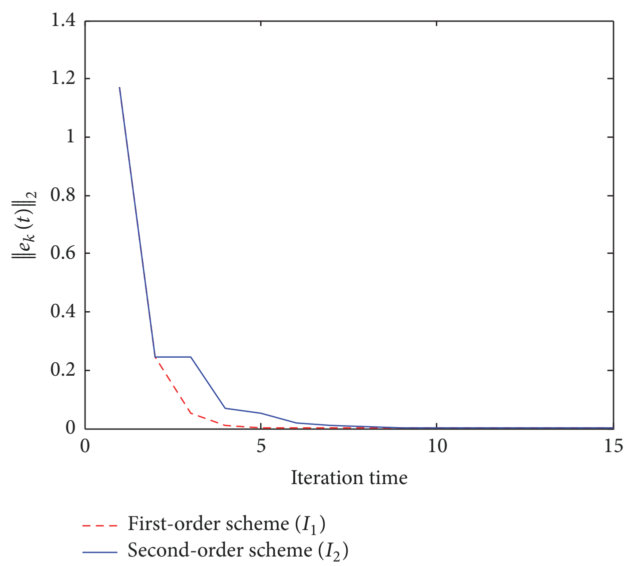

FIGURE 4: Comparison of tracking errors.

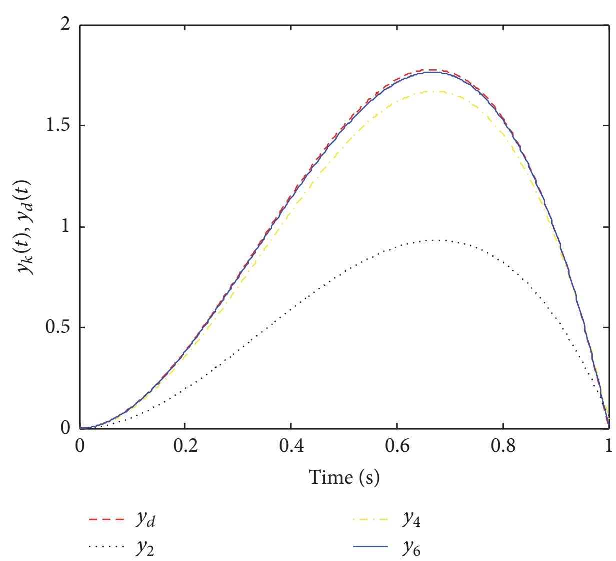

FIGURE 5: System outputs of (I_2).

and $\rho_2 = 0.5316 < 1$, which belongs to the case that $\rho_2 < \rho_1^2$. The outputs by scheme (I_2) at the 2nd, 4th, and 6th iterations are exhibited in Figure 5. Figure 6 shows that the tracking error of the updating law (I_1) convergence is slower than that of (I_2) in the sense of L^2-norm.

6. Conclusion

In this paper, for a class of fractional-order nonlinear systems, the first-order and second-order PD$^\alpha$-type ILC strategies are developed and the sufficiency for convergence is analyzed by means of evaluating the tracking error in the sense of L^p-norm. For analysis, it is found that the sufficient conditions of convergence not only depend on all of system dynamics, but also rely on all of learning gains. Moreover, the convergence

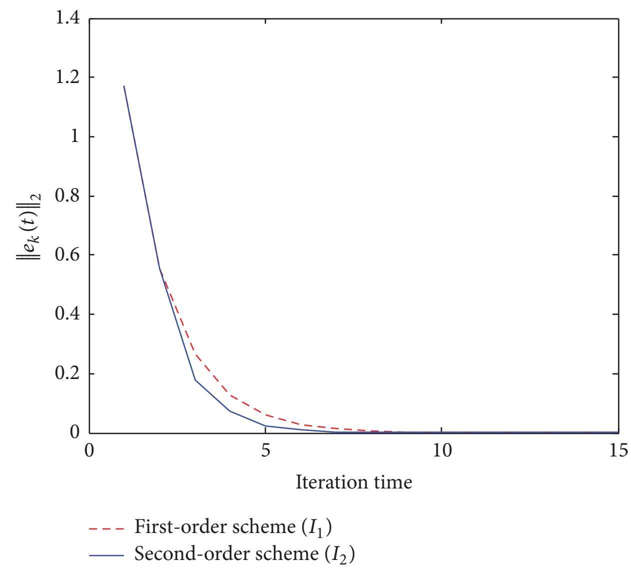

FIGURE 6: Comparison of tracking errors.

speed comparison of the second-order law with the first-order one has been affirmed. We have clarified that, for a fractional-order nonlinear system, the results of the second-order law, that is, its convergence being faster than, equivalent to, or slower than first-order scheme, are validated. All theoretical results were conducted by simulations.

