Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 6935095, 10 pages
https://doi.org/10.1155/2018/6935095

Research Article

Stochastic P-Bifurcation of a Bistable Viscoelastic Beam with Fractional Constitutive Relation under Gaussian White Noise

Yajie Li,[1,2] **Zhiqiang Wu**,[1,2] **Guoqi Zhang**,[1,2] **and Feng Wang**[1,2]

[1] *Department of Mechanics, School of Mechanical Engineering, Tianjin University, Tianjin 300072, China*
[2] *Tianjin Key Laboratory of Nonlinear Dynamics and Chaos Control, Tianjin University, Tianjin 300072, China*

Correspondence should be addressed to Zhiqiang Wu; zhiqwu@tju.edu.cn

Received 28 July 2018; Revised 26 October 2018; Accepted 5 November 2018; Published 2 December 2018

Academic Editor: Dorota Mozyrska

In this paper, we study the stochastic P-bifurcation problem for axially moving of a bistable viscoelastic beam with fractional derivatives of high order nonlinear terms under Gaussian white noise excitation. First, using the principle for minimum mean square error, we show that the fractional derivative term is equivalent to a linear combination of the damping force and restoring force, so that the original system can be simplified to an equivalent system. Second, we obtain the stationary Probability Density Function (PDF) of the system's amplitude by stochastic averaging, and using singularity theory, we find the critical parametric condition for stochastic P-bifurcation of amplitude of the system. Finally, we analyze the types of the stationary PDF curves of the system qualitatively by choosing parameters corresponding to each region within the transition set curve. We verify the theoretical analysis and calculation of the transition set by showing the consistency of the numerical results obtained by Monte Carlo simulation with the analytical results. The method used in this paper directly guides the design of the fractional order viscoelastic material model to adjust the response of the system.

1. Introduction

Fractional calculus is a generalization of integer-order calculus, it extends the order of calculus operation from the traditional integer order to the case of noninteger order, and it has a history of more than 300 years as so far. Due to the limitation of the definition of integer-order derivative, it cannot express the memory property of viscoelastic substances. The definition of fractional derivative contains convolution, which can well express the memory effect and show the cumulative effect over time. Compared with the traditional integer-order calculus, fractional calculus has more advantages and is a suitable mathematical tool for describing the memory characteristics [1–11] and in recent years, it has become the powerful mathematical tool in many disciplines, especially in the study of viscoelastic materials.

The fractional derivative can accurately describe the constitutive relation of viscoelastic materials with fewer parameters, so the studies of fractional differential equations on the typical mechanical properties and the influences of fractional order parameters on the system are very necessary and have important significance. In recent years, many scholars have done a lot of work and achieved fruitful results in this field: Li and Tang studied the nonlinear parametric vibration of an axially moving string made by rubber-like materials, a new nonlinear fractional mathematical model governing transverse motion of the string is derived based on Newton's second law, the Euler beam theory, and the Lagrangian strain, and the principal parametric resonance is analytically investigated via applying the direct multiscale method [12]. Liu et al. introduced a transfer entropy and surrogate data algorithm to identify the nonlinearity level of the system by using a numerical solution of nonlinear response of beams, the Galerkin method was applied to discretize the dimensionless differential governing equation of the forced vibration, and then the fourth-order Runge-Kutta method was used to obtain the time history response of the lateral displacement [13]. Liu et al. investigated the stochastic stability of coupled viscoelastic system with non-viscously damping driven by white noise through moment Lyapunov exponents and Lyapunov exponents, obtained the coupled Itô stochastic differential equations of the norm of

the response and angles process by using the coordinate transformation, and discussed the effects of various physical quantities of stochastic coupled system on the stochastic stability [14]. Nutting, Gemant and Scott-Blair et al. [15–17] first proposed the fractional derivative models to study the constitutive relation of viscoelastic materials and the research on the viscoelastic materials with fractional derivative is also increasing, and so far, it is still a research hotspot [18–25]. Rodr Guez et al. calculated the correlation function of transverse wave in linear and homogeneous viscoelastic liquid by the Generalized Langevin Equation (GLE) method and the influence of fractional correlation function on the dynamic behavior of the system is analyzed [26]. Bagley and Torvik used fractional calculus to study the dynamic behavior of viscoelastic damping structure and the responses of the system under general load as well as step load are analyzed respectively [27, 28]. Pakdemirli and Ulsoy studied the primary parametric resonance and combined resonance of the axial acceleration rope based on the discrete perturbation method and the multiscale method [29]. Zhang and Zhu analyzed the stability and dynamic response of viscoelastic belt under parametric excitation by the multiscale method [30, 31]. Chen et al. studied the dynamic behavior and steady-state response of axially accelerating viscoelastic beam by the Galerkin method [32–35], derived the differential equation of nonlinear vibration for axially moving viscoelastic rope, and then pointed out that the damping of viscoelastic rope only exists in the nonlinear term [36, 37]. Leung et al. studied the steady-state response of a simply supported viscoelastic column under the axial harmonic excitation based on the fractional derivative constitutive model of cubic nonlinear and derived the generalized Mathieu-Duffing equation with time delay by the Galerkin discrete method, then the bifurcation behavior of the system caused by the order of the fractional derivative is analyzed [38]. Ghayesh and Moradian developed the Kelvin-Voigt viscoelastic model of the axially moving and the tensile belt, and then found the existence of nontrivial limit cycle in this system [39]. Liu et al. studied the dynamic response of an axially moving viscoelastic beam under random disorder periodic excitation, the first order expression of the solution is obtained by the multiscale method, and the stochastic jump phenomenon between the steady-state solutions is carried out [40]. Yang and Fang derived the system equation based on Newton's second law and the fractional Kelvin constitutive relation and then studied the stability of the axially moving beam under the parametric resonance condition [41]. Leung et al. studied the single mode dynamic characteristics of the nonlinear arch with the fractional derivative, the steady-state solution of the system is obtained based on the residual harmonic homotopy method, and the influence of the parametric variation on the dynamic behaviors of the viscoelastic damping material is analyzed [42]. Galucio et al. obtained the fractional derivative model to describe the viscoelasticity of the system based on the Timoshenko theory and Euler-Bernoulli hypothesis and proposed a finite element formula for analyzing the sandwich beam of viscoelastic material with fractional derivative and the results were verified numerically [43].

