Research Article

Investigating a Coupled Hybrid System of Nonlinear Fractional Differential Equations

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We study sufficient conditions for existence of solutions to the coupled systems of higher order hybrid fractional differential equations with three-point boundary conditions. For this motive, we apply the coupled fixed point theorem of Krasnoselskii type to form adequate conditions for existence of solutions to the proposed system. We finish the paper with suitable illustrative example.

1. Introduction

Fractional calculus is found to be more practical and effective than the classical calculus in the mathematical modeling of several phenomena. Fractional differential equations are very important and significant part of the mathematics and have various applications in viscoelasticity, electroanalytical chemistry, and many physical problems [1–6]. A systematic presentation of the applications of fractional differential equations can be found in the book of Balachandran and Park [7]. In recent years, many works have been devoted to the study of the mathematical aspects of fractional order differential equations [8–12]. There are numerous advanced and efficient methods, which have been focusing on the existence of solution to fractional differential equations. One of the powerful tools for obtaining the existence of solutions to such equations is the fixed point methods. Many authors use fixed point theorems to prove the existence and uniqueness of solution to nonlinear fractional differential equations; see, for example, [13–17].

On the other hand, the study for coupled systems of fractional differential equations is also important as such systems occur in various problems of applied nature, for instance, [18–25]. Additionally, fixed point theory can be used to develop the existence theory for the coupled systems of

fractional hybrid differential equations [13, 16, 17]. Bashiri et al. [17] discussed the existence of solution to the following system of fractional hybrid differential equations of order $p \in (0, 1)$:

$$\begin{aligned} D^p [x(t) - f(t, x(t))] &= g(t, y(t), I^\alpha y(t)), \\ \text{a.e. } t &\in [0, T], \quad T > 0, \\ D^p [y(t) - f(t, y(t))] &= g(t, x(t), I^\alpha x(t)), \\ \text{a.e. } t &\in [0, T], \quad T > 0, \\ x(0) &= 0, \\ y(0) &= 0, \end{aligned} \quad (1)$$

where $\alpha > 0$, and the functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $f(0, 0) = 0$ and $g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy certain conditions. D^p is the R-L fractional derivative of order p .

Recently, the existence of solutions for fractional differential equations involving the Caputo fractional derivative was studied in [13, 26–28]. Motivated by the work of Bashiri et al. [17], in this paper we are concerned with the existence of solutions to three-point boundary value problem for a

coupled system of hybrid fractional differential equations of order $p \in [n-1, n]$ given by

$$\begin{aligned} D^p [x(t) - f(t, x(t))] &= g(t, y(t), I^\alpha y(t)), \\ &\text{a.e. } t \in [0, 1], \\ D^p [y(t) - f(t, y(t))] &= g(t, x(t), I^\alpha x(t)), \\ &\text{a.e. } t \in [0, 1], \\ x(0) &= \delta_1 x(\eta_1), \\ y(0) &= \delta_1 y(\eta_1), \\ x(1) &= \delta_2 x(\eta_2), \\ y(1) &= \delta_2 y(\eta_2), \\ x^{(i)}(0) &= y^{(i)}(0) = 0, \\ &\text{for } i = 1, 2, 3, \dots, (n-2), \end{aligned} \quad (2)$$

where $\alpha > 0$, $0 < \eta_1 < \eta_2 < 1$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, $f(0, x(0)) = 0$, and $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. D^p is the Caputo fractional derivative of order p . Moreover, an example is given to illustrate the validity of the existence result.

2. Preliminaries

Throughout this manuscript $\Phi = \{\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that } \varphi(r) < r \text{ for } r > 0 \text{ and } \varphi(0) = 0\}$, $C([0, T] \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, and $C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that

- (i) the map $t \rightarrow g(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$,
- (ii) the map $x \rightarrow g(t, x, y)$ is continuous for each $x \in \mathbb{R}$,
- (iii) the map $y \rightarrow g(t, x, y)$ is continuous for each $y \in \mathbb{R}$.

We need the following definitions which can be found in [9].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (3)$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2. Let α be a positive real number, such that $m-1 \leq \alpha < m$, $m \in \mathbb{N}$, and $f^m(x)$ exists, a function of class C . Then Caputo fractional derivative of f is defined as

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \quad (4)$$

provided that the right side is pointwise defined on $(0, \infty)$, where $m = [\alpha] + 1$ and $[\alpha]$ represents the integer part of α .

Definition 3 (see [29]). The mapping $F : X \times X \rightarrow X$ has a coupled fixed point $(x, y) \in X \times X$ if $F(x, y) = x$ and $F(y, x) = y$.

Theorem 4 (see [17]). Let S be a nonempty, closed, convex, and bounded subset of the Banach space X and $\tilde{S} = S \times S$. Suppose that $A : X \rightarrow X$ and $B : S \rightarrow X$ are two operators such that

$$\begin{aligned} (C_1) \text{ there exists } \varphi_A \in \Phi \text{ such that, for all } x, y \in X, \text{ one has} \\ \|Ax - Ay\| &\leq \sigma \varphi_A(\|x - y\|), \\ &\text{for some constant } \sigma > 0, \end{aligned} \quad (5)$$

(C_2) B is completely continuous,

(C_3) $x = Ax + By \Rightarrow x \in S$, for all $y \in S$.

