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Research Article

Chover-Type Laws of the Iterated Logarithm for Kesten-Spitzer Random Walks in Random Sceneries Belonging to the Domain of Stable Attraction

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Let $X = \{X_i, i \geq 1\}$ be a sequence of real valued random variables, $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$ $(k \geq 1)$. Let $\sigma = \{\sigma(x), x \in \mathbb{Z}\}$ be a sequence of real valued random variables which are independent of X's. Denote by $K_n = \sum_{k=0}^{n} \sigma(\lfloor S_k \rfloor)$ $(n \geq 0)$ Kesten-Spitzer random walk in random scenery, where $\lfloor a \rfloor$ means the unique integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$. It is assumed that σ's belong to the domain of attraction of a stable law with index $0 < \beta < 2$. In this paper, by employing conditional argument, we investigate large deviation inequalities, some sufficient conditions for Chover-type laws of the iterated logarithm and the cluster set for random walk in random scenery K_n. The obtained results supplement to some corresponding results in the literature.

1. Introduction

Let $X = \{X_i, i \geq 1\}$ be a sequence of real valued random variables, $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$ $(k \geq 1)$. Let $\sigma = \{\sigma(x), x \in \mathbb{Z}\}$ be a sequence of $\mathbb{R}$-valued random variables which are independent of X's. We refer to $S = \{S_k, k \geq 0\}$ as the *random walk* and σ as the *random scenery*. Then the process $K = \{K_n, n \in \mathbb{N}\}$ is defined by

$$K_n = \sum_{k=0}^{n} \sigma(\lfloor S_k \rfloor), \quad n \in \mathbb{N}, \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\lfloor a \rfloor$ means the unique integer satisfying $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1$, called a *random walk in random scenery* (RWRS, in short), sometimes also referred to as the *Kesten-Spitzer random walk in random scenery*; see Kesten and Spitzer [1]. An interpretation is as follows. If a random walker has to pay $\sigma(x)$ units at any time he/she visits the site x, then K_n is the total amount he/she pays by time n.

RWRS was first introduced by Kesten and Spitzer [1] and Borodin [2, 3] in order to construct new self-similar stochastic processes. Kesten and Spitzer [1] proved that when the random walk and the random scenery belong to the domains of attraction of different stable laws of indices $1 < \alpha \leq 2$ and $0 < \beta \leq 2$, respectively, then there exists $\delta > 1/2$ such that $\{n^{-\delta} K_{\lfloor nt \rfloor}, t \geq 0\}$ converges weakly as $n \longrightarrow \infty$ to a continuous δ-self-similar process with stationary increments, δ being related to α and β by $\delta = 1 - \alpha^{-1} + (\alpha\beta)^{-1}$. The limiting process can be seen as a mixture of β-stable processes, but it is not a stable process. When $0 < \alpha < 1$ and for arbitrary β, the sequence $\{n^{-1/\beta} K_{\lfloor nt \rfloor}, t \geq 0\}$ converges weakly, as $n \longrightarrow \infty$, to a stable process with index β (see Castell et al. [4]). Bolthausen [5] (see also Deligiannidis and Utev [6]) gave a method to solve the case $\alpha = 1$ and $\beta = 2$ and, especially, he proved that when S is a recurrent $\mathbb{Z}^2$-random walk, the sequence $\{(n \log n)^{-1/2} K_{\lfloor nt \rfloor}, t \geq 0\}$ satisfies a functional central limit theorem. More recently, the case S one- or two-dimensional random walks and $\beta \in (0, 2)$ was solved in Castell et al. [4]; the authors prove that the sequence $\{n^{-1/\beta} (\log n)^{1/\beta - 1} K_{\lfloor nt \rfloor}, t \geq 0\}$ converges weakly to a stable process with index β. Finally for any arbitrary transient random walk, it can be shown that the sequence $\{n^{-1/2} K_n, n \in \mathbb{N}\}$ is asymptotically normal (see for instance Spitzer [7] page 53).

Among others, we can cite strong approximation results [8–10], laws of the iterated logarithm [11–13], limit theorems for correlated sceneries or walks [14–17], large and moderate deviations results [18–22], and ergodic and mixing properties (see the survey [23]).