Due to the complexity of fractional derivative, the analysis method of it becomes more difficult, the study on the vibration characteristics of the parameters can only be qualitatively analyzed, and the critical conditions of the parametric influences cannot be found, which affect the analysis and design of such systems, as well as the stochastic P-bifurcation of bistability for the viscoelastic beam with fractional derivatives of high order nonlinear terms under random noise excitation has not been reported. In view of the above situation, the nonlinear vibration of viscoelastic beam with fractional constitutive relation under Gaussian white noise excitation is taken as an example, the transition set curve of the fractional order system as well as the critical parametric condition for stochastic P-bifurcation of the system is obtained by the singularity theory, and then the types of stationary PDF curves of the system in each region in the parametric plane divided by the transition set are analyzed. By the method of Monte Carlo simulation, the numerical results are compared with the analytical results obtained in this paper, it can be seen that the numerical solutions are in good agreements with the analytical solutions, and thus the correctness of the theoretical analysis in this paper is verified.

2. Equation of Axially Moving Viscoelastic Beam

There are many definitions of fractional derivatives, and the Riemann-Liouville derivative and Caputo derivative are commonly used. The initial conditions corresponding to the Riemann-Liouville derivative have no physical meanings, however, the initial conditions of the systems described by the Caputo derivative have clear physical meanings and their forms are the same as the initial conditions for the differential equations of integer order. So in this paper, the Caputo-type fractional derivative is adopted as follows:

$$ {}_a^C D^p [x(t)] = \frac{1}{\Gamma(m-p)} \int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} du, \tag{1} $$

where $m-1 < p \le m, m \in N, t \in [a,b]$, $\Gamma(m)$ is the Euler Gamma function, and $x^{(m)}(t)$ is the m order derivative of $x(t)$.

For a given physical system, due to the fact that the initial moment of the oscillator is $t = 0$, the following form of the Caputo derivative is often used:

$$ {}_0^C D^p [x(t)] = \frac{1}{\Gamma(m-p)} \int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} du, \tag{2} $$

In this paper, the transverse vibration $y(x,t)$ of a viscoelastic simply supported beam under lateral excitation $F(x,t)$ as shown in Figure 1 is considered; applying the d'Alambert principle, the governing equation can be written as [42]

$$ \begin{aligned} \frac{\partial Q}{\partial x} &= F(x,t) - \rho A \frac{\partial^2 y}{\partial t^2} \\ \frac{\partial M}{\partial x} &= Q - N \frac{\partial y}{\partial x}, \end{aligned} \tag{3} $$

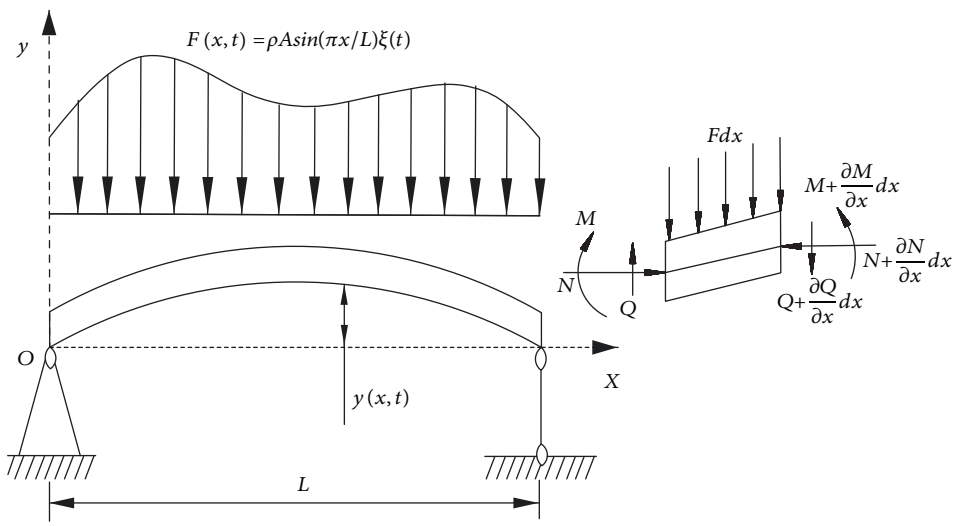


Figure 1: Schematic stress diagram of viscoelastic beam.

where ρ is the mass density of the beam, A is the area of cross-section, M is the bending moment, Q is the lateral force, and N is the horizontal force. From (3), we have

$$\frac{\partial^2 M}{\partial x^2} + N\frac{\partial^2 y}{\partial x^2} + \rho A\frac{\partial y^2}{\partial t^2} - F(x,t) = 0. \tag{4}$$

Assuming the material of the beam obeys a fractional derivative viscoelastic constitutive relation:

$$\begin{aligned}\sigma(x,z,t) &= E_0\varepsilon(x,z,t) + E_1\,{}_0^C D^p[\varepsilon(x,z,t)]\\ &= E_0\left(1+\eta\,{}_0^C D^p\right)\varepsilon(x,z,t) \triangleq S\varepsilon(x,z,t),\end{aligned} \tag{5}$$

where p is the order of fractional derivative ${}_0^C D^p[\varepsilon(x,z,t)]$ as is defined in (2), $\eta = E_1/E_0$ is the material modulus ratio, and $\varepsilon(x,z,t)$ is axial strain component.

When the deformation of the beam is small, the axial strain ε and lateral displacement $y(x,t)$ satisfy the relationship as follows:

$$\varepsilon(x,z,t) = -z\frac{\partial^2 y(x,t)}{\partial x^2}. \tag{6}$$

Substituting (6) into (5) yields

$$\sigma(x,z,t) = S\left[-z\frac{\partial^2 y(x,t)}{\partial x^2}\right]. \tag{7}$$

The relationship between bending moment $M(x,t)$ and axial stress $\sigma(x,z,t)$ of the beam can be expressed as follows:

$$M(x,t) = \int_{-h/2}^{h/2} z\sigma(x,z,t)\,dz, \tag{8}$$

where h is the thickness of the beam.

From (7) and (8), the expression of the bending moment M can be obtained as follows:

$$\begin{aligned}M(x,t) &= I\left[E_0\left(\frac{\partial^2 y}{\partial x^2}\right) + E_1\cdot{}_0^C D^p\left(\frac{\partial^2 y}{\partial x^2}\right)\right]\\ &= IS\left(\frac{\partial^2 y}{\partial x^2}\right),\end{aligned} \tag{9}$$

where $I = h^3/12$.