Appendix

Proof of Lemma 9

For the locally integrable function $r(t)$, denote $Br(t) = \int_0^t (t-s)^{\alpha-1} r(s)ds$. Then we have

$$x(t) \le c(t) + aBy(t) + bBx(t), \qquad \text{(A.1)}$$

and it can be written as

$$x(t) \le \sum_{k=0}^{n-1} b^k B^k c(t) + \sum_{k=0}^{n-1} ab^k B^{k+1} y(t) + b^n B^n x(t). \qquad \text{(A.2)}$$

Now, let us prove that

$$B^n x(t) \le \int_0^t \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} x(s)\, ds \qquad \text{(A.3)}$$

and $B^n x(t) \to 0$ as $n \to +\infty$ for each t in $[0, T_0)$.

Step 1. For $n = 1$, inequality (A.3) is true.

Step 2. Assume that when $n = k$, inequality (A.3) is true.

Step 3. If $n = k+1$, then, from induction hypothesis, we derive that

$$B^{k+1}x(t) = B\left(B^k x(t)\right) \le \int_0^t (t-s)^{\alpha-1} \cdot \left[\int_0^s \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} (s-\tau)^{k\alpha-1} x(\tau)\, d\tau\right] ds. \qquad \text{(A.4)}$$

By interchanging the order of integration, we get

$$\begin{aligned}&B^{k+1}x(t)\\&\leq \int_0^t\left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\right]x(\tau)\,d\tau\\&=\frac{(\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(t-s)^{(k+1)\alpha-1}x(s)\,ds,\end{aligned}\tag{A.5}$$

where the integral

$$\begin{aligned}&\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\\&\quad=(t-\tau)^{(k+1)\alpha-1}\int_0^1(1-z)^{\alpha-1}z^{k\alpha-1}dz\\&\quad=(t-\tau)^{(k+1)\alpha-1}B(k\alpha,\alpha)\\&\quad=\frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}(t-\tau)^{(k+1)\alpha-1}\end{aligned}\tag{A.6}$$

with $s=\tau+z(t-\tau)$.

Inequality (A.3) is proved.

Since $B^n x(t)\leq\int_0^t((\Gamma(\alpha))^n/\Gamma(n\alpha))(t-s)^{n\alpha-1}x(s)ds\rightarrow 0$ as $n\rightarrow+\infty$ for each t in $[0,T_0)$, we have

$$\begin{aligned}&x(t)\leq c(t)+\int_0^t\left[\sum_{n=1}^\infty\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}c(s)\right]ds\\&\quad+\int_0^t\left[\sum_{n=0}^\infty a\Gamma(\alpha)\right.\\&\quad\left.\cdot\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha+\alpha)}(t-s)^{(n+1)\alpha-1}y(s)\right]ds.\end{aligned}\tag{A.7}$$

Note that, the second term on the right side of (A.7) implies

$$\begin{aligned}&\int_0^t\left[\sum_{n=1}^\infty\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}c(s)\right]ds\\&\quad=\int_0^t\left[\sum_{n=1}^\infty\frac{(b\Gamma(\alpha))^{n-1}\cdot b\Gamma(\alpha)}{\Gamma((n-1)\alpha+\alpha)}(t-s)^{(n-1)\alpha+\alpha-1}\right.\\&\quad\left.\cdot c(s)\right]ds=\int_0^t\left[\sum_{n=0}^\infty\frac{(b\Gamma(\alpha))^n\cdot b\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}(t-s)^{n\alpha}\right.\\&\quad\left.\cdot(t-s)^{\alpha-1}c(s)\right]ds=\int_0^t\left[b\Gamma(\alpha)\cdot E_{\alpha,\alpha}\left(b\Gamma(\alpha)\right.\right.\\&\quad\left.\left.\cdot(t-s)^\alpha\right)\cdot(t-s)^{\alpha-1}c(s)\right]ds.\end{aligned}\tag{A.8}$$

Since the last term on the right side of (A.7) is

$$\begin{aligned}&\int_0^t\left[\sum_{n=0}^\infty a\Gamma(\alpha)\cdot\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha+\alpha)}(t-s)^{(n+1)\alpha-1}y(s)\right]ds\\&\quad=\int_0^t\left[a\Gamma(\alpha)\cdot E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^\alpha\right)\cdot(t-s)^{\alpha-1}\right.\\&\quad\left.\cdot y(s)\right]ds,\end{aligned}\tag{A.9}$$

submitting (A.8) and (A.9) into (A.7) obtains

$$\begin{aligned}x(t)\leq c(t)&+\int_0^t\left[\Gamma(\alpha)\cdot\Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s))\right.\\&\left.\cdot(bc(s)+ay(s))\right]ds,\end{aligned}\tag{A.10}$$

where $\Phi_{\alpha,\alpha}(b\Gamma(\alpha)(t-s))=(t-s)^{\alpha-1}E_{\alpha,\alpha}(b\Gamma(\alpha)(t-s)^\alpha)$.

The proof is complete.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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