# Traveling Waves in the Underdamped Frenkel-Kontorova Model 

Hengyan Li (1) ${ }^{1}$ and Shaowei Liu ( ${ }^{2}{ }^{2}$<br>${ }^{1}$ School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China<br>${ }^{2}$ School of Energy Science and Engineering, Henan Polytechnic University, Jiaozuo 454000, China<br>Correspondence should be addressed to Shaowei Liu; lsw770320@hpu.edu.cn

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This paper studies a damped Frenkel-Kontorova model with periodic boundary condition. By using Nash-Moser iteration scheme, we prove that such model has a family of smooth traveling wave solutions.

## 1. Introduction

The present work concerns the existence of traveling wave solutions for the following underdamped Frenkel-Kontorova model:

$$
\begin{equation*}
\psi_{j}^{\prime \prime}+\Gamma \psi_{j}^{\prime}+\sin \left(\psi_{j}\right)=F+K\left[\psi_{j+1}-2 \psi_{j}+\psi_{j-1}\right] \tag{1}
\end{equation*}
$$

$$
\forall j \in \mathbb{Z}
$$

.
with periodic boundary condition

$$
\begin{equation*}
\psi_{j+n}(t)=\psi_{j}(t)+2 m \pi, \quad \forall j \in \mathbb{Z} \tag{2}
\end{equation*}
$$

where the parameters $\Gamma>0, K>0, F>0$, and $m \geq 1$.
In the last decades, there has been large growth in the study of the existence and stability of traveling wave solutions for lattice systems including Frenkel-Kontorova model (discrete sine-Gordon equations), which arises from many physical systems, such as circular arrays of Josephson junctions, glassy materials, sliding friction, adsorbate layer on the surface of a crystal, ionic conductors, and mechanical interpretation as a model for a ring of pendula coupled by torsional springs (see $[1-4]$ ). When $\Gamma=0$, system (4) is conservative. Baesens and MacKay [5] proved the existence and also global stability of traveling waves. When $\Gamma>0$, system (4) is dissipative. Under the condition $\Gamma>2 \sqrt{2 K+1}$ and $\epsilon=1$, Baesens and MacKay [6] showed that the traveling wave solution is globally stable if and only if (4) and (2) do not have stationary solutions. Levi in [7] pointed out that
the local stability of traveling waves can be obtained by the monotonicity method in [8]. Under the condition $\Gamma>2 \sqrt{2 K}$ and $F \geq F_{0}>1$, Qin et al. [9] investigated the stability of single-wave-form for the underdamped Frenkel-Kontorova model (4) by the monotonicity method.

Recently, by using Schauder fixed point theorem, Mirollo and Rosen [10] and Katriel [11] have obtained a series of results about the existence of traveling waves for (4) with periodic boundary condition (2). Katriel [11] proved the following:
(1) Fixing any $\Gamma>0$ and $K>0$ and given any velocity $v>0$, there exists a traveling wave solution of (4) and (2) with velocity $v$ for an appropriate $F>0$.
(2) For any $F>1$ there exists a traveling wave solution of (4) and (2).
(3) Assume that $n$ does not divide $m$. Fixing any $\widetilde{F}>0$ and $\widetilde{\Gamma}>0$, for all $K$ sufficiently large there exists a traveling wave solution of (4) and (2) for any $F \geq \widetilde{F}>0$ and $\Gamma \geq \widetilde{\Gamma}>0$.
(4) Fixing any $\widetilde{F}>0$ and $\widetilde{\Gamma}>0$, for all $\Gamma>0$ sufficiently small there exists a traveling wave solution of (4) and (2) for any $F \geq \widetilde{F}>0$ and $0<K \leq \widetilde{K}$.

In the final of Katriel's paper, he gave several open problems. One of them is the following: Is it true that, fixing $\Gamma>0$ and $K>0$, for sufficiently small $F>0$ and small applied force, a traveling wave does exist? If $n$ divides $m$, what is the situation of the existence of traveling waves for (4) with periodic boundary condition (2)? In fact, if $n$ divides $m$, there appears the "small divisor." Then, the problem is difficult.

Levi et al. [12] showed that, for fixing $\Gamma>0$, (4) possesses a traveling wave only when $F$ exceeds a positive critical value.

In this paper, we will construct a new Nash-Moser iteration to answer the open problem mentioned above. This method has been used in solving the existence of periodic solutions for nonlinear elliptic equations [13], nonlinear wave equations [14-18], and standing waves [19]. Here, we try to use this method to study the existence of traveling wave solutions for dissipative and conservative lattice systems.

Instead of looking for solutions of (1) in a shrinking neighborhood of zero, it is a convenient device to perform the rescaling

$$
\begin{equation*}
\psi \longrightarrow \epsilon \psi, \quad \epsilon>0 \tag{3}
\end{equation*}
$$

having

$$
\begin{align*}
\psi_{j}^{\prime \prime} & +\Gamma \psi_{j}^{\prime}+\epsilon^{-1} \sin \left(\epsilon \psi_{j}\right)  \tag{4}\\
& =\epsilon^{-1} F+K\left[\psi_{j+1}-2 \psi_{j}+\psi_{j-1}\right], \quad \forall j \in \mathbb{Z}
\end{align*}
$$

To overcome the "small divisor" problem, we need the following nonresonance conditions:

$$
\begin{align*}
& \text { (NR1) } \mathcal{O}_{\gamma, \tau}:=\{\omega \in \Omega:|k v(\omega)-1| \\
& \left.\quad \geq \frac{\gamma}{|k|^{\tau}} \text { or }|k v(\omega)+1| \geq \frac{\gamma}{|k|^{\tau}} \forall k \in \mathbb{Z}, k \neq 0\right\} \neq \emptyset \\
& \text { (NR2) } \overline{\mathcal{O}}_{\gamma, \tau}:=\{\omega \in \Omega:|k v(\omega) \pm 1|  \tag{5}\\
& \quad \geq \frac{\gamma}{|k|^{\tau}} \text { or }|k v(\omega) \pm i \sqrt{1-4 K}| \geq \frac{\gamma}{|k|^{\tau}} \forall k \in \mathbb{Z}, k \\
& \quad \neq 0\} \neq \emptyset
\end{align*}
$$

where $\Omega \subset \mathbb{R}$ is a bounded region.
It is shown in [20] that if, for some $l>0$,

$$
\begin{equation*}
\left|v^{(l)}(\omega)\right| \geq d>0 \quad \text { on } \omega \in \Omega \tag{6}
\end{equation*}
$$

then the Lebesgue measure

$$
\begin{align*}
\left|\Omega \backslash \mathcal{O}_{\gamma, \tau}\right|=O\left(\gamma^{1 / l}\right) \longrightarrow & 0 \\
\left|\Omega \backslash \overline{\mathcal{O}}_{\gamma, \tau}\right|=O\left(\gamma^{1 / l}\right) \longrightarrow & 0  \tag{7}\\
& \text { as } \gamma \longrightarrow 0 .
\end{align*}
$$

Now, we state our main result.
Theorem 1. Under the assumption (NR1), fixing any $K>0$ and sufficient small $F>0$, there exist $\Gamma_{0}>0, \epsilon_{0}>0$, and $0<\gamma<1 \leq \tau$, such that, for any $\zeta:=v \Gamma \in\left[0, \Gamma_{0}\right], \epsilon \in$ $\left[0, \epsilon_{0}\right]$, and $\omega \in \mathcal{O}_{\gamma, \tau}$, (4) with periodic boundary condition (2) possesses a unique traveling wave solution $u(\theta ; \zeta)+\theta$, where $\theta \in \mathbb{T}:=\mathbb{R} / 2 \pi$.

When $\Gamma=0$, (4) is
$\psi_{j}^{\prime \prime}+\epsilon^{-1} \sin \left(\epsilon \psi_{j}\right)=\epsilon^{-1} F+K\left[\psi_{j+1}-2 \psi_{j}+\psi_{j-1}\right]$,
$\forall j \in \mathbb{Z}$.
The corresponding Hamiltonian of (8) is

$$
\begin{align*}
H= & \sum_{j} \frac{1}{2}\left(\frac{d \psi_{j}}{d t}\right)^{2}+\epsilon^{-1}\left(1-\cos \epsilon \psi_{j}\right)+\epsilon^{-1} F \psi_{j}  \tag{9}\\
& +W\left(\left\{\psi_{j}\right\}\right)
\end{align*}
$$

where the nearest-neighbor coupling potential is

$$
\begin{equation*}
W\left(\left\{\psi_{j}\right\}\right)=\frac{K}{2}\left(\psi_{j+1}-\psi_{j}\right)^{2} . \tag{10}
\end{equation*}
$$

We have the following result about the existence of traveling waves for (8).