The expression of the horizontal tension is

$$\begin{aligned}N &= \frac{E_0 A}{2L}\int_0^L\left[\left(\frac{\partial y}{\partial x}\right)^2 - \left(\frac{\partial y}{\partial x}\right)^4\right]dx\\ &\quad + \frac{E_1 A}{2L}\int_0^L {}_0^C D^p\left[\left(\frac{\partial y}{\partial x}\right)^2 - \left(\frac{\partial y}{\partial x}\right)^4\right]dx\\ &= \frac{A}{2L}S\int_0^L\left[\left(\frac{\partial y}{\partial x}\right)^2 - \left(\frac{\partial y}{\partial x}\right)^4\right]dx.\end{aligned} \tag{10}$$

Substituting (9) and (10) into system (4), system (4) can be rewritten as

$$\begin{aligned}&IS\left(\frac{\partial^4 y}{\partial x^4}\right)\\ &\quad + S\frac{A}{2L}\frac{\partial^2 y}{\partial x^2}\int_0^L {}_0^C D^p\left[\left(\frac{\partial y}{\partial x}\right)^2 - \left(\frac{\partial y}{\partial x}\right)^4\right]dx\\ &\quad + \rho A\frac{\partial^2 y}{\partial t^2} - F(x,t) = 0.\end{aligned} \tag{11}$$

The boundary conditions are

$$\frac{\partial^2 y}{\partial x^2} = 0, \quad \text{when } x = 0,\ x = L. \tag{12}$$

According to the boundary conditions (12), the solution of system (11) can be expressed as the Fourier series:

$$y(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi}{L}x\right). \tag{13}$$

Assume that the initial transverse vibration of the system is $y_0(x) = 0$ and the transverse load of the system satisfies the following form:

$$F(x,t) = \rho A \cdot \sin\left(\frac{\pi}{L}x\right)\xi(t), \tag{14}$$

where $\xi(t)$ is the Gaussian white noise, which satisfies $E[\xi(t)] = 0$, $E[\xi(t)\xi(t-\tau)] = 2D\delta(\tau)$, D represents the noise intensity, and $\delta(\tau)$ is the Dirac function.

By the discrete format based on Galerkin method, system (11) can be reduced to the fractional differential equation as follows:

$$\begin{aligned}\ddot{u} + k_1 u + \eta k_1 {}_0^C D^p u - k_3 u^3 - \eta k_3 u {}_0^C D^p\left(u^2\right) + k_5 u^5 \\ + \eta k_5 u {}_0^C D^p\left(u^4\right) = \xi(t),\end{aligned} \tag{15}$$

where ${}_0^C D_t^p u$ is the p $(0 < p < 1)$ order Caputo derivative of $u(t)$ as is defined in (2), and

$$\begin{aligned} k_1 &= \left(\frac{\pi}{L}\right)^4 \frac{IE_0}{\rho A}, \\ k_3 &= \left(\frac{\pi}{L}\right)^4 \frac{E_0}{4\rho}, \\ k_5 &= \left(\frac{\pi}{L}\right)^6 \frac{3E_0}{16\rho}. \end{aligned} \tag{16}$$

For convenience, system (15) can be represented as follows:

$$\begin{aligned}\ddot{u} + w^2 u + \eta w^2 {}_0^C D^p u - k_3 u^3 - \eta k_3 u {}_0^C D^p\left(u^2\right) \\ + k_5 u^5 + \eta k_5 u {}_0^C D^p\left(u^4\right) = \xi(t),\end{aligned} \tag{17}$$

where $w = \sqrt{k_1}$.

The fractional derivative term has contributions to both damping and restoring forces [44–47], hence, introducing the following equivalent system:

$$\begin{aligned}&\ddot{u} + w^2 u + \eta w^2\left[c(a)\dot{u} + w^2(a)u\right] - k_3 u^3 \\ &\quad - \eta k_3 u {}_0^C D^p\left(u^2\right) + k_5 u^5 + \eta k_5 u {}_0^C D^p\left(u^4\right) \\ &\quad = \xi(t),\end{aligned} \tag{18}$$

where $c(a)$, $w^2(a)$ are the coefficients of equivalent damping and restoring forces of fractional derivative ${}_0^C D^p u$, respectively.

The error between system (17) and (18) is

$$e = \eta w^2\left[c(a)\dot{u} + w^2(a)u - {}_0^C D^p u\right]. \tag{19}$$

The necessary conditions for minimum mean square error are [48]

$$\begin{aligned}\frac{\partial E\left[e^2\right]}{\partial(c(a))} &= 0, \\ \frac{\partial E\left[e^2\right]}{\partial(w(a))} &= 0.\end{aligned} \tag{20}$$

Substituting (19) into (20) yields

$$\begin{aligned}&E\left[c(a)\dot{u}^2 + w^2(a)u\dot{u} - \dot{u}\,{}_0^C D^p u\right] \\ &\quad = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(c(a)\dot{u}^2 + w^2(a)u\dot{u} - \dot{u}\,{}_0^C D^p u\right)dt \\ &\quad = 0 \\ &E\left[c(a)\dot{u}u + w^2(a)u^2 - u\,{}_0^C D^p u\right] \\ &\quad = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(c(a)\dot{u}u + w^2(a)u^2 - u\,{}_0^C D^p u\right)dt \\ &\quad = 0.\end{aligned} \tag{21}$$

Assume that the solution of system (18) has the following form:

$$\begin{aligned} u(t) &= a(t)\cos\varphi(t) \\ \varphi(t) &= wt + \theta, \end{aligned} \tag{22}$$

where w is the natural frequency of system (17).

Based on (22), we can obtain

$$\dot{u}(t) = -a(t)w\sin\varphi(t). \tag{23}$$

Substituting (22) and (23) into (21) yields

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(c(a)\dot{u}^2 + w^2(a)u\dot{u} - \dot{u}\,{}_0^C D^p u\right)dt \\ &\quad = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(c(a)a^2w^2\sin^2\varphi - a^2ww^2(a)\sin\varphi\right. \\ &\quad \left.\cdot\cos\varphi - aw\sin\varphi\,{}_0^C D^p u\right)dt \approx \frac{c(a)a^2w^2}{2} \\ &\quad + \frac{1}{\Gamma(1-p)}\lim_{T\to\infty}\frac{1}{T}\int_o^T\left[(aw\sin\varphi)\right. \\ &\quad \left.\cdot\int_0^t\frac{\dot{x}(t-\tau)}{\tau^\alpha}d\tau\right]dt = \frac{c(a)a^2w^2}{2} \\ &\quad - \frac{1}{\Gamma(1-p)}\lim_{T\to\infty}\frac{1}{T}\int_o^T\left(a^2w^2\sin\varphi\right. \\ &\quad \left.\cdot\int_0^t\frac{\sin\varphi\cos(w\tau) - \cos\varphi\sin(w\tau)}{\tau^\alpha}\right)dt = 0 \\ &\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(c(a)\dot{u}u + w^2(a)uu - u\,{}_0^C D^p u\right)dt \\ &\quad = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(-c(a)a^2w\sin\varphi\cos\varphi + a^2w^2(a)\right. \\ &\quad \left.\cdot\cos^2\varphi - a\cos\varphi\,{}_0^C D^p u\right)dt \approx \frac{a^2w^2(a)}{2}\end{aligned} \tag{24}$$