Then the operator $T(x, y) = Ax + By$ has at least a coupled fixed point in $\tilde{S}$ whenever $\sigma < 1$.

Lemma 5. The following result holds for fractional differential equations:

$$I^\alpha [{}^c D^\alpha h(t)] = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{m-1} t^{m-1}, \quad (6)$$

for arbitrary $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, m-1$, where $m = [\alpha] + 1$ and $[\alpha]$ represents the integer part of α .

3. Existence Results

Let us set the following notations for convenience:

$$\lambda = (1 - \delta_1) (1 - \delta_2 \eta_2^{n-1}) + (1 - \delta_2) \delta_1 \eta_1^{n-1}, \quad (7)$$

$$\begin{aligned} F_0 &= \frac{2}{|\lambda|} \left(|f(1, 0)| + \frac{1}{2} \max_{t \in [0, 1]} |f(t, 0)| \right. \\ &\quad \left. + \max_{\eta_1 \in [0, 1]} |f(\eta_1, 0)| + \max_{\eta_2 \in [0, 1]} |f(\eta_2, 0)| \right). \end{aligned} \quad (8)$$

For the forthcoming analysis, we assume that

(A_1) the function $x \rightarrow x - f(t, x)$ is increasing in $\mathbb{R}$ for all $t \in [0, 1]$;

(A_2) there exists $M \geq L > 0$ such that

$$\begin{aligned} |f(t, x(t)) - f(t, y(t))| &\leq \frac{L |x(t) - y(t)|}{8(M + |x(t) - y(t)|)}, \\ \frac{|f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))|}{|\lambda|} \\ &\leq \sup \left\{ \frac{L \delta_2 |x(t) - y(t)|}{16(M + \delta_2 |x(t) - y(t)|)} \right\}, \end{aligned}$$

$$\begin{aligned}
& \frac{|f(\eta_1, x(\eta_1)) - f(\eta_1, y(\eta_1))|}{|\lambda|} \\
& \leq \sup \left\{ \frac{L|x(t) - y(t)|}{16(M + |x(t) - y(t)|)} \right\}, \\
& \frac{|f(\eta_2, x(\eta_2)) - f(\eta_2, y(\eta_2))|}{|\lambda|} \\
& \leq \sup \left\{ \frac{L|x(t) - y(t)|}{16(M + |x(t) - y(t)|)} \right\}; \quad (9)
\end{aligned}$$

(A₃) there exists a continuous function $h \in C([0, 1], \mathbb{R})$ such that

$$g(t, x(t), y(t)) \leq h(t), \quad x, y \in \mathbb{R}, \quad t \in [0, 1]. \quad (10)$$

Lemma 6. If $f(0, x(0)) = 0$ and $(\partial^i f(t, x(t))/\partial t^i)|_{t=0} = 0$ for $i = 1, 2, \dots, (n-2)$, then integral representation of the system (2) is given by

$$\begin{aligned}
x(t) = & f(t, x(t)) + \frac{1}{\lambda} \left\{ \delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, x(\eta_2)) \right. \\
& - \delta_1 \eta_1^{n-1} f(1, x(1)) \\
& + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, x(\eta_1)) \Big\} \\
& + \frac{1}{\lambda} \left\{ (1 - \delta_1) \delta_2 f(\eta_2, x(\eta_2)) \right. \\
& - (1 - \delta_1) f(1, x(1)) - (1 - \delta_2) \delta_1 f(\eta_1, x(\eta_1)) \Big\} \\
& \cdot t^{n-1} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds, \\
y(t) = & f(t, y(t)) + \frac{1}{\lambda} \left\{ \delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, y(\eta_2)) \right. \\
& - \delta_1 \eta_1^{n-1} f(1, y(1)) \\
& + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, y(\eta_1)) \Big\} \\
& + \frac{1}{\lambda} \left\{ (1 - \delta_1) \delta_2 f(\eta_2, y(\eta_2)) \right.
\end{aligned}$$

$$\begin{aligned}
& - (1 - \delta_1) f(1, y(1)) - (1 - \delta_2) \delta_1 f(\eta_1, y(\eta_1)) \Big\} \\
& \cdot t^{n-1} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds. \quad (12)
\end{aligned}$$

Proof. Applying the operator I^p on the first equation of system (2) and using Lemma 5, we obtain

$$\begin{aligned}
x(t) = & f(t, x(t)) \\
& + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& + C_0 + C_1 t + C_2 t^2 + \dots + C_{n-1} t^{n-1}. \quad (13)
\end{aligned}$$

Applying the initial conditions $x^{(i)}(0) = 0$, for $i = 1, 2, 3, \dots, (n-2)$, we conclude that $C_1 = C_2 = \dots = C_{n-2} = 0$. Therefore (13) becomes

$$\begin{aligned}
x(t) = & f(t, x(t)) + \frac{1}{\Gamma(p)} \\
& \cdot \int_0^t (t-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds + C_0 \\
& + C_{n-1} t^{n-1}. \quad (14)
\end{aligned}$$