Theorem 2. Under the assumption (NR2), fixing any $K>0$, there exist $F_{*}>0, \epsilon_{0}>0$, and $0<\gamma<1 \leq \tau$, such that, for any $\epsilon \in\left[0, \epsilon_{0}\right]$ and $\omega \in \overline{\mathcal{O}}_{\gamma, \tau}$, (4) with periodic boundary condition (2) possesses a unique traveling wave solution $u(\theta ; \zeta)+\theta$, where $\theta \in \mathbb{T}:=\mathbb{R} / 2 \pi$.

This paper is organized as follows. In Section 2, we first establish a Nash-Moser theorem for the case of $\Gamma>0$. Then, we apply this result to prove our main results. The case of $\Gamma=$ 0 is also considered.

## 2. Proof of the Main Results

2.1. The Case of $\Gamma>0$. In numerical simulations or experimental works on (4) with periodic boundary condition (2), it is observed that solutions often converge to a traveling wave

$$
\begin{equation*}
\psi_{j}(t)=\varphi\left(t+j \frac{m}{n} T\right) \tag{11}
\end{equation*}
$$

where the waveform $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

$$
\begin{equation*}
\varphi(t+T)=\varphi(t)+2 \pi, \quad \forall t \in \mathbb{R} \tag{12}
\end{equation*}
$$

$\varphi$ is a waveform if and only if it satisfies (12) and

$$
\begin{align*}
\varphi^{\prime \prime}(t) & +\Gamma \varphi^{\prime}(t)+\epsilon^{-1} \sin (\epsilon \varphi(t)) \\
= & \epsilon^{-1} F  \tag{13}\\
& +K\left[\varphi\left(t+\frac{m}{n} T\right)-2 \varphi(t)+\varphi\left(t-\frac{m}{n} T\right)\right] .
\end{align*}
$$

Hence, as in [11], we investigate the traveling wave of the type

$$
\begin{equation*}
\varphi(t)=u(v t)+v t \tag{14}
\end{equation*}
$$

where the wave velocity $v=2 \pi / T=2 \pi \omega$ and $u$ satisfies

$$
\begin{equation*}
u(\theta+2 \pi)=u(\theta), \quad \forall \theta \in \mathbb{R} \tag{15}
\end{equation*}
$$

Inserting (14) into (13), we get

$$
\begin{align*}
v^{2} u^{\prime \prime} & (\theta)+\zeta u^{\prime}(\theta)+\epsilon^{-1} \sin (\epsilon \theta+\epsilon u(\theta)) \\
= & K\left[u\left(\theta+2 \pi \frac{m}{n}\right)-2 u(\theta)+u\left(\theta-2 \pi \frac{m}{n}\right)\right]  \tag{16}\\
& +\epsilon^{-1} F-\zeta .
\end{align*}
$$

Write

$$
\begin{equation*}
\sin (\epsilon \theta+\epsilon u(\theta))=\sin (\epsilon \theta)+2 \epsilon \cos (\epsilon \theta) u+g(u) . \tag{17}
\end{equation*}
$$

We consider the following space:

$$
\begin{align*}
X_{\sigma} & :=\left\{u(\theta):=\left.\sum_{k \in \mathbb{Z}^{2}} u_{k} e^{i k \cdot \theta}\left|\|u\|_{\sigma}^{2}:=\sum_{k \in \mathbb{Z}}\right| u_{k}\right|^{2} e^{2 \sigma|k|}\right. \\
& <\infty\} \tag{18}
\end{align*}
$$

where $u_{k}$ denotes the $k$ the Fourier coefficient.
Obviously, for a nested family of Banach spaces $\left\{X_{\sigma}: \sigma \geq\right.$ $0\}$, there holds

$$
\begin{gather*}
X_{\sigma_{2}} \subset X_{\sigma_{1}} \\
\|u\|_{\sigma_{1}} \leq\|u\|_{\sigma_{2}} \tag{19}
\end{gather*}
$$

$$
\forall 0 \leq \sigma_{1} \leq \sigma_{2}
$$

For $\sigma \geq 0$, the space $X_{\sigma}$ is Banach algebra with respect to multiplication of functions; that is, if $u_{1}, u_{2} \in X_{\sigma}$, then $u_{1} u_{2} \in X_{\sigma}$ and there exists a positive constant $C$, such that

$$
\begin{equation*}
\left\|u_{1} u_{2}\right\|_{\sigma} \leq C\left\|u_{1}\right\|_{\sigma}\left\|u_{2}\right\|_{\sigma} \tag{20}
\end{equation*}
$$

It is obviously that each function in $X_{\sigma}$ has a bounded analytic extension in the complex multistrip $|\operatorname{Im} \theta|<\sigma$, where $\theta \in$ C. By the definition of the space $X_{\sigma}$, the following inequality holds:

$$
\begin{equation*}
\left\|\partial_{\theta}^{h} u\right\|_{\sigma} \leq\left\|\partial_{\theta}^{l} u\right\|_{\sigma}, \quad \forall h, l \in N: h \leq l . \tag{21}
\end{equation*}
$$

For uniqueness, we assume that $u$ satisfies

$$
\begin{equation*}
\langle u\rangle:=\int_{\mathbb{T}} u(s) d s=0, \quad \mathbb{T}=\frac{\mathbb{R}}{2 \pi} . \tag{22}
\end{equation*}
$$

Now we define a function space with zero average by

$$
\begin{equation*}
X_{\sigma}^{0}:=\left\{u \in X_{\sigma} \text { s.t. } u^{0}=\langle u\rangle=0\right\}, \tag{23}
\end{equation*}
$$

as the closed subspace of $X_{\sigma}$.
Let $\widetilde{\sigma}>\sigma>0$. Then, we define
$Y_{\sigma}$ denotes the set of functions $u \in \mathbb{C}^{\infty}\left(\mathbb{T} \times\left[0, \Gamma_{0}\right]\right)$ such that, for all $\zeta \in\left[0, \Gamma_{0}\right], \theta \rightarrow u(\theta ; \zeta) \in X_{\sigma}^{0}$.
$\mathscr{W}_{\sigma}$ denotes the set $Y_{\sigma} \times \mathbb{C}^{\infty}\left(\left[0, \Gamma_{0}\right]\right)$.
Denote operator $\mathscr{A}:==^{\prime}$. Then (16) can be written as

$$
\begin{align*}
& v^{2} \mathscr{A}^{2} u(\theta)+\zeta \mathscr{A} u(\theta)+\sin (\theta+u(\theta)) \\
&= F-\zeta  \tag{24}\\
&+K\left[u\left(\theta+2 \pi \frac{m}{n}\right)-2 u(\theta)+u\left(\theta-2 \pi \frac{m}{n}\right)\right] .
\end{align*}
$$

We define an operator $\mathscr{L}: X_{\sigma} \rightarrow X_{\sigma}$ by $\mathscr{L} u$

$$
\begin{align*}
:= & v^{2} \mathscr{A}^{2} u(\theta)+\zeta \mathscr{A} u(\theta) \\
& -K\left[u\left(\theta+2 \pi \frac{m}{n}\right)-2 u(\theta)+u\left(\theta-2 \pi \frac{m}{n}\right)\right]  \tag{25}\\
& +(1-i \Gamma) .
\end{align*}
$$

Then, (24) can be written as

$$
\begin{align*}
\mathscr{F}(u, \zeta)= & \mathscr{L} u+\epsilon \sin (\theta+u(\theta))+(i \Gamma-1) u(\theta)+\zeta  \tag{26}\\
& -F=0 .
\end{align*}
$$

We have the following properties about operator $\mathscr{L}$.
Lemma 3. Fix the following $\Gamma>0$ and $K>0$. The "diagonal" operator $\mathscr{L}$ (on Fourier spaces) satisfies the following:
(1) $\forall u \in \mathbb{X}_{\sigma}^{0}$,

$$
\begin{equation*}
\mathscr{L} u=\mathscr{L}\left(\sum_{k \in \mathbb{Z}} a_{l} e^{i k \theta}\right)=\sum_{k \in \mathbb{Z}} \lambda a_{l} e^{i k \theta} \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
\lambda:= & (k v-1)(i \Gamma-1-k v) \\
& -2 K\left(\cos \left(\frac{2 m k \pi}{n}\right)-1\right) . \tag{28}
\end{align*}
$$

