Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 5493270, 10 pages
https://doi.org/10.1155/2018/5493270

Research Article

A Stochastic SIR Epidemic System with a Nonlinear Relapse

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Received 19 January 2018; Revised 7 May 2018; Accepted 29 May 2018; Published 25 June 2018

Academic Editor: Rigoberto Medina

We investigate the conditions that control the extinction and the existence of a unique stationary distribution of a nonlinear mathematical spread model with stochastic perturbations in a population of varying size with relapse. Numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

In medicine, relapse is the return of a disease or the signs and symptoms of a disease after a period of improvement. Relapse also refers to returning to the use of an addictive substance or behavior, such as cigarette smoking [1]. For example, for human tuberculosis, incomplete treatment can lead to relapse, but relapse can also occur in patients who take a full course of treatment and are declared cured. Recently, considerable attention has been paid to model the relapse phenomenon. In [2] Tuder developed one of the first epidemic models with relapse in a constant population with bilinear incidence rate. Moreira and Wang [3] included a nonlinear incidence rate in the model. Van Den Driessche and Zou in [4] formulated a SIRI epidemic model as an integro-differential system with the fraction $P(t)$ of recovered individuals remaining in the recovered class t units after the recovery. In [5] the displacement of recovered individuals to the infective class due to relapse is given by ηR_t. Inspired by the works cited above and the fact that relapse is due to contact with infected, it is more reasonable to consider a bilinear relapse rate $\sigma\beta R_t I_t$. We consider the following SIR compartmental model in a population of varying size with a bilinear relapse rate.

$$\begin{aligned} dS_t &= \left(\Lambda - \mu_1 S_t - \beta S_t I_t\right) dt, \\ dI_t &= \left(-\left(\mu_2 + \alpha\right) I_t + \beta S_t I_t + \sigma\beta R_t I_t\right) dt, \\ dR_t &= \left(-\mu_3 R_t + \alpha I_t - \sigma\beta R_t I_t\right) dt. \end{aligned} \tag{1}$$

In this model each letter refers to a compartment in which an individual can reside. Let S_t denote the number of members of a population susceptible to the disease at time t, I_t the number of infective members, and R_t the number of members who have been removed from the possibility of infection with permanent or temporary immunity. The parameters that occur in the model have the following meaning. Λ is the rate at which new individuals enter the population. α is the rate at which the infective individuals become recovered. $\sigma \in [0, 1]$ is the parameter that measure the intensity of the relapse. The positive constants μ_1, μ_2, and μ_3 satisfying

$$\mu_1 \leq \min\left\{\mu_2, \mu_3\right\}, \tag{2}$$

represent the natural death rate of susceptible, infected, and recovered individuals, respectively. Another addition in the modeling of population dynamics of diseases is the introduction of stochasticity into epidemic models. Many scholars have studied the effect of stochasticity on epidemic models [6–10]. For instance, to include stochastic demographic variability, Allen [6] studied SDEs for simple SIS and SIR epidemic models with constant population size that was derived

from a continuous time Markov chain model. In [7, 10], the situation of a white noise stochastic perturbations around the endemic equilibrium state was considered. Lahrouz et al. in [11] formulated a stochastic version of the classical SIS epidemic model with varying population size. The authors studied the long time behavior of the stochastic system. They also gave conditions for extinction and persistence of the disease in the population. According to the value of the threshold $R_\sigma = \beta\Lambda/\mu(\mu+\lambda+(1/2)\sigma^2)$, they showed that if $R_\sigma < 1$, the disease will die out from the population with the probability one and the disease will persist if $R_\sigma > 1$. In the case of persistence, they proved the existence of a stationary distribution.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all Prob-null sets). In this paper, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the death rates of S, I, and R. Specifically, $\mu_1 dt$ is replaced with $\mu_1 dt + \sigma_1 dB_1$; that is, the rate is perturbed by Gaussian white noise. The rates μ_2 and μ_3 are similarly perturbed by an independent Gaussian white noises. Therefore, the corresponding stochastic system to (1) can be described by the Itô equation:

$$\begin{aligned} dS_t &= (\Lambda - \mu_1 S_t - \beta S_t I_t)\, dt - \sigma_1 S_t dB_t^1, \\ dI_t &= (-(\mu_2+\alpha) I_t + \beta S_t I_t + \sigma\beta R_t I_t)\, dt - \sigma_2 I_t dB_t^2, \\ dR_t &= (-\mu_3 R_t + \alpha I_t - \sigma\beta R_t I_t)\, dt - \sigma_3 R_t dB_t^3, \end{aligned} \tag{3}$$

where B_t^1, B_t^2, and B_t^3 are independent Brownian motions; σ_1, σ_2, σ_3 represent the intensities of the white noises. By using the same method as in [10, 12], the existence and uniqueness of positive solution for system (3) hold with probability 1, if we start from any positive initial value (S_0, I_0, R_0). The main concern of the present paper is to establish a sufficient condition for the extinction and the persistence of solutions of the system (3).

2. Preliminaries

Throughout the rest of this paper, we denote

$$\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \mid x_i > 0,\ i = 1, 2, 3\}. \tag{4}$$

In general, consider the n–dimensional stochastic differential equation:

$$\begin{aligned} dX(t) &= f(X(t), t)\, dt + g(X(t), t)\, dB(t), \\ X(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $B(t)$ denotes a d-dimensional standard Brownian motion defined on the underlying probability space. If A is a vector or matrix, its transpose is denoted by A^T. The $n \times n$ matrix

$$\Sigma(x) \triangleq g(x)^T g(x) \tag{6}$$

is called the diffusion matrix. For the convenience of a later presentation, we introduce the generator $\mathscr{L}$ associated with (5) as follows. For any twice continuously differentiable $\mathscr{V}(x) \in \mathscr{C}^2(\mathbb{R}^n)$

$$\mathscr{L}\mathscr{V}(x) = \nabla\mathscr{V}(x) f(x) + \frac{1}{2} trace\left(\Sigma(x) \nabla^2\mathscr{V}(x)\right), \tag{7}$$

where $\nabla\mathscr{V}$, $\nabla^2\mathscr{V}$ denote the gradient, Hessian of $\mathscr{V}$, respectively.

The following theorem gives a criterion for positive recurrence in terms of Lyapunov function (see [13] Theorem 3.13, p. 1164, Theorem 4.3, p. 1168, and Theorem 4.4, p. 1169).