Now, to find the values of C_0 and C_{n-1} , since $x(0) = \delta_1 x(\eta_1)$, from (14) we have

$$\begin{aligned}
& f(0, x(0)) + I^p g(0, y(0), I^\alpha y(0)) + C_0 \\
& = \delta_1 \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \right. \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds + C_0 \\
& \left. + C_{n-1} \eta_1^{n-1} \right\} \Rightarrow
\end{aligned}$$

$$(1 - \delta_1) C_0 - \delta_1 C_{n-1} \eta_1^{n-1} - \delta_1 \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} = 0. \quad (15)$$

Also, since $x(1) = \delta_2 x(\eta_2)$, from (14), we have

$$f(1, x(1)) + \frac{1}{\Gamma(p)} \cdot \int_0^1 (1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds + C_0 + C_{n-1} = \delta_2 \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds + C_0 + C_{n-1} \eta_2^{n-1} \right\} \Rightarrow \quad (16)$$

$$(1 - \delta_2) C_0 + (1 - \delta_2 \eta_2^{n-1}) C_{n-1} + f(1, x(1)) + \frac{1}{\Gamma(p)} \cdot \int_0^1 (1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds - \delta_2 \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} = 0.$$

Solving (15) and (16) for C_0 , we get

$$\begin{aligned} & \left\{ (1 - \delta_2 \eta_2^{n-1}) (1 - \delta_1) + \delta_1 \eta_1^{n-1} (1 - \delta_2) \right\} C_0 \\ & + \delta_1 \eta_1^{n-1} \left\{ f(1, x(1)) + \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & - \delta_1 (1 - \delta_2 \eta_2^{n-1}) \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & - \delta_2 \delta_1 \eta_1^{n-1} \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & = 0, \end{aligned} \quad (17)$$

and using (7), we can write

$$\begin{aligned} C_0 = & \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & + \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & - \frac{\delta_1 \eta_1^{n-1}}{\lambda} \left\{ f(1, x(1)) + \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\}. \end{aligned} \quad (18)$$

Similarly, solving (15) and (16) and using (7), we obtained that

$$\begin{aligned} C_{n-1} = & -\frac{(1 - \delta_1)}{\lambda} \left\{ f(1, x(1)) + \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & - \frac{(1 - \delta_2) \delta_1}{\lambda} \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & + \frac{(1 - \delta_1) \delta_2}{\lambda} \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\}. \end{aligned} \quad (19)$$

Substituting the values of C_0 and C_1 in (14), we can write

$$\begin{aligned} x(t) = & f(t, x(t)) + \frac{1}{\Gamma(p)} \cdot \int_0^t (t - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\ & + \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} \left\{ f(\eta_1, x(\eta_1)) + \frac{1}{\Gamma(p)} \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \\ & + \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} \left\{ f(\eta_2, x(\eta_2)) + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \Big\} \\
& - \frac{\delta_1 \eta_1^{n-1}}{\lambda} \left\{ f(1, x(1)) \right. \\
& + \frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \Big\} \\
& - \frac{(1-\delta_1)}{\lambda} \left\{ f(1, x(1)) \right. \\
& + \frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \Big\} t^{n-1} \\
& - \frac{(1-\delta_2)\delta_1}{\lambda} \left\{ f(\eta_1, x(\eta_1)) \right. \\
& + \frac{1}{\Gamma(p)} \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \Big\} \\
& \cdot t^{n-1} + \frac{(1-\delta_1)\delta_2}{\lambda} \left\{ f(\eta_2, x(\eta_2)) \right. \\
& + \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \Big\} \\
& \cdot t^{n-1},
\end{aligned} \tag{20}$$

which implies that

$$\begin{aligned}
x(t) &= f(t, x(t)) + \frac{1}{\lambda} \left\{ \delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, x(\eta_2)) \right. \\
& - \delta_1 \eta_1^{n-1} f(1, x(1)) \\
& + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, x(\eta_1)) \Big\} \\
& + \frac{1}{\lambda} \left\{ (1 - \delta_1) \delta_2 f(\eta_2, x(\eta_2)) \right. \\
& - (1 - \delta_1) f(1, x(1)) - (1 - \delta_2) \delta_1 f(\eta_1, x(\eta_1)) \Big\} \\
& \cdot t^{n-1} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1-\delta_1)t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1-s)^{p-1} g(s, y(s), I^\alpha y(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1-\delta_2)\delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1-\delta_1)\delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, y(s), I^\alpha y(s)) ds.
\end{aligned} \tag{21}$$

Similarly, repeating the above process with the second equation of system (2), we obtain integral equation (12). $\square$

Now, we are in a position to present the existence theorem for the system (2).

Theorem 7. Assume that hypotheses (A_1) – (A_3) hold. Then there exists a solution for coupled systems (2) of higher order hybrid FHDEs with three-point boundary conditions.