(2) Let $0 \leq \widetilde{\sigma}<\sigma$ and $v \in \mathcal{O}_{\gamma, \tau}$. The operator $\mathscr{L}$ is bounded and invertible, and $\mathscr{L}^{-1}$ maps $X_{\sigma}^{0}$ onto $X_{\tilde{\sigma}}^{0}$,

$$
\begin{equation*}
\mathscr{L}^{-1} u=\mathscr{L}^{-1}\left(\sum_{k \in \mathbb{Z}} a_{k} e^{i k \theta}\right)=\sum_{k \in \mathbb{Z}} \lambda^{-1} a_{k} e^{i k \theta} \in X_{\tilde{\sigma}}^{0} \tag{29}
\end{equation*}
$$

If $n$ divides $m$, that is, $m / n \in \mathbb{Z}$, then,

$$
\begin{equation*}
\left\|\mathscr{L}^{-1} u\right\|_{\tilde{\sigma}} \leq \mu(\sigma-\widetilde{\sigma})\|u\|_{\sigma} \tag{30}
\end{equation*}
$$

where

$$
\begin{align*}
\mu(\sigma) & :=\mu(\sigma ; v, \Gamma) \\
& :=\sup _{k \in \mathbb{Z} \backslash\{0\}}\left(|v k-1|^{-1}|i \Gamma-1-v k|^{-1} e^{-\sigma|k|}\right) . \tag{31}
\end{align*}
$$

## If $n$ does not divide $m$, then

$$
\begin{equation*}
\left\|\mathscr{L}^{-1} u\right\|_{\tilde{\sigma}} \leq \bar{\mu}(\sigma-\widetilde{\sigma})\|u\|_{\sigma}, \tag{32}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\mu}(\sigma):=\bar{\mu}(\sigma ; v, \Gamma):=\sup _{k \in \mathbb{Z}\{\{0\}}\left(\Gamma^{-1}|v k-1|^{-1} e^{-\sigma|k|}\right) . \tag{33}
\end{equation*}
$$

Furthermore,

$$
\begin{align*}
& \mu(\sigma) \leq \frac{1}{\sigma^{2 \tau} \gamma^{2}}\left(\frac{2 \tau}{e}\right)^{2 \tau}  \tag{34}\\
& \bar{\mu}(\sigma) \leq \frac{1}{\Gamma \gamma \sigma^{\tau}}\left(\frac{\tau}{e}\right)^{\tau}
\end{align*}
$$

Proof. By the definition of operator $\mathscr{L}$, we can easily get (1) and (29). Now we prove (30), (32), and (34).

If $n$ divides $m$, that is, $m / n \in \mathbb{Z}$ and $2 K(\cos (2 m k \pi / n)-$ $1)=0$, then we have

$$
\begin{align*}
& \left\|\mathscr{L}^{-1} u\right\|_{\tilde{\sigma}} \\
& \quad=\sum_{k \in \mathbb{Z} \backslash\{0\}}\left(|v k-1|^{-1}|i \Gamma-1-v k|^{-1} e^{-(\sigma-\widetilde{\sigma}|k|}\right)\left|a_{k}\right|  \tag{35}\\
& \quad \cdot e^{|k| \sigma} \leq \mu(\sigma-\widetilde{\sigma})\|u\|_{\sigma} .
\end{align*}
$$

Since

$$
\begin{align*}
|i \Gamma-1-v k| & \geq|1+v k| \geq \frac{\gamma}{|k|^{\tau}}  \tag{36}\\
|v k-1| & \geq \frac{\gamma}{|k|^{\tau}}
\end{align*}
$$

and $\sup _{x>0}\left(x^{a} e^{-x}\right)=(a / e)^{a}, \forall a \geq 0$, we obtain

$$
\begin{equation*}
\mu(\sigma) \leq \frac{1}{\sigma^{2 \tau} \gamma^{2}}\left(\frac{2 \tau}{e}\right)^{2 \tau} \tag{37}
\end{equation*}
$$

If $n$ does not divide $m$, that is, $2 K(\cos (2 m k \pi / n)-1) \neq 0$, then operator $\mathscr{L}$ is invertible and no "small divisor" appears. We have

$$
\begin{align*}
& \left\|\mathscr{L}^{-1} u\right\|_{\tilde{\sigma}}=\sum_{k \in \mathbb{Z} \backslash\{0\}}\left(\mid i \Gamma(v k-1)+1-v^{2} k^{2}\right. \\
& \left.\quad-\left.2 K\left(\cos \left(\frac{2 m k \pi}{n}\right)-1\right)\right|^{-1} e^{-(\sigma-\widetilde{\sigma})|k|}\right)\left|a_{k}\right| e^{|k| \sigma}  \tag{38}\\
& \quad \leq \sum_{k \in \mathbb{Z} \backslash\{0\}}\left(|\Gamma(v k-1)|^{-1} e^{-(\sigma-\widetilde{\sigma})|k|}\right)\left|a_{k}\right| e^{|k| \sigma} \leq \bar{\mu}(\sigma \\
& -\widetilde{\sigma})\|u\|_{\sigma} .
\end{align*}
$$

By $\sup _{x>0}\left(x^{a} e^{-x}\right)=(a / e)^{a}, \forall a \geq 0$, and (36), we obtain

$$
\begin{equation*}
\bar{\mu}(\sigma) \leq \frac{1}{\Gamma \gamma \sigma^{\tau}}\left(\frac{\tau}{e}\right)^{\tau} \tag{39}
\end{equation*}
$$

This completes the proof.
Remark 4. By the estimate (39), we have that $\Gamma \neq 0$ in our Nash-Moser algorithm. However, when $\omega=0$, we have $v=0$ and $\zeta:=v \Gamma=0$.

Our method of finding traveling waves comes from the idea of Newton scheme, which is an approximation method. If we choose first step $\left(u_{0}, \zeta_{0}\right)$ suitable, by finding a "quadratically better approximation," we can move forward a single step to our target. Hence, the critical point is to construct "second step," that is, to get $\left(u_{1}, \zeta_{1}\right)$; then, the method of making "next step" is the same. Finally, our solution of (26) can be written as

$$
\begin{equation*}
(u, \zeta)=\left(u_{0}+\sum_{s=1}^{\infty} u_{s}, \zeta_{0}+\sum_{s=1}^{\infty} \zeta_{s}\right) \tag{40}
\end{equation*}
$$

For convenience, we define

$$
\begin{align*}
\mathscr{T}\left(u_{0}, \zeta_{0}\right) & :=\left(u_{0}+u_{1}, \zeta_{0}+\zeta_{1}\right), \quad \text { for }\left(u_{0}, \zeta_{0}\right) \in \mathscr{W}_{\sigma},  \tag{41}\\
E & :=\mathscr{F}\left(u_{0}, \zeta_{0}\right), \\
E_{1} & :=\mathscr{F}\left(u_{0}+u_{1}, \zeta_{0}+\zeta_{1}\right)=\mathscr{F}\left(\mathscr{T}\left(u_{0}, \zeta_{0}\right)\right) . \tag{42}
\end{align*}
$$

Now, we construct the "first step approximation" to find $\left(u_{1}, \zeta_{1}\right)$.