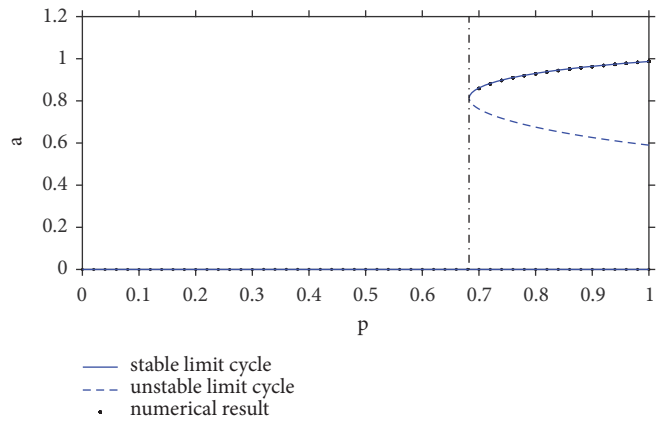


Figure 2: Bifurcation diagram of amplitude of system (28) at $D = 0$.

$$-\frac{1}{\Gamma(1-p)}\lim_{T\to\infty}\frac{1}{T}\int_o^T\left[(a\cos\varphi)\right.$$
$$\left.\cdot\int_0^t\frac{\dot{x}(t-\tau)}{\tau^\alpha}d\tau\right]dt=\frac{a^2w^2(a)}{2}+\frac{1}{\Gamma(1-p)}$$
$$\cdot\lim_{T\to\infty}\frac{1}{T}\int_o^T\left(a^2w\cos\varphi\right.$$
$$\left.\cdot\int_0^t\frac{\sin\varphi\cos(w\tau)-\cos\varphi\sin(w\tau)}{\tau^\alpha}\right)dt=0. \tag{25}$$

To simplify (24) and (25) further, asymptotic integrals are introduced as follows:

$$\begin{aligned}\int_0^t\frac{\cos(w\tau)}{\tau^p}d\tau&=w^{p-1}\left(\Gamma(1-p)\sin\left(\frac{p\pi}{2}\right)\right.\\&\left.+\frac{\sin(wt)}{(wt)^p}+o\left((wt)^{-p-1}\right)\right)\\\int_0^t\frac{\sin(w\tau)}{\tau^p}d\tau&=w^{p-1}\left(\Gamma(1-p)\cos\left(\frac{p\pi}{2}\right)\right.\\&\left.-\frac{\cos(wt)}{(wt)^p}+o\left((wt)^{-p-1}\right)\right).\end{aligned} \tag{26}$$

Substituting (26) into (24) and (25) and averaging them across φ produce the ultimate forms of $c(a)$ and $w^2(a)$ as follows:

$$\begin{aligned}c(a)&=w^{p-1}\sin\left(\frac{p\pi}{2}\right),\\w^2(a)&=w^p\cos\left(\frac{p\pi}{2}\right).\end{aligned} \tag{27}$$

Therefore, the equivalent system associated with system (18) can be expressed as follows:

$$\begin{aligned}&\ddot{u}+w_0^2u+\gamma\dot{u}-k_3u^3-\eta k_3u\,{}_0^C D^p\left(u^2\right)+k_5u^5\\&+\eta k_5u\,{}_0^C D^p\left(u^4\right)=\xi(t),\end{aligned} \tag{28}$$

where $w_0^2=w^2[1+\eta w^p\cos(p\pi/2)]$ and $\gamma=\eta w^{p+1}\sin(p\pi/2)$.

Next, we consider the stochastic P-bifurcation of system (28) which comprises the fractional derivatives of high order nonlinear terms and analyze the influence of parametric variation on the system response.

3. The Stationary PDF of Amplitude

For the system (28), the material modulus ratio is given as $\eta = 0.5$, coefficients of nonlinear terms are given as $k_3 = 7.8$, $k_5 = 5.9$, respectively, and nature frequency is given as $w = 1$. For the convenience to discuss the parametric influence, the bifurcation diagram of amplitude of the limit cycle along with variation of the fractional order p is shown in Figure 2 when $D = 0$.

As can be seen from Figure 2, the solution corresponding to the solid line is almost completely coincided with the numerical solution, which proves the correctness and accuracy of the approximate analytical result of the deterministic system; at the same time, it shows that the solution corresponding to the solid line is stable and the solution corresponding to the dotted line is unstable. And it also can be seen that there is 1 attractor in the system when p changes in the interval $[0, 0.6817]$: equilibrium, as shown in Figure 3(a); there are 2 attractors when p changes in the interval $[0.6818, 1]$: equilibrium and limit cycle, as shown in Figure 3(b).

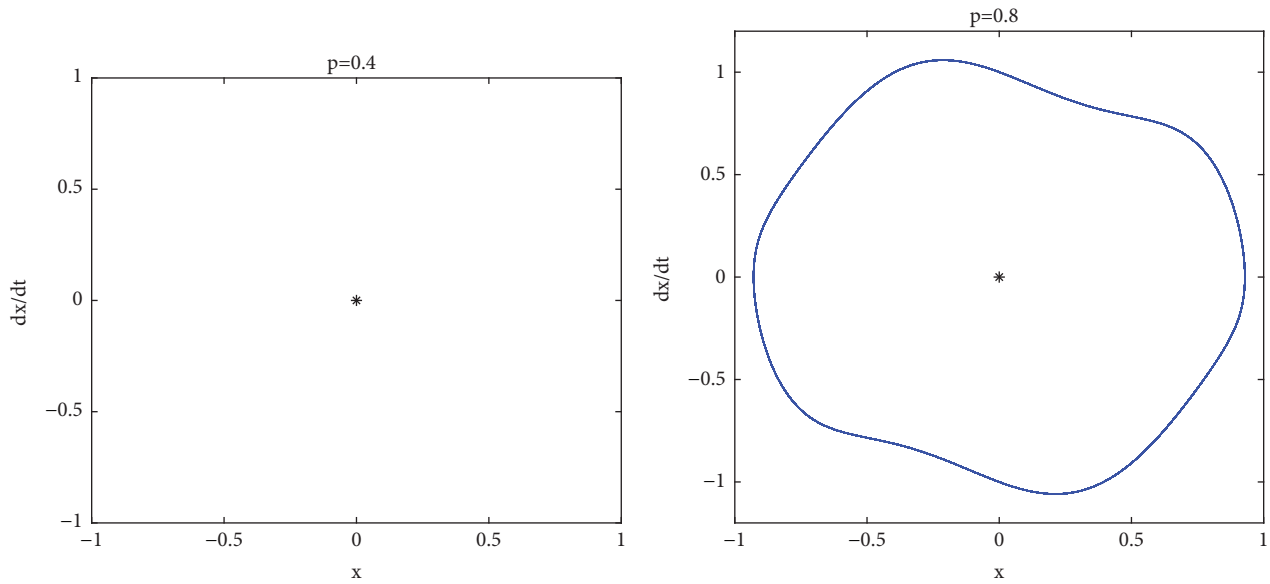
In order to obtain the stationary PDF of the amplitude of system (28), the following transformation is introduced:

$$\begin{aligned}u(t)&=a(t)\cos\Phi(t)\\\dot{u}(t)&=-a(t)w_0\sin\Phi(t)\\\Phi(t)&=w_0t+\theta(t),\end{aligned} \tag{29}$$

where w_0 is the natural frequency of the equivalent system (28), $a(t)$ and $\theta(t)$ represent the amplitude process and the phase process of the system response, respectively, and they are all random processes.

Substituting (29) into (28), we can obtain

$$\frac{da}{dt}=F_{11}(a,\theta)+G_{11}(a,\theta)\xi(t)$$



(a) Equilibrium (b) Equilibrium and limit cycle

Figure 3: Phase diagrams of system (28) at $D = 0$.

$$\frac{d\theta}{dt} = F_{21}(a,\theta) + G_{21}(a,\theta)\xi(t), \tag{30}$$

in which

$$\begin{aligned} F_{11} &= -\frac{\Delta \sin\Phi}{w_0} \\ F_{21} &= -\frac{\Delta \cos\Phi}{aw_0} \\ G_{11} &= -\frac{\sin\Phi}{w_0} \\ G_{21} &= -\frac{\cos\Phi}{aw_0}, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \Delta = {} & \eta w^{p+1} a w_0 \sin\left(\frac{p\pi}{2}\right)\sin\Phi - k_3 a^3 \cos^3\Phi - \eta k_3 \\ & \cdot \frac{a^3}{2}(2w_0)^p \cos\Phi \cos\left(2\Phi + \frac{p\pi}{2}\right) - k_5 a^5 \cos^5\Phi \\ & - \eta k_5 a^5 \cos\Phi \left[\frac{1}{8}(4w_0)^p \cos\left(4\Phi + \frac{p\pi}{2}\right)\right. \\ & \left. + \frac{1}{2}(2w_0)^p \cos\left(2\Phi + \frac{p\pi}{2}\right)\right]. \end{aligned} \tag{32}$$

Equation (30) can be regarded as a Stratonovich stochastic differential equation, by adding the corresponding Wong-

Zakai correction term: it can be transformed into the following Itô stochastic differential equation:

$$\begin{aligned} da = {} & [F_{11}(a,\theta) + F_{12}(a,\theta)]\,dt \\ & + \sqrt{2D}G_{11}(a,\theta)\,dB(t) \\ d\theta = {} & [F_{21}(a,\theta) + F_{22}(a,\theta)]\,dt \\ & + \sqrt{2D}G_{21}(a,\theta)\,dB(t), \end{aligned} \tag{33}$$

where $B(t)$ is a standard Wiener process and

$$\begin{aligned} F_{12}(a,\theta) &= D\frac{\partial G_{11}}{\partial a}G_{11} + D\frac{\partial G_{11}}{\partial \theta}G_{21} \\ F_{22}(a,\theta) &= D\frac{\partial G_{21}}{\partial a}G_{11} + D\frac{\partial G_{21}}{\partial \theta}G_{21}. \end{aligned} \tag{34}$$

By the stochastic averaging method, averaging (33) regarding Φ, the following averaged Itô equation can be obtained as follows:

$$\begin{aligned} da &= m_1(a)\,dt + \sigma_1(a)\,dB(t) \\ d\theta &= m_2(a)\,dt + \sigma_2(a)\,dB(t), \end{aligned} \tag{35}$$

where

$$\begin{aligned} m_1(a) = {} & -\frac{\eta k_5 2^p w_0^{p-1} a^5 \sin(p\pi/2)}{8} \\ & - \frac{\eta k_3 2^p w_0^{p-1} a^3 \sin(p\pi/2)}{8} \\ & - \frac{\eta w^{p+1} a \sin(p\pi/2)}{2} + \frac{D}{2aw_0^2} \\ {\sigma_1}^2(a) = {} & \frac{D}{w_0^2} \\ m_2(a) = {} & \frac{k_5\left(2\eta \cdot 2^p w_0^p \cos(p\pi/2) + 5\right)a^4}{16w_0} \end{aligned}$$

$$+\frac{k_3\left(\eta\cdot 2^p w_0^p\cos\left(p\pi/2\right)+3\right)a^2}{8w_0}$$

$$\sigma_2{}^2(a)=\frac{D}{a^2w_0^2}. \tag{36}$$

Equations (35) and (36) show that the averaged Itô equation for $a(t)$ is independent of $\theta(t)$ and the random process $a(t)$ is a one-dimensional diffusion process. The corresponding FPK equation associated with $a(t)$ can be written as

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial a}\left[m_1(a)\,p\right]+\frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\sigma_1{}^2(a)\,p\right]. \tag{37}$$

The boundary conditions are

$$p=c,\ c\in(-\infty,+\infty)\quad \text{when } a=0$$

$$p\longrightarrow 0,\ \frac{\partial p}{\partial a}\longrightarrow 0\quad \text{when } a\longrightarrow\infty. \tag{38}$$

Thus, based on these boundary conditions (38), we can obtain the stationary PDF of amplitude a as follows:

$$p(a)=\frac{C}{\sigma_1{}^2(a)}\exp\left[\int_0^a\frac{2m_1(u)}{\sigma_1{}^2(u)}du\right], \tag{39}$$

where C is a normalized constant, which satisfies

$$C=\left[\int_0^\infty\left(\frac{1}{\sigma_1{}^2(a)}\exp\left[\int_0^a\frac{2m_1(u)}{\sigma_1{}^2(u)}du\right]\right)da\right]^{-1}. \tag{40}$$

Substituting (36) into (39), the explicit expression for the stationary PDF of amplitude a can be obtained as follows:

$$p(a)=\frac{Caw_0^2}{D}\exp\left[-\frac{\eta a^2\sin\left(p\pi/2\right)\left(k_5 2^{p+1}a^4w_0^{p+1}+3k_3 2^p a^2 w_0^{p+1}+24w_0^2w^{p+1}\right)}{48D}\right], \tag{41}$$

where $w_0^2=w^2[1+\eta w^p\cos(p\pi/2)]$.