Proof. Set $X = C([0, 1], \mathbb{R})$ and a subset S of X defined by

$$\begin{aligned}
S &= \{x \in X : \|x\| \leq N\}, \\
\text{where } N &\geq 7L + F_0 + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \right\}.
\end{aligned} \tag{22}$$

Clearly S is a closed, convex, and bounded subset of the Banach space X . Now, since $x(t)$ is a solution of the FHDEs system (2) if and only if $x(t)$ satisfies the system of integral equations in Lemma 6, to show the existence solution of system (2) it is enough to show the existence solution of the integral equations in Lemma 6. For this, define two operators $A : X \rightarrow X$ and $B : S \rightarrow X$ by

$$\begin{aligned}
Ax(t) &= f(t, x(t)) + \frac{1}{\lambda} \left\{ \delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, x(\eta_2)) \right. \\
& - \delta_1 \eta_1^{n-1} f(1, x(1)) \\
& + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, x(\eta_1)) \Big\} \\
& + \frac{1}{\lambda} \left\{ (1 - \delta_1) \delta_2 f(\eta_2, x(\eta_2)) \right. \\
& - (1 - \delta_1) f(1, x(1)) - (1 - \delta_2) \delta_1 f(\eta_1, x(\eta_1)) \Big\} \\
& \cdot t^{n-1},
\end{aligned}$$

$$\begin{aligned}
Bx(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1-\delta_1)t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1-\delta_2)\delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1-\delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds.
\end{aligned} \tag{23}$$

Then the operators form of system (2) is

$$\begin{aligned}
x(t) &= Ax(t) + By(t), \quad t \in [0, 1], \\
y(t) &= Ay(t) + Bx(t), \quad t \in [0, 1].
\end{aligned} \tag{24}$$

We have to show that the operators A and B satisfy all the conditions of Theorem 4. For this, let $x, y \in X$; then we have

$$\begin{aligned}
& |Ax(t) - Ay(t)| \leq |f(t, x(t)) - f(t, y(t))| \\
& + \left| \frac{\delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, x(\eta_2)) - \delta_1 \eta_1^{n-1} f(1, x(1)) + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, x(\eta_1))}{\lambda} \right. \\
& - \left. \frac{\delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, y(\eta_2)) - \delta_1 \eta_1^{n-1} f(1, y(1)) + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, y(\eta_1))}{\lambda} \right| \\
& + \left| \frac{(1-\delta_1) \delta_2 f(\eta_2, x(\eta_2)) - (1-\delta_1) f(1, x(1)) - (1-\delta_2) \delta_1 f(\eta_1, x(\eta_1))}{\lambda} \right. \\
& - \left. \frac{(1-\delta_1) \delta_2 f(\eta_2, y(\eta_2)) - (1-\delta_1) f(1, y(1)) - (1-\delta_2) \delta_1 f(\eta_1, y(\eta_1))}{\lambda} \right| |t^{n-1}| \leq |f(t, x(t)) - f(t, y(t))| \tag{25} \\
& + \delta_1 \eta_1^{n-1} \left| \frac{f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))}{\lambda} \right| + \delta_1 (1 - \delta_2 \eta_2^{n-1}) \left| \frac{f(\eta_1, x(\eta_1)) - f(\eta_1, y(\eta_1))}{\lambda} \right| \\
& + \delta_2 \delta_1 \eta_1^{n-1} \left| \frac{f(\eta_2, x(\eta_2)) - f(\eta_2, y(\eta_2))}{\lambda} \right| + (1-\delta_1) \left| \frac{f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))}{\lambda} \right| + (1-\delta_2) \\
& \cdot \delta_1 \left| \frac{f(\eta_1, x(\eta_1)) - f(\eta_1, y(\eta_1))}{\lambda} \right| + (1-\delta_1) \delta_2 \left| \frac{f(\eta_2, x(\eta_2)) - f(\eta_2, y(\eta_2))}{\lambda} \right| < |f(t, x(t)) - f(t, y(t))| \\
& + \frac{2|f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))|}{|\lambda|} + \frac{2|f(\eta_1, x(\eta_1)) - f(\eta_1, y(\eta_1))|}{|\lambda|} + \frac{2|f(\eta_2, x(\eta_2)) - f(\eta_2, y(\eta_2))|}{|\lambda|}.
\end{aligned}$$

Taking the supremum over t and using (7), we get

$$\begin{aligned}
& \|Ax(t) - Ay(t)\| \leq \frac{1}{2} \left\{ \frac{L \|x(t) - y(t)\|}{4(M + \|x(t) - y(t)\|)} \right. \\
& + \frac{L \delta_2 \|x(t) - y(t)\|}{4(M + \delta_2 \|x(t) - y(t)\|)} \\
& + \frac{L \|x(t) - y(t)\|}{4(M + \|x(t) - y(t)\|)} \\
& + \left. \frac{L \|x(t) - y(t)\|}{4(M + \|x(t) - y(t)\|)} \right\} \\
& = \frac{1}{2} \left\{ \frac{3L \|x(t) - y(t)\|}{4(M + \|x(t) - y(t)\|)} \right. \\
& + \frac{L \delta_2 \|x(t) - y(t)\|}{4(M + \delta_2 \|x(t) - y(t)\|)} \left. \right\} \leq \frac{1}{2} \\
& \cdot \varphi(\|x(t) - y(t)\|).
\end{aligned} \tag{26}$$

Thus A satisfies condition (C_1) of Theorem 4 with $\sigma = 1/2$ and $\varphi(r) = 3Lr/4(M+r) + L\delta_2 r/4(M+\delta_2 r)$.