Theorem 1. *The system (5) is positive recurrent if there is a bounded open subset Δ of $\mathbb{R}^n$ with a regular boundary, and the following holds.*

(i) There exist some $\delta \in (0, 1]$ such that, for all $x \in \Delta$,

$$\delta |\xi|^2 \leq \xi^T \Sigma(x) \xi \leq \delta^{-1} |\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n, \tag{8}$$

(ii) There exists a nonnegative function $\mathscr{V} : \Delta^c \to \mathbb{R}$ such that $\mathscr{V}$ is twice continuously differentiable and that for some $\theta > 0$

$$\mathscr{L}\mathscr{V} \leq -\theta, \quad \text{for any } x \in \Delta^c \tag{9}$$

Moreover, the positive Markov process $X(t)$ has a unique ergodic stationary distribution π. That is, if h is a function integrable with respect to the measure π, then

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t h(X(s))\, ds = \int_{\mathbb{R}^n} h(x)\, \pi(dx)\right) = 1. \tag{10}$$

3. Extinction of the Disease

In this section, we followed the methods of Lahrouz et al. [11] to establish sufficient condition for the extinction of the disease. Before this, let us prepare two useful lemmas. In the following lemma, we show that the positive solutions to (3) have finite moments.

Lemma 2. *Let $N_t = S_t + I_t + R_t$ be the total population size in system (3) for an initial positive value (S_0, I_0, R_0). Then for any $q > 1$ such that $\mu_1 - (1/2)(q-1)\overline{\sigma}^2 > 0$, where $\overline{\sigma} = \max(\sigma_1, \sigma_2, \sigma_3)$, we have*

$$(i)\ \limsup_{t\to+\infty} \mathbb{E} N_t^q \leq \left(\frac{\Lambda}{\mu_1 - (1/2)(q-1)\overline{\sigma}^2}\right)^q, \tag{11}$$

$$(ii)\ \sup_n \mathbb{E}\left(\max_{t\in[n,n+1]} N_t^q\right) < +\infty. \tag{12}$$

Proof. (i) Let $q > 1$ such that $\mu_1 - (1/2)(q-1)\overline{\sigma}^2 > 0$ and let $\theta > 0$ where $\mu_1 - (1/2)(q-1)\overline{\sigma}^2 - \theta/q > 0$.

Applying Itô's formula leads to

$$\begin{aligned} &de^{\theta t} N_t^q \\ &\quad = \mathscr{L} e^{\theta t} N_t^q dt \\ &\qquad - q e^{\theta t} N_t^{q-1} (\sigma_1 S_t dB_1 + \sigma_2 I_t dB_2 + \sigma_3 R_t dB_3). \end{aligned} \tag{13}$$

where

$$\begin{aligned}\mathscr{L}e^{\theta t}N_t^q &= \theta e^{\theta t}N_t^q + e^{\theta t}qN_t^{q-1}\left[\Lambda-\mu_1S_t-\mu_2I_t-\mu_3R_t\right]\\ &\quad+\frac{q(q-1)}{2}e^{\theta t}N_t^{q-2}\left(\sigma_1^2S_t^2+\sigma_2^2I_t^2+\sigma_3^2R_t^2\right)\\ &\le qe^{\theta t}\left[\Lambda N_t^{q-1}-\left(\mu_1-\frac{1}{2}(q-1)\overline{\sigma}^2-\frac{\theta}{q}\right)N_t^q\right].\end{aligned} \tag{14}$$

Note that the function $\Lambda x^{q-1}-(\mu_1-(1/2)(q-1)\overline{\sigma}^2-\theta/q)x^q$ is bounded in $(0,\infty)$ with

$$\begin{aligned}&\max_{x>0}\left\{\Lambda x^{q-1}\left(\mu_1-\frac{1}{2}(q-1)\overline{\sigma}^2\right)x^q\right\}\\ &\quad=\frac{(q-1)^{q-1}\Lambda^q}{q^q\left(\mu_1-(1/2)(q-1)\overline{\sigma}^2-\theta/q\right)^{q-1}}.\end{aligned} \tag{15}$$

integrating (13); then taking expectation on both sides and using (14) and (15), we obtain

$$\begin{aligned}\mathbb{E}N_t^q &\le N_0^qe^{-\theta t}\\ &\quad+\frac{(q-1)^{q-1}\Lambda^q}{q^{q-1}\theta\left(\mu_1-(1/2)(q-1)\overline{\sigma}^2-\theta/q\right)^{q-1}}\left(1\right.\\ &\quad\left.-e^{-\theta t}\right).\end{aligned} \tag{16}$$

Therefore, for all $\theta\in(0,q(\mu_1-(1/2)(q-1)\overline{\sigma}^2))$, we have

$$\begin{aligned}&\limsup_{t\to+\infty}\mathbb{E}N_t^q\\ &\quad\le\frac{(q-1)^{q-1}\Lambda^q}{q^{q-1}\theta\left(\mu_1-(1/2)(q-1)\overline{\sigma}^2-\theta/q\right)^{q-1}}\end{aligned} \tag{17}$$

and for the fact that

$$\begin{aligned}&\min_{0<\theta<q(\mu_1-(1/2)(q-1)\overline{\sigma}^2)}\frac{(q-1)^{q-1}\Lambda^q}{q^{q-1}\theta\left(\mu_1-(1/2)(q-1)\overline{\sigma}^2-\theta/k\right)^{q-1}}\\ &\quad=\left(\frac{\Lambda}{\mu_1-(1/2)(q-1)\overline{\sigma}^2}\right)^q.\end{aligned} \tag{18}$$

we have (i).