4. Stochastic P-Bifurcation

Stochastic P-bifurcation refers to the changes of the number of peaks in the PDF curve; in order to obtain the critical parametric condition for stochastic P-bifurcation, the influences of the parameters for stochastic P-bifurcation of the system are analyzed by using the singularity theory below.

For the sake of convenience, we can write $p(a)$ as follows:

$$\begin{aligned}p(a)&=CR\left(a,D,w,\eta,p,k_3,k_5\right)\\&\quad\cdot\exp\left[Q\left(a,D,w,\eta,p,k_3,k_5\right)\right],\end{aligned} \tag{42}$$

in which

$$\begin{aligned}&R\left(a,D,w,\eta,p,k_3,k_5\right)=\frac{aw_0^2}{D}\\&Q\left(a,D,w,\eta,p,k_3,k_5\right)\\&=-\frac{\eta a^2\sin\left(p\pi/2\right)\left(k_5 2^{p+1}a^4w_0^{p+1}+3k_3 2^p a^2 w_0^{p+1}+24w_0^2w^{p+1}\right)}{48D}.\end{aligned} \tag{43}$$

According to the singularity theory, the stationary PDF needs to satisfy the following two conditions:

$$\begin{aligned}\frac{\partial p(a)}{\partial a}&=0,\\\frac{\partial^2 p(a)}{\partial a^2}&=0.\end{aligned} \tag{44}$$

Substituting (42) into (44), we can obtain the following condition [49]:

$$\begin{aligned}&H\\&=\left\{R'+RQ'=0,R''+2R'Q'+RQ''+RQ'^2=0\right\},\end{aligned} \tag{45}$$

where H represents the critical condition for the changes of the number of peaks in the PDF curve.

Substituting (43) into (45), we can get the critical parametric condition for stochastic P-bifurcation of the system as follows:

$$\begin{aligned}D&=\frac{1}{8}\eta k_5 2^{p+1}w_0^{p+1}\sin\left(\frac{p\pi}{2}\right)a^6\\&\quad+\frac{1}{8}\eta k_3 2^{p+1}w_0^{p+1}\sin\left(\frac{p\pi}{2}\right)a^4\\&\quad+\eta w_0^2w^{p+1}\sin\left(\frac{p\pi}{2}\right)a^2,\end{aligned} \tag{46}$$

where $w_0^2=w^2[1+\eta w^p\cos(p\pi/2)]$.

And amplitude a satisfies

$$3k_5 2^p w_0^{p+1}a^4+2k_3 2^p w_0^{p+1}a^2+4w_0^2w^{p+1}=0. \tag{47}$$

Taking the parameters as $\eta=0.5$, $k_3=7.8$, $k_5=5.9$, and $w=1$, according to (46) and (47), the transition set for stochastic P-bifurcation of the system with the unfolding parameters p and D can be obtained, as shown in Figure 4.

According to the singularity theory, the topological structures of the stationary PDF curves of different points (p,D) in the same region are qualitatively the same. Taking a point (p,D) in each region, all varieties of the stationary PDF curves which are qualitatively different could be obtained. It can be seen that the unfolding parameter $p-D$ plane is divided into two subregions by the transition set curve;

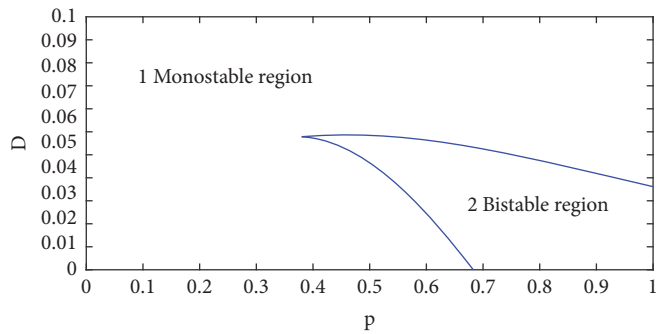
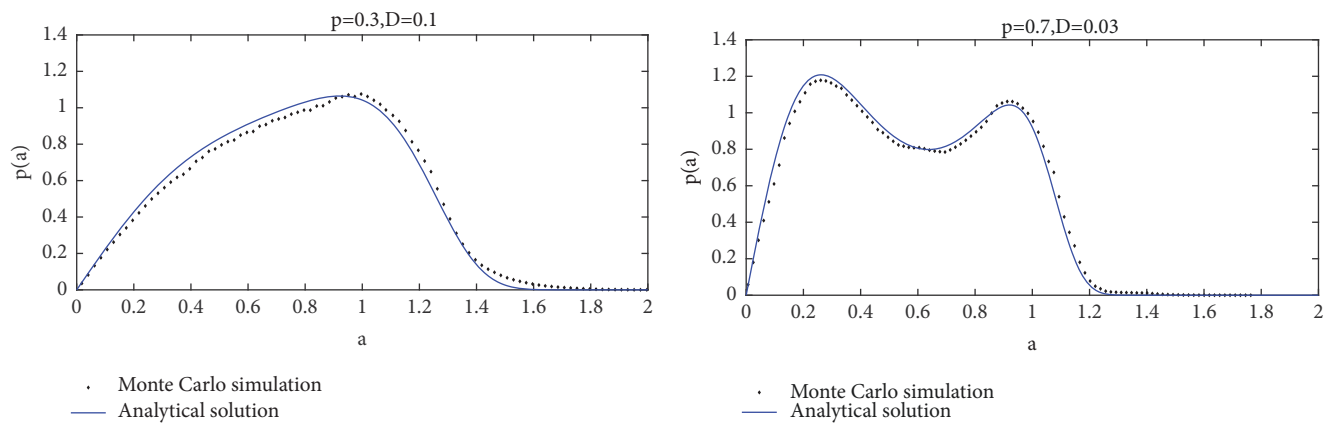


Figure 4: The transition set curve of system (28).



(a) parameter (p, D) in region 1 of Figure 4

(b) parameter (p, D) in region 2 of Figure 4

Figure 5: The stationary PDF curves for the amplitude of system (28).

for convenience, each region in Figure 4 is marked with a number.

Taking a given point (p, D) in each of the two subregions of Figure 4, the characteristics of stationary PDF curves are analyzed, and the corresponding results are shown in Figure 5.