Next, we show that B is compact and continuous operator on S . Let $\{x_m\}$ be a sequence in S such that $\{x_m\} \rightarrow x \in S$. Then for all $t \in [0, 1]$, we have

$$\begin{aligned}
& \lim_{m \rightarrow \infty} Bx_m(t) = \frac{1}{\Gamma(p)} \\
& \cdot \lim_{m \rightarrow \infty} \int_0^t (t-s)^{p-1} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1-\delta_1) t^{n-1}}{\lambda} \right\} \lim_{m \rightarrow \infty} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1-s)^{p-1} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1-\delta_2) \delta_1 t^{n-1}}{\lambda} \right\} \lim_{m \rightarrow \infty} \frac{1}{\Gamma(p)}
\end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \lim_{m \rightarrow \infty} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x_m(s), I^\alpha x_m(s)) ds = \frac{1}{\Gamma(p)} \\
& \cdot \int_0^t (t - s)^{p-1} \lim_{m \rightarrow \infty} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1 - s)^{p-1} \lim_{m \rightarrow \infty} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} \lim_{m \rightarrow \infty} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} \lim_{m \rightarrow \infty} g(s, x_m(s), I^\alpha x_m(s)) ds \\
& = \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds = Bx(t).
\end{aligned} \tag{27}$$

Thus the map B is continuous on S .

Let $x \in S$; then we have

$$\begin{aligned}
|Bx(t)| & \leq \frac{1}{\Gamma(p)} \left| \int_0^t (t - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
& + \left| \left(\frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right) \frac{1}{\Gamma(p)} \right|
\end{aligned}$$

$$\begin{aligned}
& \cdot \left| \int_0^1 (1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
& + \left| \left(\frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right) \frac{1}{\Gamma(p)} \right| \\
& \cdot \left| \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
& + \left| \left(\frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right) \frac{1}{\Gamma(p)} \right| \\
& \cdot \left| \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \leq \frac{1}{\Gamma(p)} \\
& \cdot \int_0^t (t - s)^{p-1} |g(s, x(s), I^\alpha x(s))| ds \\
& + \left\{ \frac{|\delta_1 \eta_1^{n-1}|}{|\lambda|} + \frac{|(1 - \delta_1) t^{n-1}|}{|\lambda|} \right\} \frac{1}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} \\
& \cdot |g(s, x(s), I^\alpha x(s))| ds \\
& + \left\{ \frac{|\delta_1 (1 - \delta_2 \eta_2^{n-1})|}{|\lambda|} + \frac{|(1 - \delta_2) \delta_1 t^{n-1}|}{|\lambda|} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} |g(s, x(s), I^\alpha x(s))| ds \\
& + \left\{ \frac{|\delta_2 \delta_1 \eta_1^{n-1}|}{|\lambda|} + \frac{|(1 - \delta_1) \delta_2 t^{n-1}|}{|\lambda|} \right\} \frac{1}{\Gamma(p)} \int_0^{\eta_2} (\eta_2 - s)^{p-1} \\
& |g(s, x(s), I^\alpha x(s))| ds \leq \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} \|h(s)\| ds \\
& + \frac{2}{|\lambda| \Gamma(p)} \left\{ \int_0^1 (1 - s)^{p-1} \|h(s)\| ds \right. \\
& + \int_0^{\eta_1} (\eta_1 - s)^{p-1} \|h(s)\| ds \\
& + \left. \int_0^{\eta_2} (\eta_2 - s)^{p-1} \|h(s)\| ds \right\} \\
& \leq \frac{\|h\|_{L^1}}{\Gamma(p)} \left\{ \int_0^t (t - s)^{p-1} ds + \frac{2}{|\lambda|} \left(\int_0^1 (1 - s)^{p-1} ds \right. \right. \\
& + \int_0^{\eta_1} (\eta_1 - s)^{p-1} ds + \left. \int_0^{\eta_2} (\eta_2 - s)^{p-1} ds \right\} \\
& = \frac{\|h\|_{L^1}}{\Gamma(p)} \left\{ \frac{(t - s)^p}{-p} \Big|_0^t + \frac{2}{|\lambda|} \left(\frac{(1 - s)^p}{-p} \Big|_0^1 \right. \right. \\
& + \left. \left. \frac{(\eta_1 - s)^p}{-p} \Big|_0^{\eta_1} + \frac{(\eta_2 - s)^p}{-p} \Big|_0^{\eta_2} \right) \right\} = \frac{\|h\|_{L^1}}{\Gamma(p)} \left\{ \frac{t^p}{p} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{|\lambda|} \left(\frac{1}{p} + \frac{\eta_1^p}{p} + \frac{\eta_2^p}{p} \right) \Big\} = \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ t^p + \frac{2}{|\lambda|} \left(1 + \eta_1^p + \eta_2^p \right) \right. \\
& \left. + \eta_1^p + \eta_2^p \right\} \leq \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} \left(1 + \eta_1^p + \eta_2^p \right) \right\}. \quad (28)
\end{aligned}$$

Taking the supremum over t , we get

$$\|Bx\| \leq \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} \left(1 + \eta_1^p + \eta_2^p \right) \right\}, \quad \forall x \in S. \quad (29)$$

Thus B is uniformly bounded on S .