(ii) From (13) and (14) we have, for all $t\in[n,n+1]$,

$$\begin{aligned}N_t^q &\le e^{(n-t)\theta}N_n^q+\int_n^t e^{(s-t)\theta}\left(\Lambda N_s^{q-1}\right.\\ &\quad\left.-\left(\mu_1-\frac{1}{2}(q-1)\overline{\sigma}^2-\frac{\theta}{q}\right)N_s^q\right)ds\\ &\quad-\int_n^t qe^{\theta(s-t)}N_s^{q-1}\left(\sigma_1S_sdB_1(s)+\sigma_2I_sdB_2(s)\right.\\ &\quad\left.+\sigma_3R_sdB_3(s)\right)ds\le N_n^q+\int_n^t\left|\Lambda N_s^{q-1}\right.\\ &\quad\left.-\left(\mu_1-\frac{1}{2}(q-1)\overline{\sigma}^2-\frac{\theta}{q}\right)N_s^q\right|ds\\ &\quad-\int_n^t qe^{\theta(t-s)}N_s^{q-1}\left(\sigma_1S_sdB_1(s)+\sigma_2I_sdB_2(s)\right.\\ &\quad\left.+\sigma_3R_sdB_3(s)\right)ds.\end{aligned} \tag{19}$$

so

$$\begin{aligned}&\max_{t\in[n,n+1]}N_t^q\\ &\quad\le N_n^q\\ &\quad+\int_n^{n+1}\left|\Lambda N_s^{q-1}-\left(\mu_1-\frac{1}{2}(q-1)\overline{\sigma}^2-\frac{\theta}{q}\right)N_s^q\right|ds\\ &\quad+\max_{t\in[n,n+1]}M_t,\end{aligned} \tag{20}$$

where

$$\begin{aligned}M_t &= -\int_n^t qe^{\theta(t-s)}N_s^{q-1}\left(\sigma_1S_sdB_1(s)+\sigma_2I_sdB_2(s)\right.\\ &\quad\left.+\sigma_3R_sdB_3(s)\right)ds\end{aligned} \tag{21}$$

In view of (11) and the continuity of N_t, there exists $C>0$ such that

$$\sup_{t\ge0}\mathbb{E}N_t^q\le C. \tag{22}$$

So, there exists a positive constant C_1 such that

$$\mathbb{E}\left(\max_{t\in[n,n+1]}N_t^q\right)\le C_1+\mathbb{E}\left(\max_{t\in[n,n+1]}M_t\right) \tag{23}$$

Applying Itô's formula leads to

$$\begin{aligned}dN_t^q &= qN_t^{q-1}\left(\Lambda-\mu_1S_t-\mu_2I_t-\mu_3R_t\right)dt\\ &\quad+\frac{q(q-1)}{2}N_t^{q-2}\left(\sigma_1^2S_t^2+\sigma_2^2I_t^2+\sigma_3^2R_t^2\right)dt\\ &\quad-qN_t^{q-1}\left(\sigma_1S_tdB_1(t)+\sigma_2I_tdB_2(t)+\sigma_3R_tdB_3(t)\right).\end{aligned} \tag{24}$$

From

$$\begin{aligned}&-\int_n^t qN_s^{q-1}\left(\sigma_1S_sdB_1(s)+\sigma_2I_sdB_2(s)\right.\\ &\quad\left.+\sigma_3R_sdB_3(s)\right)ds=N_t^q-N_n^q-\int_n^t qN_s^{q-1}\left(\Lambda\right.\\ &\quad\left.-\mu_1S_s-\mu_2I_s-\mu_3R_s\right)ds\\ &\quad-\int_n^t\frac{q(q-1)}{2}N_s^{q-2}\left(\sigma_1^2S_s^2+\sigma_2^2I_s^2+\sigma_3^2R_s^2\right)ds,\end{aligned} \tag{25}$$

we have

$$\begin{aligned}|M_t| \le \int_n^t \Big| & qN_s^{q-1}\left(\sigma_1 S_s dB_1(s) + \sigma_2 I_s dB_2(s)\right. \\ & \left. + \sigma_3 R_s dB_3(s)\right)\Big| ds \le N_t^q + N_n^q \\ & + \int_n^{n+1} \left|qN_s^{q-1}\left(\Lambda - \mu_1 S_s - \mu_2 I_s - \mu_3 R_s\right)\right| ds \\ & + \int_n^{n+1} \left|\frac{q(q-1)}{2} N_s^{q-2}\left(\sigma_1^2 S_s^2 + \sigma_2^2 I_s^2\right.\right. \\ & \left.\left. + \sigma_3^2 R_s^2\right)\right| ds.\end{aligned} \tag{26}$$

So, there exists a positive constant C_2 such that

$$|M_t| \le N_t^q + N_n^q + C_2 \int_n^{n+1} \left[N_s^{q-1} + N_s^q\right] ds. \tag{27}$$

Letting $q' > 1$ we have

$$\begin{aligned}\mathbb{E}\left(\max_{t\in[n,n+1]} |M_t|^{q'}\right) &\le \left(\frac{q'}{q'-1}\right)^{q'} \max_{t\in[n,n+1]} \mathbb{E}|M_t|^{q'} \\ &\le C_3 \mathbb{E}\left(N_t^{qq'} + N_n^{qq'}\right. \\ &\quad \left. + \int_n^{n+1} \left[N_s^{(q-1)q'} + N_s^{(q-1)q'}\right] ds\right),\end{aligned} \tag{28}$$

where C_3 is a positive constant independent of n, and the first inequality is derived from the maximal inequality for martingales and the second by (27) and Jensen's inequality.

Choosing $q' = 1 + \gamma$ where $0 < \gamma < 2\mu_1/\sigma_1^2$ we have

$$\begin{gathered} q' > 1, \\ qq' > 1 \\ \text{and } \mu_1 - \frac{1}{2}\left(qq' - 1\right)\overline{\sigma}^2 > 0. \end{gathered} \tag{29}$$

So, by (22) we get $\mathbb{E}(\max_{t\in[n,n+1]}|M_t|^{q'}) < +\infty$, and by (23) we have

$$\begin{aligned}\mathbb{E}\left(\max_{t\in[n,n+1]} N_t^q\right) &\le C_1 + \left(\mathbb{E}\left(\max_{t\in[n,n+1]} |M_t|^{q'}\right)\right)^{q'} \\ &< +\infty.\end{aligned} \tag{30}$$

Under the conditions of the preceding lemma we have the following lemma. □

Lemma 3. *Let (S_t, I_t, R_t) be the solution of (3) with the positive initial condition (S_0, I_0, R_0).*

Assume that $(1/2)\overline{\sigma}^2 < \mu_1$. Then

$$\begin{aligned} &(i)\ \lim_{t\to\infty} \frac{N_t}{t} = 0. \\ &(ii)\ \lim_{t\to\infty} \frac{1}{t}\int_0^t S_u dB(u) = \lim_{t\to\infty} \frac{1}{t}\int_0^t I_u dB(u) \\ &\qquad = \lim_{t\to\infty} \frac{1}{t}\int_0^t R_u dB(u) = 0. \end{aligned} \tag{31}$$

Proof. (i) In view of (12) and Markov's inequality there exists a positive constant C such that

$$\begin{aligned}\mathbb{P}\left(\sup_{n\le t\le n+1} \frac{N_t}{t} > n^{-q'}\right) &= \mathbb{P}\left(\sup_{n\le t\le n+1} \frac{N_t^q}{t^q} > n^{-qq'}\right) \\ &\le \frac{C}{n^{q-qq'}},\end{aligned} \tag{32}$$

where q verifies the conditions of Lemma 3 and q' is a positive constant such that $q' < 1 - 1/q$.