The problem we investigate in the present paper has already been studied in Lewis [24] in the case that random sceneries σ's satisfy $\mathbb{E}[\sigma(0)] = 0$ and $\mathbb{E}[\sigma^2(0)] = 1$, and the random walk S (which can be $\mathbb{Z}^d$-valued) satisfies some mild conditions. Lewis [24] established the following LIL:

$$\limsup_{n\longrightarrow\infty} \frac{\left|V_{2,n}^{-1}K_n\right|}{\sqrt{2\log\log n}} = 1 \quad \text{a.s.}, \tag{2}$$

where $\xi_n(x)$ is the number of visits of the random walk to the point $x \in \mathbb{Z}$ in the time interval $[0, n]$, i.e.,

$$\xi_n(x) := \#\{0 \le k \le n : \lfloor S_k \rfloor = x\} = \sum_{k=0}^{n} I\{\lfloor S_k \rfloor = x\}, \qquad n \ge 0,\ x \in \mathbb{Z}. \tag{3}$$

Here and in the sequel, the following notation is used: for $a > 0$ and $n \ge 0$,

$$V_{a,n} = \left(\sum_{x\in\mathbb{Z}} \mathbb{E}\left[\xi_n^a(x)\right]\right)^{1/a}. \tag{4}$$

It is therefore natural to investigate limit behavior of RWRS K_n when the sceneries σ's do not have finite second moment. For the sake of convenience, we are summarizing here the main assumptions we are making on the sceneries σ's. Assume that the sceneries σ's belong to the domain of attraction of a stable law G_β $(0 < \beta < 2)$; that is, σ's satisfy that

$$\lim_{n\longrightarrow\infty} \mathbb{P}\left(n^{-1/\beta}\sum_{k=1}^{n}\sigma(k) \le x\right) = G_\beta(x), \tag{5}$$

where G_β is a stable distribution of index $0 < \beta < 2$, with characteristic function

$$\exp\left\{-|\theta|^\beta\left(A_1 + iA_2 \operatorname{sgn}\theta\right)\right\} \tag{6}$$

for some $0 < A_1 < \infty$, $|A_1^{-1}A_2| \le \tan(\pi/2)\beta$. From the known characterization of the domain of attraction of a stable law G_β (Feller [25], II, Chap. 17) it follows that, for $0 < \beta < 2$, (5) and (6) are equivalent to

$$\mathbb{P}(\sigma(0) \ge x) \sim \frac{c_{1,1}}{x^\beta}, \qquad \mathbb{P}(\sigma(0) \le -x) \sim \frac{c_{1,2}}{x^\beta} \tag{7}$$

as $x \longrightarrow \infty$ for suitable constants $c_{1,1}$ and $c_{1,2}$. Note that (5) and (6) imply

$$\mathbb{E}[\sigma(0)] = 0 \quad \text{if } \beta > 1. \tag{8}$$

For $\beta = 1$ we impose an additional condition (stronger than (5) and (6)), namely, that for some positive constant c_0,

$$\left|\mathbb{E}\left[\sigma(0) I\left(|\sigma(0)| \le \rho\right)\right]\right| \le c_0 < \infty \quad \forall \rho > 0. \tag{9}$$

It is well known that LILs for heavy tailed random variables are different from those for random variables attracted to the normal law. We have to use power norming and the resulting limit theorem is called Chover-type LIL (see Chover [26]). The main results of this paper read as follows.

Theorem 1. *Let $\sigma = \{\sigma(x), x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables satisfying (5) and (9), and $X = \{X_i, i \ge 1\}$ be a sequence of i.i.d. random variables with a common distribution F and independent of σ's. Assume that F is supported on $[0, \infty)$, absolutely continuous, and $1 - F(x) \sim c_{1,1}x^{-\alpha}$, $0 < \alpha < 2$. Then*

$$\limsup_{n\longrightarrow\infty} \left|V_{\beta,n}^{-1}K_n\right|^{1/\log\log n} = e^{1/\beta} \quad \text{a.s.} \tag{10}$$

Theorem 1 gives the following information about the maximal growth rate of RWRS K_n.