Now, let $t_1, t_2 \in [0, 1]$ such that $t_1 \neq t_2$; then for any $x \in S$, we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left| \frac{1}{\Gamma(p)} \right. \\
& \cdot \int_0^{t_1} (t_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& - \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds - \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{t_2} (t_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^1 (1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& + \left\{ \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)} \\
& \cdot \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \\
& - \left\{ \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t_1^{n-1}}{\lambda} \right\} \frac{1}{\Gamma(p)}
\end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \Big| \\
& \leq \frac{1}{\Gamma(p)} \left| \int_0^{t_1} (t_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right. \\
& \quad \left. - \int_0^{t_2} (t_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
& \leq \frac{1}{\Gamma(p)} \left| \int_0^{t_1} (t_1 - s)^{p-1} h(s) ds \right. \\
& \quad \left. - \int_0^{t_1} (t_2 - s)^{p-1} h(s) ds + \int_0^{t_1} (t_2 - s)^{p-1} h(s) ds \right. \\
& \quad \left. - \int_0^{t_2} (t_2 - s)^{p-1} h(s) ds \right| \\
& \leq \frac{\|h\|_{L^1}}{\Gamma(p)} \left(\left| \int_0^{t_1} [(t_1 - s)^{p-1} - (t_2 - s)^{p-1}] ds \right| \right. \\
& \quad \left. + \left| \int_{t_1}^{t_2} (t_2 - s)^{p-1} ds \right| \right) \\
& = \frac{\|h\|_{L^1}}{\Gamma(p)} \left(\left| \frac{(t_1 - s)^p}{-p} - \frac{(t_2 - s)^p}{-p} \right|_0^{t_1} \right. \\
& \quad \left. + \left| \frac{(t_2 - s)^p}{-p} \right|_{t_1}^{t_2} \right) = \frac{\|h\|_{L^1}}{\Gamma(p)} \left(\left| \frac{(t_2 - t_1)^p}{p} + \frac{t_1^p}{p} - \frac{t_2^p}{p} \right| \right. \\
& \quad \left. + \left| \frac{(t_2 - t_1)^p}{p} \right| \right) \leq \frac{\|h\|_{L^1}}{\Gamma(p+1)} (|t_2 - t_1|^p + |t_1^p - t_2^p| \\
& \quad + |t_2 - t_1|^p). \quad (30)
\end{aligned}$$

Since t^p is uniformly continuous on $[0, 1]$ for $n - 1 < p < n$, for any $\varepsilon > 0$ we can find $\delta_1^* > 0$ such that

$$|t_1^p - t_2^p| < \frac{\Gamma(p+1)}{3 \|h\|_{L^1}} \varepsilon, \quad \text{whenever } |t_1 - t_2| < \delta_1^*. \quad (31)$$

Let $\delta^* = \min \{\delta_1^*, (\Gamma(p+1)/3 \|h\|_{L^1})^{1/p}\}$; then, we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &\leq \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left(\frac{\Gamma(p+1)}{3 \|h\|_{L^1}} \varepsilon \right. \\
& \quad \left. + \frac{\Gamma(p+1)}{3 \|h\|_{L^1}} \varepsilon + \frac{\Gamma(p+1)}{3 \|h\|_{L^1}} \varepsilon \right) = \varepsilon, \quad (32)
\end{aligned}$$

whenever $|t_1 - t_2| < \delta^*$.

Thus $B(S)$ is equicontinuous and hence B is completely continuous on S .

To prove hypothesis (C_3) of Theorem 4, let $x \in X$ and $y \in S$ such that $x = Ax + By$; using (7) and (8), we have