An application of Borel-Cantelli lemma yields to for almost $\omega \in \Omega$ and there is a random integer $n_0(\omega)$ such that for $n \ge n_0$

$$\sup_{n\le t\le n+1} \frac{N_t}{t} \le n^{-q'}, \tag{33}$$

hence for all $t \in [n, n+1]$ and $n > n_0$

$$\frac{N_t}{t} \le n^{-q'}. \tag{34}$$

Let $n \to \infty$, then $t \to \infty$ so

$$\lim_{\infty} \frac{N_t}{t} = 0. \tag{35}$$

(ii) We shall prove that $\lim_{t\to\infty}(1/t)\int_0^t S_u dB(u) = 0$, and the other limits can be obtained in the same way. Put $X_t = \int_0^t S_u dB(u)$. In view of the assumption $(1/2)\overline{\sigma}^2 < \mu_1$ and (11), one can derive that there exists $C > 0$ such that

$$\sup_{t\ge 0} \mathbb{E}\left(N_t^2\right) \le C. \tag{36}$$

Therefore

$$\mathbb{E}X_t^2 = \mathbb{E}\left(\int_0^t S_u^2 du\right) \le Ct. \tag{37}$$

Beside, let δ_1, δ_2 such that $\delta_1 > 2$ and $\delta_1 + 2 \le \delta_2 < 2\delta_1$. Thanks to Doob's martingale inequality, and using (37), we get

$$\mathbb{P}\left(\sup_{(k-1)^{\delta_1}\le t\le k^{\delta_1}} X_t^2 \ge k^{\delta_2}\right) \le \frac{\mathbb{E}\left(X_{k^{\delta_1}}^2\right)}{k^{\delta_2}} \le \frac{C}{k^{\delta_2-\delta 1}}, \tag{38}$$

Since $\Sigma(1/(k^{\delta_2-\delta_1})) < \infty$, the Borel-cantelli lemma implies that for almost $\omega \in \Omega$, there is $k_0(\omega)$ such that for $k \ge k_0$

$$\sup_{(k-1)^{\delta_1}\le t\le k^{\delta_1}} X_t^2 < k^{\delta_2}. \tag{39}$$

Hence, for all $t \in [(k-1)^{\delta_1}, k^{\delta_1}]$ and $k \geq k_0$

$$\frac{X^2(t)}{t^2} \leq \frac{k^{\delta_2}}{(k-1)^{2\delta_1}}. \tag{40}$$

Since $2\delta_1 > \delta_2$, we have $k^{\delta_2}/(k-1)^{2\delta_1} \sim 1/k^{2\delta_1-\delta_2}$ and $\lim_{k\to\infty}(1/k^{2\delta_1-\delta_2}) = 0$ leads to

$$\lim_{t\to\infty} \frac{X^2(t)}{t^2} = 0. \tag{41}$$

That is

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S_u dB(u) = 0 \tag{42}$$

Now, we are in the position to establish the threshold criterion for disease extinction in terms of the positive number:

$$\mathscr{R}_T \triangleq \frac{\Lambda\beta}{\mu_1(\alpha+\mu_2+\sigma_2^2/2)}. \tag{43}$$

□

Theorem 4. *Let $(S(t), I(t), R(t))$ be the solution of system (3) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. if $(1/2)\overline{\sigma}^2 < \mu_1$ and $\mathscr{R}_T < 1$, then we have the following property:*

$$\lim_{t\to\infty} I_t = \lim_{t\to\infty} R_t = 0. \tag{44}$$

Proof. Applying Itô's formula leads to

$$\begin{aligned} d\log I_t &= \left[-\left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)+\beta S_t+\sigma\beta R_t\right]dt \\ &\quad -\sigma_2 dB_2 \\ &\leq -\left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)+\beta(S_t+R_t)-\sigma_2 dB_2 \\ &\leq -\left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)+\beta N_t-\sigma_2 dB_2, \end{aligned} \tag{45}$$

and

$$\begin{aligned} dN_t &= [\Lambda-\mu_1 S_t-\mu_2 I_t-\mu_3 R_t]\,dt-\sigma_1 S_t dB_1 \\ &\quad -\sigma_2 I_t dB_2-\sigma_3 R_t dB_3 \\ &\leq (\Lambda-\mu_1)N_t-\sigma_1 S_t dB_1-\sigma_2 I_t dB_2-\sigma_3 R_t dB_3, \end{aligned} \tag{46}$$

so

$$\begin{aligned} d\left(\log I_t+\frac{\beta}{\mu_1}N_t\right) &\leq \frac{\beta\Lambda}{\mu_1}-\left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)-\sigma_2 dB_2 \\ &\quad -\sigma_1 S_t dB_1-\sigma_2 I_t dB_2 \\ &\quad -\sigma_3 R_t dB_3. \end{aligned} \tag{47}$$

Integrating (47) and dividing by $t > 0$ we obtain

$$\begin{aligned} &\frac{\log I_t}{t}-\frac{\log I_0}{t}+\frac{\beta}{\mu_1}\frac{N_t-N_0}{t} \\ &\quad \leq \left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)(\mathscr{R}_T-1)-\sigma_2 B_2 \\ &\quad -\frac{\sigma_1}{t}\int_0^t S_u dB_1(u)-\frac{\sigma_2}{t}\int_0^t I_u dB_2(u) \\ &\quad -\frac{\sigma_3}{t}\int_0^t R_u dB_3(u), \end{aligned} \tag{48}$$

and using Lemma 3 yields

$$\limsup_{t\to\infty} \frac{\log I_t}{t} \leq \left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)(\mathscr{R}_T-1). \tag{49}$$

Hence by condition $\mathscr{R}_T < 1$ we have $\lim_{t\to\infty} I_t = 0$ a.s.