Lemma 5. Fix any $K>0, F>0$, and $\Gamma_{0}>0$. Assume that $\omega \in \mathcal{O}_{\gamma, \tau}$. Then, for any $\zeta \in\left[0, \Gamma_{0}\right]$, one obtains the "first step approximation":

$$
\begin{align*}
& u_{1}:=-\mathscr{L}^{-1} E+\left\langle\mathscr{L}^{-1} E\right\rangle \\
& \zeta_{1}:=(i \Gamma-1)\left\langle\mathscr{L}^{-1} E\right\rangle \tag{43}
\end{align*}
$$

Proof. We define

$$
\begin{equation*}
R:=\epsilon \sin \left(\theta+u_{0}+u_{1}\right)-\epsilon \sin \left(\theta+u_{0}\right)+(i \Gamma-1) u_{1} . \tag{44}
\end{equation*}
$$

Then we have

$$
\begin{align*}
\mathscr{F}\left(u_{0}+u_{1}, \zeta_{0}+\zeta_{1}\right)= & \mathscr{L}\left(u_{0}+u_{1}\right) \\
& +\epsilon \sin \left(\theta+u_{0}+u_{1}\right) \\
& +(i \Gamma-1)\left(u_{0}+u_{1}\right)+\zeta_{0}+\zeta_{1} \\
= & \mathscr{L} u_{0}+\epsilon \sin \left(\theta+u_{0}\right)  \tag{45}\\
& +(i \Gamma-1) u_{0}+\zeta_{0}+\mathscr{L} u_{1} \\
& +\epsilon \sin \left(\theta+u_{0}+u_{1}\right) \\
& -\epsilon \sin \left(\theta+u_{0}\right)+(i \Gamma-1) u_{1} \\
& +\zeta_{1}=E+\mathscr{L} u_{1}+\zeta_{1}+R .
\end{align*}
$$

Based on our approximation method, we need to solve the following equation:

$$
\begin{equation*}
E+\mathscr{L} u_{1}+\zeta_{1}=0 \tag{46}
\end{equation*}
$$

If $n$ divides $m$, that is, $m / n \in \mathbb{Z}$, operator $\mathscr{L}$ is not invertible, the "small divisor" appears. Therefore, the removing of a "small set" (in Lebesgue measure sense) is needed; that is, we require $\omega \in \mathcal{O}_{\gamma, \tau}$. Then, we construct

$$
\begin{align*}
& u_{1}:=-\mathscr{L}^{-1} E+\left\langle\mathscr{L}^{-1} E\right\rangle \\
& \zeta_{1}:=(i \Gamma-1)\left\langle\mathscr{L}^{-1} E\right\rangle \tag{47}
\end{align*}
$$

If $n$ dose not divide $m$, operator $\mathscr{L}$ is invertible. Then we can also construct $\left(u_{1}, \zeta_{1}\right)$ as the same form.

It is easy to verify that $\left(u_{1}, \zeta_{1}\right)$ is the solution of (46) and satisfies condition (22). This completes the proof.

Remark 6. In fact, to obtain $s$ th step approximation ( $u_{s}$, $\left.\zeta_{s}\right)(s \geq 1)$, we need to solve

$$
\begin{equation*}
E_{s}+\mathscr{L} u_{s}+\zeta_{s}=0 \tag{48}
\end{equation*}
$$

where

$$
\begin{equation*}
E_{s}=\mathscr{F}\left(\mathscr{T}^{s}\left(u_{0}, \zeta_{0}\right)\right) \tag{49}
\end{equation*}
$$

By the method in Lemma 3, we can construct sth step solution for (48) as

$$
\begin{align*}
& u_{s}:=-\mathscr{L}^{-1} E_{s}+\left\langle\mathscr{L}^{-1} E_{s}\right\rangle, \\
& \zeta_{s}:=(i \Gamma-1)\left\langle\mathscr{L}^{-1} E_{s}\right\rangle . \tag{50}
\end{align*}
$$

Now, in order to prove the convergence of our algorithm, we need the following KAM estimates.

Lemma 7 (KAM estimates). Assume that $\left(u_{0}, \zeta_{0}\right) \in \mathscr{W}_{\sigma}$. Then there exist $\beta:=\beta(\tau)>1$ and $C_{0}:=C_{0}\left(\Gamma_{0}, \tau\right)>1$ such that, for any $0<\alpha<\sigma$ and any $\Gamma \in\left[0, \Gamma_{0}\right]$, the following estimates hold:

$$
\begin{equation*}
\left\|u_{1}\right\|_{\sigma-(2 / 3) \alpha},\left|\zeta_{1}\right|,\left\|E_{1}\right\|_{\sigma-\alpha} \leq C_{0} \alpha^{-\beta}\|E\|_{\sigma} \tag{51}
\end{equation*}
$$

Proof. We first estimate the case that $n$ divides $m$. It follows from (30) that

$$
\begin{equation*}
\sup _{\mathbb{T}}\left|\mathscr{L}^{-1} E\right| \leq\left\|\mathscr{L}^{-1} E\right\|_{0} \leq \mu(\sigma)\|E\|_{\sigma} . \tag{52}
\end{equation*}
$$

By the definition of $\left(u_{1}, \zeta_{1}\right)$ and (52), we have

$$
\begin{align*}
\left\|u_{1}\right\|_{\sigma-(2 / 3) \alpha} & \leq\left\|\mathscr{L}^{-1} e\right\|_{\sigma-(2 / 3) \alpha}+\mu\left(\frac{\alpha}{3}\right)\|E\|_{\sigma} \\
& \leq 2 \mu\left(\frac{\alpha}{3}\right)\|E\|_{\sigma} \leq C_{1} \alpha^{-\sigma \tau}\|E\|_{\sigma}  \tag{53}\\
\left|\zeta_{1}\right| & \leq C_{2} \sigma^{-\tau}\|E\|_{\sigma} \leq C_{2} \alpha^{-\tau}\|E\|_{\sigma} \tag{54}
\end{align*}
$$

By (53) and the definition $E_{1}$ in (42), we get

$$
\begin{align*}
\left\|E_{1}\right\|_{\sigma-\alpha}= & \|R\|_{\sigma-\alpha} \\
\leq & \epsilon\left\|\sin \left(\theta+u_{0}+u_{1}\right)-\sin \left(\theta+u_{0}\right)\right\|_{\sigma-\alpha} \\
& +\sqrt{\Gamma^{2}+1}\left\|u_{1}\right\|_{\sigma-\alpha} \leq C_{3}\left\|u_{1}\right\|_{\sigma-\alpha}  \tag{55}\\
\leq & C_{3} \sigma^{-6 \tau}\|E\|_{\sigma} .
\end{align*}
$$

By (52), (53), and (55), there exist $\beta:=\beta(\tau)>1$ and $C_{0}:=$ $C_{0}\left(\Gamma_{0}, \tau\right)>1$ such that

$$
\begin{equation*}
\left\|u_{1}\right\|_{\sigma-(2 / 3) \alpha},\left|\zeta_{1}\right|,\left\|E_{1}\right\|_{\sigma-\alpha} \leq C_{0} \alpha^{-\beta}\|E\|_{\sigma} \tag{56}
\end{equation*}
$$

For the case of $n$ not dividing $m$, we can also get the estimate (51). The method is the same. So we omit it. This completes the proof.

Now, we will give a sufficient condition on the convergence of our algorithm. For $s \geq 0$ and $0<\bar{\sigma}<\sigma<\tilde{\sigma}$, we set

$$
\begin{align*}
\sigma_{s} & :=\bar{\sigma}+\frac{\sigma-\bar{\sigma}}{2^{s}},  \tag{57}\\
\alpha_{s+1} & :=\sigma_{s}-\sigma_{s+1}=\frac{\sigma-\bar{\sigma}}{2^{s+1}} .
\end{align*}
$$

Then, we have the following result about the convergence of Nash-Moser algorithm.

Lemma 8. Assume that $C_{4} \geq C_{0} 4^{\beta}(\sigma-\bar{\sigma})^{-\beta}$ and $C_{4}\|E\|_{\sigma} \leq$ $\iota<1$. Then, $(u, \zeta) \in \mathscr{W}_{\bar{\sigma}}$ defined in (40) is a solution of (26); that is, $\mathscr{F}(u ; \zeta)=0, \forall \zeta \in\left[0, \Gamma_{0}\right]$.