$$\begin{aligned}
|x(t)| &\leq |Ax(t)| + |Bx(t)| \\
&= |f(t, x(t))| + \left| \frac{\delta_2 \delta_1 \eta_1^{n-1} f(\eta_2, x(\eta_2)) - \delta_1 \eta_1^{n-1} f(1, x(1)) + \delta_1 (1 - \delta_2 \eta_2^{n-1}) f(\eta_1, x(\eta_1))}{\lambda} \right| \\
&\quad + \left| \frac{(1 - \delta_1) \delta_2 f(\eta_2, x(\eta_2)) - (1 - \delta_1) f(1, x(1)) - (1 - \delta_2) \delta_1 f(\eta_1, x(\eta_1))}{\lambda} \right| |t^{n-1}| \\
&\quad + \frac{1}{\Gamma(p)} \left| \int_0^t (t-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
&\quad + \left| \frac{\delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) t^{n-1}}{\lambda} \right| \frac{1}{\Gamma(p)} \left| \int_0^1 (1-s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
&\quad + \left| \frac{\delta_1 (1 - \delta_2 \eta_2^{n-1})}{\lambda} - \frac{(1 - \delta_2) \delta_1 t^{n-1}}{\lambda} \right| \frac{1}{\Gamma(p)} \left| \int_0^{\eta_1} (\eta_1 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
&\quad + \left| \frac{\delta_2 \delta_1 \eta_1^{n-1}}{\lambda} + \frac{(1 - \delta_1) \delta_2 t^{n-1}}{\lambda} \right| \frac{1}{\Gamma(p)} \left| \int_0^{\eta_2} (\eta_2 - s)^{p-1} g(s, x(s), I^\alpha x(s)) ds \right| \\
&\leq |f(t, x(t))| + \frac{|f(\eta_2, x(\eta_2)) + f(1, x(1))| + |f(\eta_1, x(\eta_1))|}{|\lambda|} + \frac{|f(\eta_2, x(\eta_2)) + f(1, x(1))| + |f(\eta_1, x(\eta_1))|}{|\lambda|} \\
&\quad + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \right\}.
\end{aligned} \tag{33}$$

This implies

$$\begin{aligned}
|x(t)| &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
&\quad + 2 \left(\frac{|f(1, 0)|}{|\lambda|} + \frac{|f(\eta_1, 0)|}{|\lambda|} + \frac{|f(\eta_2, 0)|}{|\lambda|} \right) \\
&\quad + 2 \left(\frac{|f(1, \delta_2 x(\eta_2)) - f(1, 0)|}{|\lambda|} \right. \\
&\quad + \frac{|f(\eta_1, x(\eta_1)) - f(\eta_1, 0)|}{|\lambda|} \\
&\quad + \left. \frac{|f(\eta_2, x(\eta_2)) - f(\eta_2, 0)|}{|\lambda|} \right) + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 \right. \\
&\quad + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \left. \right\} \leq L + 2(L + L + L) \\
&\quad + \frac{2}{|\lambda|} \left(\frac{1}{2} |f(t, 0)| + |f(1, 0)| + |f(\eta_1, 0)| \right. \\
&\quad + \left. |f(\eta_2, 0)| \right) + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 \right. \\
&\quad + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \left. \right\} \leq 7L + F_0 + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 \right. \\
&\quad + \left. \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \right\}.
\end{aligned} \tag{34}$$

Taking the supremum over t on $[0, 1]$, we can write

$$\begin{aligned}
\|x(t)\| &\leq 7L + F_0 \\
&\quad + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \right\} \leq N.
\end{aligned} \tag{35}$$

That is, $x \in S$. Thus condition (C_3) of Theorem 4 holds. Therefore, all the conditions of Theorem 4 are satisfied; hence the operator $T(x, y) = Ax + By$ has a coupled fixed point on $\tilde{S}$. Consequently, system (2) has a solution defined on $[0, 1]$. $\square$

To illustrate Theorem 7, we construct the following example.

Example 8. We discuss the following hybrid fractional differential equations with three-point boundary conditions:

$$\begin{aligned}
D^{5/2} \left[x(t) - \frac{e^{-t} |x(t)|}{11 + |x(t)|} \right] \\
&= \frac{t^3}{3} - (\cos |y(t)| + \cos |I^{3/2} y(t)|), \\
D^{5/2} \left[y(t) - \frac{e^{-t} |y(t)|}{11 + |y(t)|} \right] \\
&= \frac{t^3}{3} - (\cos |x(t)| + \cos |I^{3/2} x(t)|),
\end{aligned}$$

$$\begin{aligned}
x(0) &= \frac{1}{4}x\left(\frac{1}{4}\right), \\
y(0) &= \frac{1}{4}y\left(\frac{1}{4}\right), \\
x(1) &= \frac{1}{2}x\left(\frac{1}{2}\right), \\
y(1) &= \frac{1}{2}y\left(\frac{1}{2}\right), \\
x'(0) &= y'(0) = 0,
\end{aligned} \tag{36}$$

where $t \in [0, 1]$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, $f(0, x(0)) = 0$, and $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. $D^{5/2}$ is the Caputo fractional derivative of order $5/2$.

Here

$$\begin{aligned}
\delta_1 &= \eta_1 = \frac{1}{4}, \\
\delta_2 &= \eta_2 = \frac{1}{2}, \\
f(t, x(t)) &= \frac{e^{-t}|x(t)|}{11 + |x(t)|}, \\
g(t, x(t), I^\alpha x(t)) &= \frac{t^3}{3} \\
&\quad - \left(\cos |x(t)| + \cos \left| I^{3/2} x(t) \right| \right).
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
\lambda &= \frac{19}{32}, \\
F_0 &= \frac{2}{|\lambda|} \left(|f(1, 0)| + \frac{1}{2} \max_{t \in [0, 1]} |f(t, 0)| \right. \\
&\quad \left. + \max_{\eta_1 \in [0, 1]} |f(\eta_1, 0)| + \max_{\eta_2 \in [0, 1]} |f(\eta_2, 0)| \right) = 0.
\end{aligned} \tag{38}$$