(*ii*) Let

$$\psi_t = \exp\left(\left(\mu_3+\frac{\sigma_3^2}{2}\right)t+\sigma_3 B_3\right). \tag{50}$$

We know that $\lim_{t\to\infty}(B_3/t) = 0$. Then

$$\lim_{t\to\infty} \frac{\log\psi_t}{t} = \mu_3+\frac{\sigma_3^2}{2} \triangleq h'. \tag{51}$$

From the fact that

$$\limsup_{t\to\infty} \frac{\log I_t}{t} \leq \left(\mu_2+\alpha+\frac{\sigma_2^2}{2}\right)(\mathscr{R}_T-1) \triangleq h, \tag{52}$$

we deduce that for any $\epsilon > 0$ and for almost all $\omega \in \Omega$ there exists $T(\omega)$ such that for all $t \geq T$

$$\begin{gathered} I_t \leq K_1 e^{(h+\epsilon)t} \\ \text{and } K_2 e^{(h'-\epsilon)t} \leq \psi_t \leq K_3 e^{(h'+\epsilon)t} \end{gathered} \tag{53}$$

where K_i, $i = 1, 2, 3$ are almost surely finite positive random variables. By (53) we have

$$dR_t \leq \left(-\mu_3 R_t+\alpha K_1 e^{(h+\epsilon)t}\right)dt-\sigma_3 R_t dB_3. \tag{54}$$

Let ϕ be the solution of stochastic equation:

$$d\phi_t = \left(-\mu_3\phi_t+\alpha K_1 e^{(h+\epsilon)t}\right)dt-\sigma_3\phi_t dB_3, \tag{55}$$

and we can write

$$\phi_t = \phi_0\psi_t^{-1}+\alpha K_1\phi_t^{-1}\int_0^t \psi_s e^{(h+\epsilon)s}ds. \tag{56}$$

So from (53) we have

$$\begin{aligned}\phi_t \le{}& \frac{\phi_0}{K_2}e^{-(h'-\epsilon)t} + \frac{\alpha}{K_2}e^{-(h'-\epsilon)t}\int_0^T \psi_s e^{(h+\epsilon)s}ds \\ &+ \frac{\alpha K_1 K_3}{K_2}e^{-(h'-\epsilon)}\int_T^t e^{(h'+\epsilon)s}e^{(h+\epsilon)s}ds = \left(\frac{\phi_0}{K_2}\right. \\ &+ \frac{\alpha}{K_2}\int_0^T \psi_s e^{(h+\epsilon)s}ds \\ &\left.- \frac{\alpha K_1 K_3}{K_2(h+h'+2\epsilon)}e^{(h+h'+2\epsilon)T}\right)e^{-(h'-\epsilon)t} \\ &+ \frac{\alpha K_1 K_3}{K_2(h+h'+2\epsilon)}e^{(h+h'+2\epsilon)t}e^{(h+3\epsilon)t}.\end{aligned} \tag{57}$$

By comparison theorem for stochastic differential equations we have

$$\begin{aligned}R_t \le{}& \left(\frac{\phi_0}{K_2} + \frac{\alpha}{K_2}\int_0^T \psi_s e^{(h+\epsilon)s}ds \right. \\ &\left.- \frac{\alpha K_1 K_3}{K_2(h+h'+2\epsilon)}e^{(h+h'+2\epsilon)T}\right)e^{-(h'-\epsilon)t} \\ &+ \frac{\alpha K_1 K_3}{K_2(h+h'+2\epsilon)}e^{(h+h'+2\epsilon)t}e^{(h+3\epsilon)t}.\end{aligned} \tag{58}$$

Since ϵ is arbitrary, we get

$$\begin{aligned}R_t \le{}& \left(\frac{\phi_0}{K_2} + \frac{\alpha}{K_2}\int_0^T \psi_s e^{hs}ds - \frac{\alpha K_1 K_3}{K_2(h+h')}e^{(h+h')T}\right) \\ &\cdot e^{-h't} + \frac{\alpha K_1 K_3}{K_2(h+h')}e^{(h+h')t}e^{ht}.\end{aligned} \tag{59}$$

From $h' > 0$, $R_T < 1$ and (59) we obtain $\lim_{t\to\infty} R_t = 0$. □

4. Stationary Distribution and Positive Recurrence

In many papers, the existence of stationary distribution needs the construction of suitable Lyapunov functions that are based on the positive equilibrium state of the deterministic system, which gives strong sufficient conditions [14]. In the following theorem, to investigate the existence of an asymptotically invariant distribution for the solution of model (3), we did not use equilibrium state to construct Lyapunov functions.

Theorem 5. *The solutions (S_t, I_t, R_t) are positive recurrent and admit a unique ergodic stationary distribution, provided that the following conditions hold:*

$$\mathscr{R}_T > 1 \tag{60}$$

and

$$\mu_i > \frac{\sigma_i^2}{2}, \quad (i = 1, 2, 3) \tag{61}$$

Proof. We construct a positive function $\mathscr{W}$ such that

$$\sup_{(S,I,R)\in\mathbb{R}_+^3\setminus\Delta_\rho} \mathscr{L}\mathscr{W}(S, I, R) \le -1 \tag{62}$$

for some $\rho > 0$, where $\Delta_\rho = (1/\rho, \rho) \times (1/\rho, \rho) \times (1/\rho, \rho)$.