Proof. We claim that, for $s \geq 1$,

$$
\begin{align*}
\left(u_{s}, \zeta_{s}\right) & :=\mathscr{T}^{s}\left(u_{0}, \zeta_{0}\right) \in \mathscr{W}_{\sigma_{s}} \\
E_{s}(\theta) & :=\mathscr{F}\left(\mathscr{T}^{s}\left(u_{0}, \zeta_{0}\right)\right) \in X_{\sigma_{s}} \tag{58}
\end{align*}
$$

$$
\max \left\{\left\|u_{s}\right\|_{\sigma_{s}},\left|\zeta_{s}\right|,\left\|E_{s}\right\|_{\sigma_{s}}\right\} \leq \frac{\left(C_{4}\|E\|_{\sigma}\right)^{2^{s-1}}}{2^{\beta}}
$$

In fact, if (58) holds, then by the decay of $\left(C_{4}\|E\|_{\sigma}\right)^{2^{s}}$, we obtain that

$$
\mathscr{T}^{s}\left(u_{0}, \zeta_{0}\right) \longrightarrow(u, \zeta) \in \mathscr{W}_{\sigma_{s}}
$$

$$
\begin{equation*}
\text { uniformly, as } s \longrightarrow \infty \tag{59}
\end{equation*}
$$

$$
\mathscr{F}(u, \zeta)=\lim _{s \rightarrow \infty} \mathscr{F}\left(\mathscr{T}^{s}\left(u_{0}, \zeta_{0}\right)\right)=\lim _{s \rightarrow \infty} E_{s}=0 .
$$

In the following, we will prove (58) by induction. Firstly, we check (58) for the case of $s=1$. Let $\alpha_{1}:=\alpha$ and $\sigma_{1}:=\sigma-\alpha$. By (51), we have

$$
\begin{equation*}
\left\|u_{1}\right\|_{\sigma_{1}},\left|\zeta_{1}\right|,\left\|E_{1}\right\|_{\sigma_{1}} \leq C_{0} e^{-\beta}\|E\|_{\sigma}=\frac{C_{0}}{2^{\beta}}\|E\|_{\sigma} \tag{60}
\end{equation*}
$$

which implies that $\left(u_{1}, \zeta_{1}\right) \in \mathscr{V}_{\sigma_{1}}$, so, (58) holds for $s=1$.
Let $s^{\prime} \geq 1$. Assume that (58) holds true for $1 \leq s \leq s^{\prime}$. Now we will prove that it also holds for $s=s^{\prime}+1$. Let $\alpha:=\alpha_{s+1}$ and $\sigma:=\sigma_{s}$. Note that $C \alpha_{s+1}^{-\beta}=\left(C / 2^{\beta}\right) 2^{s \beta}$. By (51), we get

$$
\begin{align*}
& \left\|u_{s+1}\right\|_{\sigma_{s+1}},\left|\zeta_{s+1}\right|,\left\|E_{s+1}\right\|_{\sigma_{s+1}} \leq C_{4} \alpha_{s+1}^{-\beta}\left\|E_{s}\right\|_{\sigma_{s}} \\
& =\frac{C_{4}}{2^{\beta}} 2^{s \beta}\left\|E_{s}\right\|_{\sigma_{s}} \leq \frac{\left(C_{4}\left\|E_{s}\right\|_{\sigma_{s}}\right)^{2^{s}}}{2^{\beta}}, \tag{61}
\end{align*}
$$

which shows that $\left(u_{s+1}, \zeta_{s+1}\right) \in \mathscr{W}_{\sigma_{s+1}}$. Hence, our claim holds. This completes the proof.

Remark 9. In this lemma, we do not care whether $n$ divides $m$ or not. Because the convergence of Nash-Moser algorithm is the same.

Lemma 10 (uniqueness). Assume that $(\bar{u}, \bar{\zeta})$ and $(\widetilde{\mathcal{u}}, \widetilde{\zeta})$ are solutions of (26) in the domain $C_{5} 4^{\beta \tau} \bar{\sigma}^{-\beta \tau}\|u\|_{\bar{\sigma}}<1$. Then, $(\bar{u}$, $\bar{\zeta})=(\widetilde{u}, \widetilde{\zeta})$; that is, the solution of $(26)$ is unique.

Proof. Let $u:=\bar{u}-\widetilde{u}$ and $\zeta:=\bar{\zeta}-\widetilde{u}$. Then

$$
\begin{align*}
\mathscr{F}(u+\widetilde{u}, \zeta+\widetilde{\zeta})= & \mathscr{L}(u+\widetilde{u})+\epsilon \sin (\theta+u+\widetilde{u}) \\
& +(i \Gamma-1)(u+\widetilde{u})+\zeta+\widetilde{\zeta} \\
= & \mathscr{L} \widetilde{u}+\epsilon \sin (\theta+\widetilde{u})+(i \Gamma-1) \widetilde{u}  \tag{62}\\
& +\widetilde{\zeta}+\mathscr{L} u+\epsilon \sin (\theta+u+\widetilde{u}) \\
& -\epsilon \sin (\theta+\widetilde{u})+(i \Gamma-1) u+\zeta \\
= & E+\mathscr{L} u+\zeta+\bar{R},
\end{align*}
$$

where

$$
\begin{equation*}
\bar{R}=\epsilon \sin (\theta+u+\widetilde{u})-\epsilon \sin (\theta+\widetilde{u})+(i \Gamma-1) u . \tag{63}
\end{equation*}
$$

Note that $\mathscr{F}(\bar{u}, \bar{\zeta})=0$. Therefore, by (62), we have

$$
\begin{equation*}
\mathscr{L} u+\zeta+\bar{R}=0 . \tag{64}
\end{equation*}
$$

It follows from condition (22) that

$$
\begin{align*}
\zeta & =(i \Gamma-1)\left\langle\mathscr{L}^{-1} \bar{R}\right\rangle  \tag{65}\\
u & :=-\mathscr{L}^{-1} \bar{R}+\left\langle\mathscr{L}^{-1} \bar{R}\right\rangle . \tag{66}
\end{align*}
$$

Note that $\|\bar{R}\|_{\sigma} \leq C_{6}\|u\|_{\sigma}$, so we have that

$$
\begin{equation*}
|\zeta| \leq C_{7} \alpha^{-\beta}\|u\|_{\sigma} . \tag{67}
\end{equation*}
$$

Next we will estimate (66). By (63) and the similar estimates in (53), for $0<\alpha<\sigma \leq \bar{\sigma}$, we have

$$
\begin{align*}
\|u\|_{\sigma-\alpha} & =\left\|-\mathscr{L}^{-1} \bar{R}+\left\langle\mathscr{L}^{-1} \bar{R}\right\rangle\right\|_{\sigma-\alpha} \leq C_{8} \alpha^{-\beta \tau}\|\bar{R}\|_{\sigma}  \tag{68}\\
& \leq C_{8} \alpha^{-\beta \tau}\|u\|_{\sigma} .
\end{align*}
$$

If we take $\sigma_{s}:=\bar{\sigma} / 2^{s}$ and $\alpha_{s}:=\bar{\sigma} / 2^{s+1}(s \geq 0)$. Then, it follows from (68) that

$$
\begin{align*}
\|u\|_{\sigma_{s+1}} & \leq C_{5} \alpha_{s}^{-\beta \tau}\|u\|_{\sigma_{s}}=C_{5} \bar{\sigma}^{-\beta \tau} 2^{\beta \tau(s+1)}\|u\|_{\sigma_{s}}  \tag{69}\\
& \leq a b^{s}\|u\|_{\sigma_{s}},
\end{align*}
$$

where $a:=C_{5} 2^{\beta \tau} \bar{\sigma}^{-\beta \tau}$ and $b:=2^{\beta \tau}$.
By (69), we have

$$
\begin{align*}
a b^{s+1}\|u\|_{\sigma_{s+1}} b & \leq\left(a b^{s+1}\right)^{2}\|u\|_{\sigma_{s}}=a b^{s+1}\left(a b^{s}\|u\|_{\sigma_{s}} b\right) \\
& \leq\left(a b^{s+1}\right)\left(a b^{s}\right)\left(a b^{s-1}\|u\|_{\sigma_{s-1}} b\right) \leq \cdots  \tag{70}\\
& \leq\left(a b^{s+1}\right)\left(a b^{s}\right) \cdots(a b)\|u\|_{\bar{\sigma}} \\
& =a^{s+1} b^{(s+1)^{2}}\|u\|_{\bar{\sigma}}
\end{align*}
$$

which shows that

$$
\begin{equation*}
\|u\|_{\sigma_{s+1}} \leq a^{s} b^{s^{2}+s-1}\|u\|_{\bar{\sigma}} . \tag{71}
\end{equation*}
$$

There exists $\chi:=\chi(s) \in C^{1}(1, \infty)$ such that

$$
\begin{aligned}
\chi^{\prime}(s) & >0 \\
\chi^{\prime \prime}(s) & =0
\end{aligned}
$$

$$
\text { for } s \geq 1, \lim _{s \rightarrow \infty} \chi(s)=\infty .
$$

Then, it follows from (71) that

$$
\begin{equation*}
\|u\|_{0} \leq\|u\|_{\sigma_{s}} \leq\left(a b\|u\|_{\bar{\sigma}}^{1 / \chi(s)}\right)^{s^{2}+s-1} . \tag{73}
\end{equation*}
$$

Note our assumption $C_{5} 4^{\beta \tau} \bar{\sigma}^{-\beta \tau}\|u\|_{\bar{\sigma}}<1$ and $\|u\|_{0}=$ $\sum_{k \in \mathbb{Z}}\left|u_{k}\right|$. Therefore, by (73), we obtain

$$
\begin{equation*}
\|u\|_{0}=0 \tag{74}
\end{equation*}
$$

which implies that

$$
\begin{align*}
u & =0, \\
\text { that is., } \bar{u} & =\tilde{u} . \tag{75}
\end{align*}
$$

This together with (65) means the uniqueness of solutions for (26). This completes the proof.