Corollary 2. *We have for all $\varepsilon > 0$, with probability one,*

$$\left|K_n\right| > V_{\beta,n}(\log n)^{(1+\varepsilon)/\beta} \quad \text{for at most finitely many } n \tag{11}$$

and

$$\left|K_n\right| > V_{\beta,n}(\log n)^{(1-\varepsilon)/\beta} \quad \text{for infinitely many } n. \tag{12}$$

Remark 3. It follows from Corollary 2 that the maximal growth rate of K_n is of the order $V_{\beta,n}(\log n)^{1/\beta}$. Equation (10) is equivalent to (11) and (12). In fact, (11) implies that

$$\log\left(V_{\beta,n}^{-1}\left|K_n\right|\right) - \left(\frac{(1+\varepsilon)}{\beta}\right)\log\log n \le 0 \quad \text{a.s.} \tag{13}$$

for all large n. Letting $\varepsilon \downarrow 0$, it yields that the limit superior on left-hand side of (10) is less than $e^{1/\beta}$. Equation (12) implies that

$$\log\left(V_{\beta,n}^{-1}\left|K_n\right|\right) - \left(\frac{(1-\varepsilon)}{\beta}\right)\log\log n \ge 0 \quad \text{a.s.} \tag{14}$$

for infinitely many n. Letting $\varepsilon \downarrow 0$, it yields that the limit superior on left-hand side of (10) is greater than $e^{1/\beta}$. Moreover, from the proof of Theorem 1 below, the upper bound of (10) does not need the assumptions that F is supported on $[0, \infty)$ and absolutely continuous.

Complementary to Theorem 1 we have the following clustering statement, which gives additional information about the path behavior of RWRS K_n.

Theorem 4. *Under the assumptions of Theorem 1, with probability one, every point in the interval $(1, e^{1/\beta}]$ is a cluster point of the sequence:*

$$\left\{\left|V_{\beta,n}^{-1}K_n\right|^{1/\log\log n} : n \ge 1\right\}. \tag{15}$$

Throughout this paper, we use the notations: $a_n = o(b_n)$ if $a_n/b_n \longrightarrow 0$, $a_n \sim b_n$ if $\lim a_n/b_n = 1$ and $a_n \asymp b_n$ if $c_{1,2} \le \lim\inf a_n/b_n \le \lim\sup a_n/b_n \le c_{1,3}$. Let i.o. mean infinitely often, a.s. mean almost surely, $\mathbb{E}[\cdot]$ mean expectation, and $\mathbb{E}_{\mathcal{F}}[\cdot]$ mean conditional expectation given σ-field $\mathcal{F}$. An unspecified positive and finite constant will be denoted by c, which may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$. The sign $\lfloor\cdot\rfloor$ sometimes denotes the integer part anf at other times denotes usual brackets; it will be clear from the context. Since we shall deal with index n which ultimately tends to infinity, our statements, sometimes without further mention, are valid only when n is sufficiently large.

2. Preliminaries

In this section we investigate some technical results necessary for our argumentation. We will first present a version of the Borel-Cantelli lemma to sums of conditional probabilities (see, e.g., Theorem 2.8.5 in Stout [27]).

Lemma 5. *Let $\{E_n, n \ge 1\}$ be a sequence of arbitrary events and $\{\mathcal{G}_n, n \ge 1\}$ be an increasing sequence of σ-fields such that $E_n \in \mathcal{G}_n$ for each $n \ge 1$. Then*

$$[E_n \ \ i.o.] = \left[\sum_{n=1}^{\infty} \mathbb{P}\left(E_n \mid \mathcal{G}_{n-1}\right) = \infty\right], \tag{16}$$

that is, $\sum_{n=1}^{\infty} \mathbb{P}(E_n \mid \mathcal{G}_{n-1}) < \infty$ implies that E_n occur at most finitely often and $\sum_{n=1}^{\infty} \mathbb{P}(E_n \mid \mathcal{G}_{n-1}) = \infty$ implies that E_n occur infinitely often.

We will need the following large deviation inequalities for RWRS, which may be of independent interest.