First of all we have

$$\mathscr{L}(-\log S) = -\frac{\Lambda}{S} + \beta I + \mu_1 + \frac{\sigma_1^2}{2}. \tag{63}$$

Second, we have

$$\begin{aligned}\mathscr{L}\left(\frac{1}{2}N^2\right) &= N(\Lambda - \mu_1 S + \mu_2 I + \mu_3 R) + \frac{\sigma_1^2}{2}S^2 \\ &\quad + \frac{\sigma_2^2}{2}I^2 + \frac{\sigma_3^2}{2}R^2 \\ &\le \Lambda N - \left(\mu_1 - \frac{\sigma_1^2}{2}\right)S^2 - \left(\mu_1 - \frac{\sigma_2^2}{2}\right)I^2 \\ &\quad - \left(\mu_1 - \frac{\sigma_3^2}{2}\right)R^2.\end{aligned} \tag{64}$$

By using the elementary inequality $S \le \epsilon S^2 + 1/4\epsilon^2$, where ϵ is a positive constant to be choosen later, we have

$$\begin{aligned}\mathscr{L}\left(\frac{1}{2}N^2\right) \le{}& \frac{3\Lambda}{4\epsilon} - \left(\mu_1 - \frac{\sigma_1^2}{2} - \epsilon\Lambda\right)S^2 \\ &- \left(\mu_2 - \frac{\sigma_2^2}{2} - \epsilon\Lambda\right)I^2 \\ &- \left(\mu_3 - \frac{\sigma_3^2}{2} - \epsilon\Lambda\right)R^2.\end{aligned} \tag{65}$$

Now we compute

$$\begin{aligned}\mathscr{L}(S\log I) &= \Lambda\log I - \mu_1 S\log I - \beta SI\log I \\ &\quad - S(\mu_2 + \alpha) + \sigma\beta RS + \beta S^2 - \frac{\sigma_2^2}{2}S.\end{aligned} \tag{66}$$

By using the elementary inequalities $-1/e \le I\log I \le I^2$ and $SR \le (1/2)(S^2 + R^2)$ we have

$$\begin{aligned}\mathscr{L}(S\log I) \le{}& \Lambda\log I - \mu_1 S\log I \\ &+ \left(\frac{\beta}{e} - \mu_2 - \alpha - \frac{\sigma_2^2}{2}\right)S + \frac{\sigma\beta}{2}R^2 \\ &+ \left(\frac{\sigma\beta}{2} + \beta\right)S^2.\end{aligned} \tag{67}$$

Next, we compute

$$\begin{aligned}\mathscr{L}\frac{1}{2}(\log I)^2 &= \log I\left(-(\mu_2+\alpha)+\beta S+\sigma\beta R\right)\\ &\quad+\frac{\sigma_2^2}{2}(1-\log I)\\ &= -\left(\frac{\sigma_2^2}{2}+\mu_2+\alpha\right)\log I+\beta S\log I\\ &\quad+\sigma\beta R\log I+\frac{\sigma_2^2}{2}\end{aligned}\tag{68}$$

and

$$\mathscr{L}(-\log R)=\mu_3-\alpha\frac{I}{R}+\sigma\beta I+\frac{\sigma_3^2}{2}\tag{69}$$

and

$$\begin{aligned}\mathscr{L}(R\log I) &= \log I\left(-\mu_3 R+\alpha I-\sigma\beta IR\right)\\ &\quad+R\left(-(\mu_2+\alpha)+\beta S+\sigma\beta R\right)-\frac{\sigma_2^2}{2}R\\ &\le -\mu_3 R\log I+\alpha I^2+\left(\frac{\sigma\beta}{e}-\frac{\sigma_2^2}{2}\right)R\\ &\quad+\frac{\beta}{2}S^2+\left(\frac{\beta}{2}+\sigma\beta\right)R^2.\end{aligned}\tag{70}$$

Now let

$$\mathscr{W}_1(S,I,R)=-\log S-\log R+\frac{1}{4}(\log I)^2+\frac{\overline{\omega}}{4}N^2\tag{71}$$

and

$$\begin{aligned}\mathscr{W}_2(S,I,R) &= \frac{\overline{\omega}}{4}N^2+\frac{\beta}{\mu_1}S\log I+\frac{\sigma\beta}{\mu_3}R\log I\\ &\quad+\frac{1}{4}(\log I)^2,\end{aligned}\tag{72}$$

where $\overline{\omega}$ is a positive constant. It is easy to check that $\mathscr{W}_1(S,I,R)$ has a minimum point $(\widehat{S},\widehat{I},\widehat{R})$ in $\mathbb{R}_+^3$, and by choosing $\overline{\omega}$ sufficiently large we can assure the positivity of $\mathscr{W}_2(S,I,R)$.

Finally consider the positive function:

$$\mathscr{W}(S,I,R)=\mathscr{W}_1(S,I,R)-\mathscr{W}_1\left(\widehat{S},\widehat{I},\widehat{R}\right)+\mathscr{W}_2.\tag{73}$$

Combining (63), (65), (67), (68), (69), and (70) we get

$$\begin{aligned}\mathscr{L}\mathscr{W}(S,I,R) &\le -\frac{\Lambda}{S}+\beta I+\mu_1+\frac{\sigma_1^2}{2}+\mu_3-\alpha\frac{I}{R}\\ &\quad+\sigma\beta I+\frac{\sigma_3^2}{2}+\overline{\omega}\frac{3\Lambda}{4\epsilon}\\ &\quad-\overline{\omega}\left(\mu_1-\frac{\sigma_1^2}{2}-\epsilon\Lambda\right)S^2\\ &\quad-\overline{\omega}\left(\mu_2-\frac{\sigma_2^2}{2}-\epsilon\Lambda\right)I^2\\ &\quad-\overline{\omega}\left(\mu_3-\frac{\sigma_3^2}{2}-\epsilon\Lambda\right)R^2+\frac{\beta\Lambda}{\mu_1}\log I\\ &\quad-\beta S\log I\\ &\quad+\frac{\beta}{\mu_1}\left(\frac{\beta}{e}-\mu_2-\alpha-\frac{\sigma_2^2}{2}\right)S\\ &\quad+\frac{\sigma\beta^2}{2\mu_1}R^2+\frac{\beta}{\mu_1}\left(\frac{\sigma\beta}{2}+\beta\right)S^2\\ &\quad-\sigma\beta R\log I+\frac{\sigma\beta\alpha}{\mu_3}I^2\\ &\quad+\left(\frac{\sigma^2\beta^2}{e\mu_3}-\frac{\sigma\beta\sigma_2^2}{2\mu_3}\right)R+\frac{\sigma\beta^2}{2\mu_3}S^2\\ &\quad+\frac{\sigma\beta}{\mu_3}\left(\frac{\beta}{2}+\sigma\beta\right)R^2\\ &\quad-\left(\frac{\sigma_2^2}{2}+\mu_2+\alpha\right)\log I+\beta S\log I\\ &\quad+\sigma\beta R\log I+\frac{\sigma_2^2}{2}.\end{aligned}\tag{74}$$