The following result can be seen as a Nash-Moser theorem for dissipative lattice systems.

Theorem 11. Let $0<\gamma<1 \leq \tau, 0<\bar{\sigma}<\sigma \leq 1, \omega \in$ $\mathcal{O}_{\gamma, \tau}$, and $\zeta \in\left[0, \Gamma_{0}\right]$ for some $\Gamma_{0}>0$. Assume that "initial approximate solution" $\left(u_{0}, \zeta_{0}\right) \in C^{\infty}\left(\mathbb{T} \times\left[0, \Gamma_{0}\right]\right) \times C^{\infty}\left[0, \Gamma_{0}\right]$ and $\theta \rightarrow u_{0}(\theta, \Gamma) \in X_{\sigma}^{0}, \varrho\|E\|_{\sigma} \leq 1, \forall \zeta \in\left[0, \Gamma_{0}\right]$, and $\varrho>$ 1. Then, (26) possesses solutions $(u, \zeta) \in C^{\infty}\left(\mathbb{T} \times\left[0, \Gamma_{0}\right]\right) \times$ $C^{\infty}\left(\left[0, \Gamma_{0}\right]\right)$ and $u \in X_{\bar{\sigma}}^{0}$. Moreover, if $(\bar{u}, \bar{\zeta}) \in C^{\infty}\left(\mathbb{T} \times\left[0, \Gamma_{0}\right]\right) \times$ $C^{\infty}\left(\left[0, \Gamma_{0}\right]\right)$ is also the solution of (26) and satisfies $\varrho\|u-\bar{u}\|_{\bar{\sigma}}<$ 1, then, $(u, \zeta)=(\bar{u}, \bar{\zeta})$; that is, the solution of (26) is unique.

Proof. This result is the conclusion of Lemmas 5-14. Let $\beta(\tau)$ and $C_{0}$ be defined in Lemma 7 . We choose $\varrho$ such that $C_{0} 4^{\beta}(\sigma-\bar{\sigma})^{-\beta} \leq \varrho \leq \min \left\{C_{4}, C_{5} 4^{\beta \tau} \bar{\sigma}^{-\beta \tau}\right\}$. Then, by our assumption, Lemmas 8 and 10 , we can get the existence and uniqueness of solutions of (26). This completes the proof.

Remark 12. In fact, in this abstract result, we do not need any assumption on $\epsilon>0$ in the case of $\epsilon=1$. Then, the problem of finding traveling wave solutions for (4) with periodic boundary condition (2) is another open problem in [11]. By Theorem 11, we can see that, for fixing $K>0$, $\Gamma>0$ and sufficient small $F>0$, there is a unique traveling wave solution for (4). However, it is difficult to find the initial approximation solution $\left(u_{0}, \zeta_{0}\right)$ which must make the error function $E$ satisfying $\varrho\|E\|_{\sigma} \leq 1(\varrho>1)$.

Now, we will use Theorem 11 to prove our main result.
Proof of Theorem 1. Let $0<\bar{\sigma}<\sigma, \forall \bar{\sigma}, \sigma \in \mathbb{R}^{+}$. We choose the initial approximation solution

$$
\begin{equation*}
\left(u_{0}, \zeta_{0}\right)=(0,0) \tag{76}
\end{equation*}
$$

Let $\epsilon_{0} \leq 1 / \varrho-F$ and $\epsilon \in\left[0, \epsilon_{0}\right](\varrho>1)$. Then, the error function $E$ defined in (42) is given by

$$
\begin{align*}
E & :=\mathscr{F}(0,0)=\epsilon \sin (\theta)-F, \\
\|E\|_{\sigma} & \leq \epsilon+F \leq \frac{1}{\varrho} \tag{77}
\end{align*}
$$

Here, we require that $F>0$ be sufficiently small so that $1 / \varrho-$ $F>0$.

It follows from Theorem 11 that our result holds. This completes the proof.

Remark 13. By the proof of Theorem 1, we can see that our result also holds for the case of $F=0$. It suffices to take $\epsilon_{0} \leq$ $1 / \varrho$ and $\epsilon \in\left[0, \epsilon_{0}\right]$.
2.2. The Case of $\Gamma=0$. We now focus on the proof of Theorem 2 by the same method.

By strong monotonicity arguments, Baesens and MacKay have obtained the existence and stability of traveling waves for (8) with periodic boundary condition (2). Here, we will use Nash-Moser iteration to study the existence and uniqueness of traveling wave solutions for (8) with periodic boundary condition (2).

Note that a waveform $\varphi$ satisfies the following equation:

$$
\begin{align*}
\varphi^{\prime \prime}(t) & -K\left[\varphi\left(t+\frac{m}{n} T\right)-2 \varphi(t)+\varphi\left(t-\frac{m}{n} T\right)\right]  \tag{78}\\
& +\epsilon \sin (\varphi(t))=F
\end{align*}
$$

Hence, as in [11], we investigate the traveling wave of the type

$$
\begin{equation*}
\varphi(t)=u(v t)+v t \tag{79}
\end{equation*}
$$

where the wave velocity $v=2 \pi / T=2 \pi \omega$ and $u$ satisfies

$$
\begin{equation*}
u(\theta+2 \pi)=u(\theta), \quad \forall \theta \in \mathbb{R} \tag{80}
\end{equation*}
$$

Inserting (79) into (78), we get

$$
\begin{align*}
v^{2} u^{\prime \prime} & (\theta) \\
& -K\left[u\left(\theta+2 \pi \frac{m}{n}\right)-2 u(\theta)+u\left(\theta-2 \pi \frac{m}{n}\right)\right]  \tag{81}\\
& +\epsilon \sin (\theta+u(\theta))=F .
\end{align*}
$$

Define the operator $\mathscr{M}: X_{\sigma} \rightarrow X_{\sigma}$ as

$$
\begin{aligned}
\mathscr{M}:= & v^{2} \mathscr{A}^{2} u(\theta) \\
& -K\left[u\left(\theta+2 \pi \frac{m}{n}\right)-2 u(\theta)+u\left(\theta-2 \pi \frac{m}{n}\right)\right] \\
& +1 .
\end{aligned}
$$

Then, (81) can be written as

$$
\begin{equation*}
\mathscr{G}(u, F)=\mathscr{M} u+\epsilon \sin (\theta+u(\theta))-u(\theta)-F=0 . \tag{83}
\end{equation*}
$$

We will also use the idea of Newton scheme to obtain the solution of (83). Firstly, we need to give some notations:

$$
\begin{align*}
& \overline{\mathscr{T}}\left(u_{0}, F_{0}\right):=\left(u_{0}+u_{1}, F_{0}+F_{1}\right), \\
& \text { for }\left(u_{0}, F_{0}\right) \in \mathscr{W}_{\sigma}, \\
& \bar{E}:=\mathscr{G}\left(u_{0}, F_{0}\right),  \tag{84}\\
& \bar{E}_{s}:=\mathscr{G}\left(\sum_{i=0}^{s} u_{i}, \sum_{i=0}^{s} F_{i}\right)=\mathscr{G}\left(\overline{\mathscr{T}}^{s}\left(u_{0}, F_{0}\right)\right) \text {, }
\end{align*}
$$

where $\left(u_{s}, F_{s}\right)$ denotes the $s$ th step approximation solution.
Next, the spectrum analysis of operator $\mathscr{M}$ is essential.