As can be seen from Figure 4, the parametric region where the PDF curve appears bimodal is surrounded by an approximately triangular region. When the parameter (p, D) is taken in the region 1, the PDF curve has only a distinct peak, as shown in Figure 5(a); in region 2, the PDF curve has a distinct peak near the origin, but the probability is obviously not zero far away from the origin; there are both the equilibrium and limit cycle in the system simultaneously, as shown in Figure 5(b).

The analysis results above show that the stationary PDF curves of the system amplitude in any two adjacent regions in Figure 4 are qualitatively different. No matter the values of the unfolding parameters cross any line in the figure, the system will occur stochastic P-bifurcation behaviors, so the transition set curve is just the critical parametric condition for the stochastic P-bifurcation of the system, and the analytic results in Figure 5 are in good agreement with the numerical results by Monte Carlo simulation, which further verify the correctness of the theoretical analysis.

5. Conclusion

In this paper, we studied the stochastic P-bifurcation for axially moving of a bistable viscoelastic beam model with fractional derivatives of high order nonlinear terms under Gaussian white noise excitation. According to the minimum mean square error principle, we transformed the original system into an equivalent and simplified system and obtained the stationary PDF of the system amplitude using stochastic averaging. In addition, we obtained the critical parametric condition for stochastic P-bifurcation of the system using singularity theory; based on this, the system response can be maintained at the monostability or small amplitude near the equilibrium by selecting the appropriate unfolding parameters, providing theoretical guidance for system design in practical engineering, and avoiding the instability and damage caused by the large amplitude vibration or nonlinear jump phenomenon of the system. Finally, the numerical results by Monte Carlo simulation of the original system also verify the theoretical results obtained in this paper. We conclude that the order p of fractional derivative and the noise intensity D can both cause stochastic P-bifurcation of the system and the number of peaks in the stationary PDF curve of system amplitude can change from 2 to 1 by selecting the appropriate unfolding parameters (p, D); it also shows that the method used in this paper is feasible to analyze

the stochastic P-bifurcation behaviors of viscoelastic material system with fractional constitutive relation.

Data Availability

Data that support the findings of this study are included within the article (and its additional files). All other data are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National 973 Project of China (Grant no. 2014CB046805) and the Natural Science Foundation of China (Grant no. 11372211).

References

[1] W. Tan and M. Xu, "Plane surface suddenly set in motion in a viscoelastic fluid with fractional Maxwell model," *Acta Mechanica Sinica. English Series. The Chinese Society of Theoretical and Applied Mechanics*, vol. 18, no. 4, pp. 342–349, 2002.

[2] J. Sabatier, O. P. Agrawal, and J. A. Machado, *Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[3] I. Podlubny, "Fractional order systems and PID controller," *IEEE Transactions on Automatic Control*, vol. 44, no. 1, pp. 208–214, 1999.

[4] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls: Fundamentals and Applications*, Advances in Industrial Control, Springer, London, UK, 2010.

[5] Q. Li, Y. L. Zhou, X. Q. Zhao, and X. Y. Ge, "Fractional order stochastic differential equation with application in European option pricing," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 621895, 12 pages, 2014.

[6] S. Nie, H. Sun, Y. Zhang et al., "Vertical distribution of suspended sediment under steady flow: existing theories and fractional derivative model," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 5481531, 11 pages, 2017.

[7] X. Liu, L. Hong, H. Dang, and L. Yang, "Bifurcations and synchronization of the fractional-order Bloch system," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 4694305, 10 pages, 2017.

[8] W. Li and Y. Pang, "Asymptotic solutions of time-space fractional coupled systems by residual power series method," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 7695924, 10 pages, 2017.

[9] Z. Liu, T. Xia, and J. Wang, "Image encryption technology based on fractional two-dimensional triangle function combination discrete chaotic map coupled with Menezes-Vanstone elliptic curve cryptosystem," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 4585083, 24 pages, 2018.

[10] C. Sánchez-López, V. H. Carbajal-Gómez, M. A. Carrasco-Aguilar, and I. Carro-Pérez, "Fractional-Order Memristor Emulator Circuits," *Complexity*, vol. 2018, Article ID 2806976, 10 pages, 2018.

[11] K. Rajagopal, A. Karthikeyan, P. Duraisamy, and R. Weldegiorgis, "Bifurcation and Chaos in Integer and Fractional Order Two-Degree-of-Freedom Shape Memory Alloy Oscillators," *Complexity*, vol. 2018, 9 pages, 2018.

[12] Ying Li and Ye Tang, "Analytical Analysis on Nonlinear Parametric Vibration of an Axially Moving String with Fractional Viscoelastic Damping," *Mathematical Problems in Engineering*, vol. 2017, Article ID 1393954, 9 pages, 2017.

[13] G. Liu, C. Ye, X. Liang, Z. Xie, and Z. Yu, "Nonlinear Dynamic Identification of Beams Resting on Nonlinear Viscoelastic Foundations BASed on the Time-DELayed TRAnsfer Entropy and IMProved Surrogate DATa Algorithm," *Mathematical Problems in Engineering*, vol. 2018, Article ID 6531051, 14 pages, 2018.

[14] D. Liu, Y. Wu, and X. Xie, "Stochastic Stability Analysis of Coupled Viscoelastic SYStems with Nonviscously Damping DRIven by White NOIse," *Mathematical Problems in Engineering*, vol. 2018, Article ID 6532305, 16 pages, 2018.

[15] P. G. Nutting, "A new general law of deformation," *Journal of The Franklin Institute*, vol. 191, no. 5, pp. 679–685, 1921.

[16] A. Gemant, "A method of analyzing experimental results obtained from elasto-viscous bodies," *Journal of Applied Physics*, vol. 7, no. 8, pp. 311–317, 1936.

[17] G. S. Blair and J. Caffyn, "VI. An application of the theory of quasi-properties to the treatment of anomalous strain-stress relations," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 40, no. 300, pp. 80–94, 2010.

[18] C. Mote, "Dynamic Stability of Axially Moving Materals," *Shock and Vibration*, vol. 4, no. 4, pp. 2–11, 1972.

[19] A. Ulsoy and C. Mote, "Band Saw Vibration and Stability," *Shock and Vibration*, vol. 10, no. 11, pp. 3–15, 1978.

[20] J. A. Wickert and C. D. Mote Jr., "Current research on the vibration and stability of axially-moving materials," *Shock and Vibration*, vol. 20, no. 1, pp. 3–13, 1988.

[21] K. W. Wang and S. P. Liu, "On the noise and vibration of chain drive systems," *Shock and Vibration Digest*, vol. 23, no. 4, pp. 8–13, 1991.