Simplification and reorganization of the above inequality give

$$\begin{aligned}\mathscr{L}\mathscr{W}(S,I,R) &\le -\frac{\Lambda}{S}\\ &\quad-\left(\overline{\omega}\left(\mu_1-\frac{\sigma_1^2}{2}-\epsilon\Lambda\right)-\frac{\beta}{\mu_1}\left(\frac{\sigma\beta}{2}+\beta\right)-\frac{\sigma\beta^2}{2\mu_3}\right)S^2\\ &\quad+\frac{\beta}{\mu_1}\left(\frac{\beta}{e}-\mu_2-\alpha-\frac{\sigma_2^2}{2}\right)S\\ &\quad+\left(\frac{\beta\Lambda}{\mu_1}-\frac{\sigma_2^2}{2}-\mu_2-\alpha\right)\log I+(\beta+\sigma\beta)I\\ &\quad-\left(\overline{\omega}\left(\mu_2-\frac{\sigma_2^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta\alpha}{\mu_3}\right)I^2\\ &\quad+\left(\frac{\sigma^2\beta^2}{e\mu_3}-\frac{\sigma\beta\sigma_2^2}{2\mu_3}\right)R\\ &\quad-\left(\overline{\omega}\left(\mu_3-\frac{\sigma_3^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta}{\mu_3}\left(\frac{\beta}{2}+\sigma\beta\right)\right)R^2-\alpha\\ &\quad\cdot\frac{I}{R}+\overline{\omega}\frac{3\Lambda}{4\epsilon}+\mu_1+\mu_3+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}\triangleq\Upsilon_1(S)\\ &\quad+\Upsilon_2(I)+\Upsilon_3(R)-\alpha\frac{I}{R}+C\triangleq\Upsilon(S,I,R),\end{aligned}\tag{75}$$

where

$$\begin{aligned}\Upsilon_1(S) &= -\frac{\Lambda}{{}_tS} \\ &\quad -\left(\overline{\omega}\left(\mu_1-\frac{\sigma_1^2}{2}-\epsilon\Lambda\right)-\frac{\beta}{\mu_1}\left(\frac{\sigma\beta}{2}+\beta\right)-\frac{\sigma\beta^2}{2\mu_3}\right)S^2 \\ &\quad +\frac{\beta}{\mu_1}\left(\frac{\beta}{e}-\mu_2-\alpha-\frac{\sigma_2^2}{2}\right)S, \\ \Upsilon_2(I) &= \left(\frac{\beta\Lambda}{\mu_1}-\frac{\sigma_2^2}{2}-\mu_2-\alpha\right)\log I+(\beta+\sigma\beta)I \\ &\quad -\left(\overline{\omega}\left(\mu_2-\frac{\sigma_2^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta\alpha}{\mu_3}\right)I^2, \\ \Upsilon_3(R) &= \left(\frac{\sigma^2\beta^2}{e\mu_3}-\frac{\sigma\beta\sigma_2^2}{2\mu_3}\right)R \\ &\quad -\left(\overline{\omega}\left(\mu_3-\frac{\sigma_3^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta}{\mu_3}\left(\frac{\beta}{2}+\sigma\beta\right)\right)R^2 \\ \text{and } C &= \overline{\omega}\frac{3\Lambda}{4\epsilon}+\mu_1+\mu_3+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}\end{aligned} \tag{76}$$

Since (61), then we can choose $\epsilon > 0$ such that

$$\mu_i-\frac{\sigma_i^2}{2}-\epsilon\Lambda>0, \quad (i=1,2,3), \tag{77}$$

so, for sufficiently large $\overline{\omega}$, we have

$$\begin{gathered}\left(\overline{\omega}\left(\mu_1-\frac{\sigma_1^2}{2}-\epsilon\Lambda\right)-\frac{\beta}{\mu_1}\left(\frac{\sigma\beta}{2}+\beta\right)-\frac{\sigma\beta^2}{2\mu_3}\right)>0, \\ \left(\overline{\omega}\left(\mu_2-\frac{\sigma_2^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta\alpha}{\mu_3}\right)>0\end{gathered} \tag{78}$$

and

$$\left(\overline{\omega}\left(\mu_3-\frac{\sigma_3^2}{2}-\epsilon\Lambda\right)-\frac{\sigma\beta}{\mu_3}\left(\frac{\beta}{2}+\sigma\beta\right)\right)>0. \tag{79}$$

From (75) we have $\Upsilon(S,I,R)\le\Upsilon_1(S)+\Upsilon_2(I)+\Upsilon_3(R)+C$ and by (79) we get

$$\begin{aligned}\lim_{S\to\infty}\Upsilon(S,I,R) &= \lim_{I\to\infty}\Upsilon(S,I,R)=\lim_{R\to\infty}\Upsilon(S,I,R) \\ &= -\infty,\end{aligned} \tag{80}$$

and from condition (60) which is equivalent to $\beta\Lambda/\mu_1-\sigma_2^2/2-\mu_2-\alpha>0$ we have

$$\lim_{S\to 0}\Upsilon(S,I,R)=\lim_{I\to 0}\Upsilon(S,I,R)=-\infty. \tag{81}$$

In the following, we will prove (62).

Firstly, since $\lim_{S\to\infty}\Upsilon_1(S)=\lim_{S\to 0}\Upsilon_1(S)=-\infty$ and continuity of $\Upsilon_1(S)$, $\Upsilon_1(S)$ has a maximum point in $\mathbb{R}_+^3$. So, let

$$M_1=\sup_{S\in\mathbb{R}_+^3}\Upsilon_1(S). \tag{82}$$

Similarly let

$$\begin{aligned}M_2 &= \sup_{I\in\mathbb{R}_+^3}\Upsilon_2(I) \\ \text{and } M_3 &= \sup_{R\in\mathbb{R}_+^3}\Upsilon_3(R)\end{aligned} \tag{83}$$

Case 1 ($1/\rho\le I\le\rho$). From (75), we have $\Upsilon(S,I,R)\le M_1+M_2+M_3+C-1/\rho R$, hence

$$\lim_{R\to 0}\Upsilon(S,I,R)=-\infty \tag{84}$$

Case 2 ($I>\rho$). From (75), we have $\Upsilon(S,I,R)\le M_1+M_2+C+\Upsilon_3(R)-\rho/R$, hence

$$\lim_{R\to 0}\Upsilon(S,I,R)=-\infty \tag{85}$$

Case 3 ($I<1/\rho$). From (75), we have $\Upsilon(S,I,R)\le M_1+M_3+C+\Upsilon_2(I)$ and since $\lim_{I\to 0}\Upsilon_2(I)=-\infty$, then we can choose ρ sufficiently large such that $\Upsilon_2(I)<-(M_1+M_3+C)-2$. Hence for a sufficiently large ρ we have

$$\Upsilon(S,I,R)\le -2. \tag{86}$$

Thereby and from (80) and (81) we deduce that, for a sufficiently large ρ,

$$\mathscr{LW}(S,I,R)\le -1 \quad \forall(S,I,R)\in\mathbb{R}_+^3\setminus\Delta_\rho, \tag{87}$$

So the condition (ii) of Theorem 1 holds.