Lemma 14. Fix $K>0$. The "diagonal" operator $\mathscr{M}$ (on Fourier spaces) satisfies the following:
(1) $\forall u \in X_{\sigma}^{0}$,

$$
\begin{equation*}
\mathscr{M}(u)=\mathscr{M}\left(\sum_{k \in \mathbb{Z}} a_{l} e^{i k \theta}\right)=\sum_{k \in \mathbb{Z}} \bar{\lambda} a_{l} e^{i k \theta} \tag{85}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\lambda}:=-k^{2} v^{2}+1-2 K\left(\cos \left(\frac{2 m k \pi}{n}\right)-1\right) . \tag{86}
\end{equation*}
$$

(2) Let $0 \leq \widetilde{\sigma}<\sigma$ and $v \in \overline{\mathcal{O}}_{\gamma, \tau}$. The operator $\mathscr{M}$ is bounded and invertible, and $\mathscr{M}^{-1}$ maps $X_{\sigma}^{0}$ onto $X_{\tilde{\sigma}}^{0}$,

$$
\begin{equation*}
\mathscr{M}^{-1} u=\mathscr{M}^{-1}\left(\sum_{k \in \mathbb{Z}} a_{k} e^{i k \theta}\right)=\sum_{k \in \mathbb{Z}} \bar{\lambda}^{-1} a_{k} e^{i k \theta} \in X_{\tilde{\sigma}}^{0} \tag{87}
\end{equation*}
$$

If $n$ divides $m$, that is, $m / n \in \mathbb{Z}$, then

$$
\begin{equation*}
\left\|M^{-1} u\right\|_{\tilde{\sigma}} \leq \mathcal{Y}(\sigma-\widetilde{\sigma})\|u\|_{\sigma} \tag{88}
\end{equation*}
$$

where

$$
\begin{equation*}
\vartheta(\sigma):=\sup _{k \in \mathbb{Z} \backslash\{0\}}\left(|v k+1|^{-1}|v k-1|^{-1} e^{-\sigma|k|}\right) . \tag{89}
\end{equation*}
$$

If $n$ does not divide $m$, then

$$
\begin{equation*}
\left\|M^{-1} u\right\|_{\tilde{\sigma}} \leq \bar{\vartheta}(\sigma-\tilde{\sigma})\|u\|_{\sigma}, \tag{90}
\end{equation*}
$$

where

$$
\begin{equation*}
\overline{\mathcal{Y}}(\sigma):=\sup _{k \in \mathbb{Z} \backslash\{0\}}\left(|v k-i \sqrt{1-4 K}|^{-1} e^{-\sigma|k|}\right) . \tag{91}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
\mathcal{\vartheta}(\sigma), \quad \bar{\vartheta}(\sigma) \leq \frac{1}{\sigma^{2 \tau} \gamma^{2}}\left(\frac{2 \tau}{e}\right)^{2 \tau} \tag{92}
\end{equation*}
$$

Proof. The idea of this proof is similar to the proof of Lemma 3. Here, we only need to verify (88), (90), and (92).

Note that $\sup _{x>0}\left(x^{a} e^{-x}\right)=(a / e)^{a}, \forall a \geq 0$, and

$$
\begin{gather*}
|k v(\omega) \pm 1| \\
|k v(\omega) \pm i \sqrt{1-4 K}| \geq \frac{\gamma}{|k|^{\tau}} \tag{93}
\end{gather*}
$$

$$
\forall k \in \mathbb{Z}
$$

Hence, in the case that $n$ divides $m$,

$$
\begin{align*}
& \left\|\mathscr{M}^{-1} u\right\|_{\tilde{\sigma}} \\
& \quad=\sum_{k \in \mathbb{Z} \backslash\{0\}}\left(|v k+1|^{-1}|v k-1|^{-1} e^{-(\sigma-\widetilde{\sigma})|k|}\right)\left|a_{k}\right| e^{|k| \sigma}  \tag{94}\\
& \quad \leq \mathcal{\vartheta}(\sigma-\widetilde{\sigma})\|u\|_{\sigma} ;
\end{align*}
$$

in the case that $n$ does not divide $m$,

$$
\begin{align*}
& \left\|M^{-1} u\right\|_{\tilde{\sigma}} \\
& =\sum_{k \in \mathbb{Z}\{0\}}\left(\left|-v^{2} k^{2}+1-2 K\left(\cos \left(\frac{2 m k \pi}{n}\right)-1\right)\right|^{-1}\right. \\
& \left.\cdot e^{-(\sigma-\tilde{\sigma})|k|}\right)\left|a_{k}\right| e^{|k| \sigma} \leq \sum_{k \in \mathbb{Z} \backslash\{0\}}\left(|v k-i \sqrt{1-4 K}|^{-1}\right.  \tag{95}\\
& \left.\cdot|v k+i \sqrt{1-4 K}|^{-1} e^{-(\sigma-\tilde{\sigma}| | k \mid}\right)\left|a_{k}\right| e^{|k| \sigma} \leq \bar{\vartheta}(\sigma-\widetilde{\sigma}) \\
& \cdot\|u\|_{\sigma},
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{\vartheta}(\sigma), \quad \overline{\mathcal{\vartheta}}(\sigma) \leq \frac{1}{\sigma^{2 \tau} \gamma^{2}}\left(\frac{2 \tau}{e}\right)^{2 \tau} . \tag{96}
\end{equation*}
$$

This completes the proof.
Lemma 15. Fix any $K>0$ and $F_{0}>0$. Assume that $\omega \in \overline{\mathcal{O}}_{\gamma, \tau}$. Then, for any $F \in\left[0, F_{0}\right]$, one obtains the "first step approximation":

$$
\begin{align*}
& \bar{u}_{1}:=-\mathscr{M}^{-1} \bar{E}+\left\langle\mathscr{M}^{-1} \bar{E}\right\rangle, \\
& F_{1}:=\left\langle\mathscr{M}^{-1} \bar{E}\right\rangle . \tag{97}
\end{align*}
$$

Proof. Define

$$
\begin{equation*}
\widetilde{R}:=\epsilon \sin \left(\theta+u_{0}+u_{1}\right)-\epsilon \sin \left(\theta+u_{0}\right)+u_{1}(\theta) . \tag{98}
\end{equation*}
$$

Then we have

$$
\begin{align*}
\mathscr{G}\left(u_{0}+u_{1}, \zeta_{0}+\zeta_{1}\right)= & \mathscr{M}\left(u_{0}+u_{1}\right) \\
& +\epsilon \sin \left(\theta+u_{0}+u_{1}\right)+u_{0}+u_{1} \\
& +F_{0}+F_{1} \\
= & M u_{0}+\epsilon \sin \left(\theta+u_{0}\right)+u_{0}+F_{0}  \tag{99}\\
& +M u_{1}+\epsilon \sin \left(\theta+u_{0}+u_{1}\right) \\
& -\epsilon \sin \left(\theta+u_{0}\right)+u_{1}+F_{1} \\
= & \bar{E}+M u_{1}+F_{1}+\widetilde{R} .
\end{align*}
$$

For getting $\left(u_{1}, F_{1}\right)$, we need to solve the following equation:

$$
\begin{equation*}
\bar{E}+M u_{1}+F_{1}=0 . \tag{100}
\end{equation*}
$$

By condition (22), we can construct "the first approximation solution":

$$
\begin{align*}
& \bar{u}_{1}:=-\mathscr{M}^{-1} \bar{E}+\left\langle\mathscr{M}^{-1} \bar{E}\right\rangle, \\
& F_{1}:=\left\langle\mathscr{M}^{-1} \bar{E}\right\rangle . \tag{101}
\end{align*}
$$

This completes the proof.

Remark 16. In fact, we can construct the sth step approximation solution as

$$
\begin{align*}
& \bar{u}_{s}:=-\mathscr{M}^{-1} \bar{E}_{s}+\left\langle\mathscr{M}^{-1} \bar{E}_{s}\right\rangle,  \tag{102}\\
& F_{s}:=\left\langle\mathcal{M}^{-1} \bar{E}_{s}\right\rangle,
\end{align*}
$$

by solving the following equation:

$$
\begin{equation*}
\bar{E}_{s}+\mathscr{M} u_{s}+F_{s}=0 . \tag{103}
\end{equation*}
$$

Lemma 17 (KAM estimates). Assume that $\left(u_{0}, F_{0}\right) \in \mathscr{W}_{\sigma}$. Then, there exist $\bar{\beta}:=\bar{\beta}>1$ and $C_{9}:=C_{9}\left(F_{*}, \tau, \gamma\right)>1$ such that, for any $0<\alpha<\sigma$ and any $F \in\left[0, F_{*}\right]$, the following estimates hold:

$$
\begin{equation*}
\left\|u_{1}\right\|_{\sigma-(2 / 3) \alpha},\left|F_{1}\right|,\left\|\bar{E}_{1}\right\|_{\sigma-\alpha} \leq C_{9} \alpha^{-\beta}\|\bar{E}\|_{\sigma} . \tag{104}
\end{equation*}
$$

Proof. The proof is the same as Lemma 7, so we omitted it.