[22] F. Pellicano and F. Vestroni, "Complex dynamics of high-speed axially moving systems," *Journal of Sound and Vibration*, vol. 258, no. 1, pp. 31–44, 2002.

[23] N.-H. Zhang and L.-Q. Chen, "Nonlinear dynamical analysis of axially moving viscoelastic strings," *Chaos, Solitons & Fractals*, vol. 24, no. 4, pp. 1065–1074, 2005.

[24] M. H. Ghayesh, H. A. Kafiabad, and T. Reid, "Sub- and supercritical nonlinear dynamics of a harmonically excited axially moving beam," *International Journal of Solids and Structures*, vol. 49, no. 1, pp. 227–243, 2012.

[25] M. H. Ghayesh, "Stability and bifurcations of an axially moving beam with an intermediate spring support," *Nonlinear Dynamics*, vol. 69, no. 1-2, pp. 193–210, 2012.

[26] R. Rodríguez, E. Salinas-Rodríguez, and J. Fujioka, "Fractional Time Fluctuations in Viscoelasticity: A Comparative Study of Correlations and Elastic Moduli," *Entropy*, vol. 20, no. 1, p. 28, 2018.

[27] R. L. Bagley and P. J. Torvik, "Fractional calculus-a different approach to the analysis of viscoelastically damped structures," *AIAA Journal*, vol. 21, no. 5, pp. 741–748, 2012.

[28] R. L. Bagley and P. J. Torvik, "Fractional calculus in the transient analysis of viscoelastically damped structures," *AIAA Journal*, vol. 23, no. 6, pp. 918–925, 1985.

[29] M. Pakdemirli and A. G. Ulsoy, "Stability analysis of an axially accelerating string," *Journal of Sound and Vibration*, vol. 203, no. 5, pp. 815–832, 1997.

[30] L. Zhang and J. W. Zu, "Nonlinear vibration of parametrically excited moving belts, part I: Dynamic response," *Journal of Applied Mechanics*, vol. 66, no. 2, pp. 396–402, 1999.

[31] L. Zhang and J. W. Zu, "Nonlinear vibration of parametrically excited viscoelastic moving belts, part II: Stability analysis," *Journal of Applied Mechanics*, vol. 66, no. 2, pp. 403–409, 1999.

[32] L. Q. Chen, X. D. Yang, and C. J. Cheng, "Dynamic stability of an axially accelerating viscoelastic beam," *European Journal of Mechanic*, vol. 23, no. 4, pp. 659–666, 2004.

[33] L. Q. Chen and X. D. Yang, "Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models," *International Journal of Solids and Structures*, vol. 42, no. 1, pp. 37–50, 2005.

[34] L. Q. Chen and X. D. Yang, "Transverse nonlinear dynamics of axially accelerating viscoelastic beams based on 4-term Galerkin truncation," *Chaos Soliton and Fractal*, vol. 27, no. 3, pp. 748–757, 2006.

[35] H. Ding and L.-Q. Chen, "Galerkin methods for natural frequencies of high-speed axially moving beams," *Journal of Sound and Vibration*, vol. 329, no. 17, pp. 3484–3494, 2010.

[36] L.-Q. Chen, J. Wu, and J. W. Zu, "Asymptotic nonlinear behaviors in transverse vibration of an axially accelerating viscoelastic string," *Nonlinear Dynamics*, vol. 35, no. 4, pp. 347–360, 2004.

[37] L.-Q. Chen, N.-H. Zhang, and J. W. Zu, "The regular and chaotic vibrations of an axially moving viscoelastic string based on fourth order Galerkin truncation," *Journal of Sound and Vibration*, vol. 261, no. 4, pp. 764–773, 2003.

[38] A. Y. T. Leung, H. X. Yang, and J. Y. Chen, "Parametric bifurcation of a viscoelastic column subject to axial harmonic force and time-delayed control," *Computers & Structures*, vol. 136, pp. 47–55, 2014.

[39] M. H. Ghayesh and N. Moradian, "Nonlinear dynamic response of axially moving, stretched viscoelastic strings," *Archive of Applied Mechanics*, vol. 81, no. 6, pp. 781–799, 2011.

[40] D. Liu, W. Xu, and Y. Xu, "Dynamic responses of axially moving viscoelastic beam under a randomly disordered periodic excitation," *Journal of Sound and Vibration*, vol. 331, no. 17, pp. 4045–4056, 2012.

[41] T. Z. Yang and B. Fang, "Stability in parametric resonance of an axially moving beam constituted by fractional order material," *Archive of Applied Mechanics*, vol. 82, no. 12, pp. 1763–1770, 2012.

[42] A. Y. T. Leung, H. X. Yang, P. Zhu, and Z. J. Guo, "Steady state response of fractionally damped nonlinear viscoelastic arches by residue harmonic homotopy," *Computers & Structures*, vol. 121, pp. 10–21, 2013.

[43] A. C. Galucio, J.-F. Deüi, and R. Ohayon, "Finite element formulation of viscoelastic sandwich beams using fractional derivative operators," *Computational Mechanics*, vol. 33, no. 4, pp. 282–291, 2004.

[44] L. Chen, W. Wang, Z. Li, and W. Zhu, "Stationary response of Duffing oscillator with hardening stiffness and fractional derivative," *International Journal of Non-Linear Mechanics*, vol. 48, pp. 44–50, 2013.

[45] L. C. Chen, Z. S. Li, Q. Zhuang, and W. Q. Zhu, "First-passage failure of single-degree-of-freedom nonlinear oscillators with fractional derivative," *Journal of Vibration and Control*, vol. 19, no. 14, pp. 2154–2163, 2013.

[46] Y. Shen, S. Yang, H. Xing, and G. Gao, "Primary resonance of Duffing oscillator with fractional-order derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 7, pp. 3092–3100, 2012.

[47] Y. G. Yang, W. Xu, Y. H. Sun, and X. D. Gu, "Stochastic response of van der Pol oscillator with two kinds of fractional derivatives under Gaussian white noise excitation," *Chinese Physics B*, vol. 25, no. 2, pp. 13–21, 2016.

[48] L. Chen and W. Zhu, "Stochastic response of fractional-order van der Pol oscillator," *Theoretical & Applied Mechanics Letters*, vol. 4, no. 1, pp. 68–72, 2014.

[49] W. Li, M. Zhang, and J. Zhao, "Stochastic bifurcations of generalized Duffing–van der Pol system with fractional derivative under colored noise," *Chinese Physics B*, vol. 26, no. 9, pp. 62–69, 2017.