Now, for $M = 11$ and $L = 512/19e < M$, we have

$$\begin{aligned}
|f(t, x(t)) - f(t, y(t))| &= \left| \frac{e^{-t}|x(t)|}{11 + |x(t)|} \right. \\
&\quad \left. - \frac{e^{-t}|y(t)|}{11 + |y(t)|} \right| \leq \left| \frac{|x(t) - y(t)| + |y(t)|}{11 + |x(t) - y(t)| + |y(t)|} \right. \\
&\quad \left. - \frac{|y(t)|}{11 + |x(t) - y(t)| + |y(t)|} \right| \\
&\leq \left| \frac{|x(t) - y(t)|}{11 + |x(t) - y(t)| + |y(t)|} \right| \\
&\leq \frac{|x(t) - y(t)|}{11 + |x(t) - y(t)|} \leq \frac{(512/19e)|x(t) - y(t)|}{16(11 + |x(t) - y(t)|)},
\end{aligned}$$

$$\begin{aligned}
&\frac{|f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))|}{|\lambda|} \\
&= \frac{|f(1, (1/2)x(1/2)) - f(1, (1/2)y(1/2))|}{19/32} \\
&= \frac{32}{19} \left| \frac{e^{-1}|(1/2)x(1/2)|}{11 + |(1/2)x(1/2)|} - \frac{e^{-1}|(1/2)y(1/2)|}{11 + |(1/2)y(1/2)|} \right|, \\
&\frac{|f(1, \delta_2 x(\eta_2)) - f(1, \delta_2 y(\eta_2))|}{|\lambda|} \\
&\leq \frac{32}{19e} \left| \frac{\delta_2 |x(1/2) - y(1/2)| + \delta_2 |y(1/2)|}{11 + \delta_2 |x(1/2) - y(1/2)| + \delta_2 |y(1/2)|} \right. \\
&\quad \left. - \frac{\delta_2 |y(1/2)|}{11 + \delta_2 |x(1/2) - y(1/2)| + \delta_2 |y(1/2)|} \right| \\
&\leq \frac{32}{19e} \left| \frac{\delta_2 |x(1/2) - y(1/2)|}{11 + \delta_2 |x(1/2) - y(1/2)| + \delta_2 |y(1/2)|} \right| \\
&\leq \frac{32}{19e} \times \frac{\delta_2 |x(1/2) - y(1/2)|}{11 + \delta_2 |x(1/2) - y(1/2)|} \\
&\leq \sup \left\{ \frac{(512/19e)\delta_2 |x(t) - y(t)|}{16(11 + \delta_2 |x(t) - y(t)|)} \right\}.
\end{aligned} \tag{39}$$

Similarly,

$$\begin{aligned}
&\frac{|f(\eta_1, x(\eta_1)) - f(\eta_1, y(\eta_1))|}{|\lambda|} \\
&\leq \sup \left\{ \frac{(512/19e)\delta_2 |x(t) - y(t)|}{16(11 + \delta_2 |x(t) - y(t)|)} \right\}, \\
&\frac{|f(\eta_2, x(\eta_2)) - f(\eta_2, y(\eta_2))|}{|\lambda|} \\
&\leq \sup \left\{ \frac{(512/19e)\delta_2 |x(t) - y(t)|}{16(11 + \delta_2 |x(t) - y(t)|)} \right\}.
\end{aligned} \tag{40}$$

Next,

$$\begin{aligned}
g(t, x(t), I^\alpha x(t)) &= \frac{t^3}{3} \\
&\quad - \left(\cos |x(t)| + \cos \left| I^{3/2} x(t) \right| \right) \\
&\leq \frac{t^3}{3} = h(t).
\end{aligned} \tag{41}$$

That is, there exists a continuous function $h \in C([0, 1], \mathbb{R})$ such that

$$g(t, x(t), y(t)) \leq h(t), \quad x, y \in \mathbb{R}, \quad t \in [0, 1]. \tag{42}$$

Finally, since

$$\begin{aligned}\|h(t)\| &= \sup \int_0^1 \frac{t^3}{3} dt = \frac{1}{12}, \\ L &= \frac{512}{19e}, \\ F_0 &= 0, \\ \lambda &= \frac{19}{32},\end{aligned}\quad (43)$$

we can write

$$7L + F_0 + \frac{\|h\|_{L^1}}{\Gamma(p+1)} \left\{ 1 + \frac{2}{|\lambda|} (1 + \eta_1^p + \eta_2^p) \right\} < 70. \quad (44)$$

Thus $N \geq 70$. It follows that assumptions (A_1) and (A_2) are satisfied. Therefore, by Theorem 7 we conclude that problem (36) has a solution.

4. Conclusion

We have successfully developed appropriate conditions for existence of at least one solution to a complicated higher order coupled system of nonlinear hybrid fractional differential equations. The respective conditions have been derived by using coupled fixed point theorem of Krasnoselskii type. The obtained results were also demonstrated by a suitable example.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have equal contribution. The first two authors designed the problem and the last two authors read and corrected its style and language and prepared the final version.

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