Lemma 18. Assume that $C_{10} \geq C_{9} 4^{\beta}(\sigma-\bar{\sigma})^{-\beta}$ and $C_{10}\|E\|_{\sigma} \leq$ $t<1$. Then, $(u, F)=\left(\sum_{s=0}^{\infty} u_{s}, \sum_{s=0}^{\infty} F_{s}\right) \in \mathscr{W}_{\bar{\sigma}}$ is a solution of (83); that is, $\mathscr{F}(u ; F)=0, \forall F \in\left[0, F_{*}\right]$. Furthermore, in the domain

$$
\begin{align*}
& \left\{(u, F) \in C^{\infty}\left(\mathbb{T} \times\left[0, F_{*}\right]\right)\right. \\
& \left.\quad \times C^{\infty}\left(\left[0, F_{*}\right]\right) \mid C_{11} 4^{\beta \tau} \bar{\sigma}^{-\beta \tau}\|u\|_{\bar{\sigma}}<1\right\}, \tag{105}
\end{align*}
$$

(83) admits a unique solution ( $u, F)$.

Proof. This proof is also similar to Lemmas 8 and 10 , so we omitted it.

Based on Lemma 18, we show the following Nash-Moser theorem for the conservative lattice systems.

Theorem 19. Let $0<\gamma<1 \leq \tau, 0<\bar{\sigma}<\sigma \leq 1, \omega \in \overline{\mathcal{O}}_{\gamma, \tau}$, and $F \in\left[0, F_{*}\right]$ for some $F_{*}>0$. Assume that "initial approximate solution" $\left(u_{0}, F_{0}\right) \in C^{\infty}\left(\mathbb{T} \times\left[0, \Gamma_{0}\right]\right) \times C^{\infty}\left[0, F_{*}\right]$ and $\theta \rightarrow$ $u_{0}(\theta, F) \in X_{\sigma}^{0}, \bar{\varrho}\|\bar{E}\|_{\sigma} \leq 1, \forall F \in\left[0, F_{*}\right]$, and $\bar{\varrho}>1$. Then, (83) possesses solutions $(u, F) \in C^{\infty}\left(\mathbb{T} \times\left[0, F_{*}\right]\right) \times C^{\infty}\left(\left[0, F_{*}\right]\right)$ and $u \in X_{\bar{\sigma}}^{0}$. Moreover, the solution of (83) is unique in the domain $\left\{(u, F) \in C^{\infty}\left(\mathbb{T} \times\left[0, F_{*}\right]\right) \times C^{\infty}\left(\left[0, F_{*}\right]\right) \mid \bar{\varrho}\|u\|_{\sigma}<1, \forall \bar{\varrho}>1\right\}$.

Proof of Theorem 2. Let $0<\bar{\sigma}<\sigma, \forall \bar{\sigma}, \sigma \in \mathbb{R}^{+}$. We choose the initial approximation solution

$$
\begin{equation*}
\left(u_{0}, F_{0}\right)=(0,0) . \tag{106}
\end{equation*}
$$

Let $\epsilon_{0} \leq 1 / \bar{\varrho}$ and $\epsilon \in\left[0, \epsilon_{0}\right](\bar{\varrho}>1)$. Then, the error function $\bar{E}$ defined in (84) is given by

$$
\begin{align*}
\bar{E} & :=\mathscr{G}(0,0)=\epsilon \sin (\theta), \\
\|\bar{E}\|_{\sigma} & \leq \epsilon \leq \frac{1}{\varrho} . \tag{107}
\end{align*}
$$

It follows from Theorem 19 that our result holds. This completes the proof.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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## References

[1] O. M. Braun and Y. S. Kivshar, "Nonlinear dynamics of the Frenkel-Kontorova model," Physics Reports, vol. 306, no. 1-2, 108 pages, 1998.
[2] T. Strunz and F.-J. Elmer, "Driven Frenkel-Kontorova model. I. Uniform sliding states and dynamical domains of different particle densities," Physical Review E: Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics, vol. 58, no. 2, pp. 1601-1611, 1998.
[3] T. Strunz and F.-J. Elmer, "Driven Frenkel-Kontorova model. II. Chaotic sliding and nonequilibrium melting and freezing," Physical Review E: Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics, vol. 58, no. 2, pp. 1612-1620, 1998.
[4] Z. Zheng, B. Hu, and G. Hu, "Resonant steps and spatiotemporal dynamics in the damped dc-driven Frenkel-Kontorova chain," Physical Review B: Condensed Matter and Materials Physics, vol. 58, no. 9, pp. 5453-5461, 1998.
[5] C. Baesens and R. S. MacKay, "Gradient dynamics of tilted Frenkel-Kontorova models," Nonlinearity, vol. 11, no. 4, pp. 949964, 1998.
[6] C. Baesens and R. S. MacKay, "A novel preserved partial order for cooperative networks of units with overdamped second order dynamics, and application to tilted Frenkel-Kontorova chains," Nonlinearity, vol. 17, no. 2, pp. 567-580, 2004.
[7] M. Levi, "Caterpillar solutions in coupled pendula," Ergodic Theory and Dynamical Systems, vol. 8, no. Charles Conley Memorial Issue, pp. 153-174, 1988.
[8] M. Levi, "Dynamics of discrete Frenkel-Kontorova models," in Analysis, et cetera, P. Rabinowitz and E. Zehnder, Eds., pp. 471494, Academic Press, Boston, MA, USA, 1990.
[9] W.-X. Qin, C.-L. Xu, and X. Ma, "Stability of single-waveform solutions in the underdamped Frenkel-Kontorova model," SIAM Journal on Mathematical Analysis, vol. 40, no. 3, pp. 952967, 2008.
[10] R. Mirollo and N. Rosen, "Existence, uniqueness, and nonuniqueness of single-wave-form solutions to Josephson junction systems," SIAM Journal on Applied Mathematics, vol. 60, no. 5, pp. 1471-1501, 2000.
[11] G. Katriel, "Existence of travelling waves in discrete sineGordon rings," SIAM Journal on Mathematical Analysis, vol. 36, no. 5, pp. 1434-1443, 2005.
[12] M. Levi, F. C. Hoppensteadt, and W. L. Miranker, "Dynamics of the Josephson junction," Quarterly of Applied Mathematics, vol. 36, no. 2, pp. 167-198, 1978/79.
[13] P. H. Rabinowitz, "A rapid convergence method for a singular perturbation problem," Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 1, no. 1, pp. 1-17, 1984.
[14] M. Berti and P. Bolle, "Periodic solutions of nonlinear wave equations with general nonlinearities," Communications in Mathematical Physics, vol. 243, no. 2, pp. 315-328, 2003.
[15] M. Berti and P. Bolle, "Cantor families of periodic solutions for completely resonant nonlinear wave equations," Duke Mathematical Journal, vol. 134, no. 2, pp. 359-419, 2006.
[16] M. Berti and P. Bolle, "Cantor families of periodic solutions for wave equations via a variational principle," Advances in Mathematics, vol. 217, no. 4, pp. 1671-1727, 2008.
[17] M. Berti and M. Procesi, "Quasi-periodic solutions of completely resonant forced wave equations," Communications in Partial Differential Equations, vol. 31, no. 4-6, pp. 959-985, 2006.
[18] W. Craig and C. E. Wayne, "Newton's method and periodic solutions of nonlinear wave equations," Communications on Pure and Applied Mathematics, vol. 46, no. 11, pp. 1409-1498, 1993.
[19] G. Iooss, P. I. Plotnikov, and J. F. Toland, "Standing waves on an infinitely deep perfect fluid under gravity", Archive for Rational Mechanics and Analysis, vol. 177, no. 3, pp. 367-478, 2005.
[20] J. Xu, J. You, and Q. Qiu, "Invariant tori for nearly integrable Hamiltonian systems with degeneracy," Mathematische Zeitschrift, vol. 226, no. 3, pp. 375-387, 1997.


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