The diffusion matrix associated with the system (3) is given by

$$\Sigma(S,I,R)=\begin{bmatrix}\sigma_1^2S^2 & 0 & 0\\ 0 & \sigma_2^2I^2 & 0\\ 0 & 0 & \sigma_3^2R^2\end{bmatrix} \tag{88}$$

It is easy to see that $\Sigma(S,I,R)$ satisfy the uniform ellipticity condition (i) of Theorem 1. □

5. Conclusion and Numerical Simulation

In the current paper, a stochastic SIR epidemic model with nonlinear relapse rate is considered to model the influence of infected individuals on the recovered ones. The threshold $\mathscr{R}_T$ which determines the dynamical behavior of the system (3) is found. Precisely, if $\mathscr{R}_T<1$ the disease will die out from the population with the probability one, while $\mathscr{R}_T>1$ leads to the persistence of the disease with a unique positive stationary distribution. We remark that $\mathscr{R}_T$ is independent of the relapse coefficient σ. Hence, relapse under the pressure of the infected individuals has no influence on the dynamics of the stochastic model (3). However, one can remark from numerical simulation that the nonlinear relapse phenomenon plays a crucial role in the speed of the extinction and the spread of the disease in the population. Indeed, in Figure 1, we show that disease dies out quickly as long as σ is small enough. Furthermore, in Figure 2, we observe that, for higher coefficient σ, the modes of the invariant stationary distributions of infected individuals become larger.

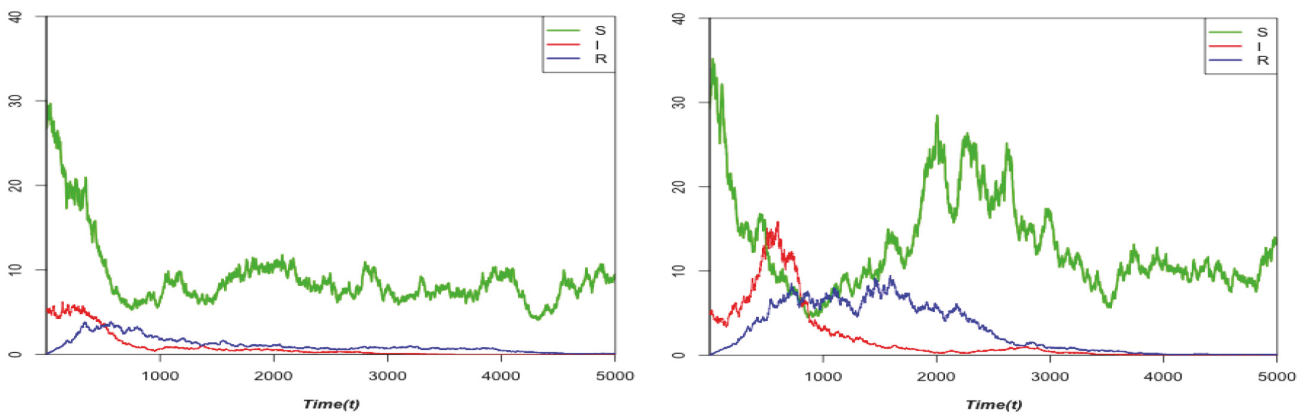

Figure 1: The numerical simulation of one path of the solution (S_t, I_t, R_t) of the system (3) using the Milstein scheme with ($\sigma = 0.035$ left), ($\sigma = 0.35$ right) and the parameters values $\Lambda = 1$, $\beta = 0.01$, $\mu_1 = 0.066$, $\mu_2 = 0.066$, $\mu_3 = 0.0666$, $\alpha = 0.16$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, and $\sigma_3 = 0.21$. In this situation $\mathscr{R}_T = 0.59 < 1$. Hence, the disease will die out.

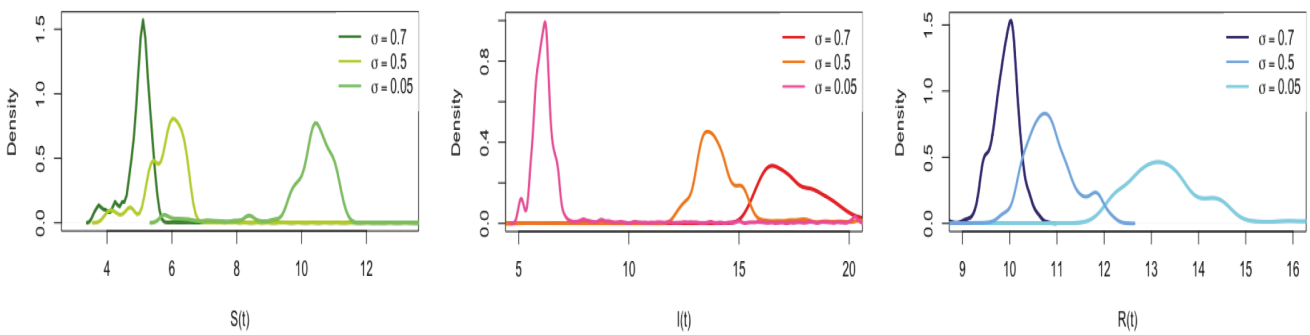

Figure 2: Computer simulation of the density function of the invariant stationary distribution of system (3) with different values of σ and the parameters values $\Lambda = 2$, $\beta = 0.02$, $\mu_1 = 0.066$, $\mu_2 = 0.066$, $\mu_3 = 0.0666$, $\alpha = 0.16$, $\sigma_1 = 0.021$, $\sigma_2 = 0.02$, and $\sigma_3 = 0.023$. In this situation $\mathscr{R}_T = 2,66 > 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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