Lemma 6. *Let $\{\sigma(x), x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables satisfying (5) and (9), and $\{X_i, i \ge 1\}$ be a sequence of arbitrary random variables and independent of σ's. Let $\{t_n, n \ge 1\}$ be a sequence of positive numbers such that $t_n \longrightarrow \infty$. Then*

$$\begin{aligned} 0 &< \lim_{n\longrightarrow\infty}\inf t_n^{\beta}\mathbb{P}\left(\left|K_n\right| \ge t_n V_{\beta,n}\right) \\ &\le \lim_{n\longrightarrow\infty}\sup t_n^{\beta}\mathbb{P}\left(\left|K_n\right| \ge t_n V_{\beta,n}\right) < \infty. \end{aligned} \tag{17}$$

Proof. We denote by $\mathcal{F} = \sigma(X_1, X_2, \ldots)$ the σ-field generated by the random walk and

$$\tilde{\eta}_n(x) = \xi_n(x)\,\sigma(x)\, I\left(|\sigma(x)| \le t_n V_{\beta,n}\xi_n^{-1}(x)\right), \qquad n \ge 0,\ \ x \in \mathbb{Z}. \tag{18}$$

By (7), for all $u > 0$ and $0 < \beta < 2$,

$$c_{2,1}u^{-\beta} \le \mathbb{P}\left(|\sigma(0)| > u\right) \le c_{2,2}u^{-\beta}. \tag{19}$$

Thus,

$$\begin{aligned} &\sum_{x\in\mathbb{Z}} \mathbb{P}\left(|\sigma(x)| \ge t_n V_{\beta,n}\xi_n^{-1}(x)\right) \\ &\quad = \sum_{x\in\mathbb{Z}} \mathbb{E}\left[\mathbb{P}\left(|\sigma(x)| \ge t_n V_{\beta,n}\xi_n^{-1}(x) \mid \mathcal{F}\right)\right] \\ &\quad \le c_{2,3}t_n^{-\beta}. \end{aligned} \tag{20}$$

By (19), for all $x \in \mathbb{Z}$,

$$\begin{aligned} &\mathbb{E}_{\mathcal{F}}\left[\left(\tilde{\eta}_n(x)\right)^2\right] \\ &\quad \le \mathbb{E}_{\mathcal{F}}\left[\xi_n^2(x)\,\sigma^2(x)\, I\left(|\sigma(x)| \le t_n V_{\beta,n}\xi_n^{-1}(x)\right)\right] \\ &\quad = \xi_n^2(x)\int_0^{t_n V_{\beta,n}\xi_n^{-1}(x)} u\mathbb{P}\left(|\sigma(x)| \ge u\right) du \\ &\quad \le c_{2,4}\xi_n^2(x)\int_0^{t_n V_{\beta,n}\xi_n^{-1}(x)} u^{1-\beta} du \\ &\quad \le c_{2,5}t_n^{2-\beta}V_{\beta,n}^{2-\beta}\xi_n^{\beta}(x). \end{aligned} \tag{21}$$

It follows that

$$\mathbb{E}\left[\left(\tilde{\eta}_n(x)\right)^2\right] \le c_{2,5}t_n^{2-\beta}V_{\beta,n}^{2-\beta}\mathbb{E}\left[\xi_n^{\beta}(x)\right]. \tag{22}$$

On the other hand, if $\beta \in (0,1)$,

$$\begin{aligned} &\mathbb{E}_{\mathcal{F}}\left[\tilde{\eta}_n(x)\right] \\ &\quad \le \mathbb{E}_{\mathcal{F}}\left[\xi_n(x)\,|\sigma(x)|\, I\left(|\sigma(x)| \le t_n V_{\beta,n}\xi_n^{-1}(x)\right)\right] \\ &\quad = \xi_n(x)\int_0^{t_n V_{\beta,n}\xi_n^{-1}(x)} \mathbb{P}\left(|\sigma(x)| > u\right) du \\ &\quad \le c_{2,6}\xi_n(x)\int_0^{t_n V_{\beta,n}\xi_n^{-1}(x)} u^{-\beta} du \le c_{2,7}t_n^{1-\beta}V_{\beta,n}^{1-\beta}\xi_n^{\beta}(x) \end{aligned} \tag{23}$$

for all $x \in \mathbb{Z}$; if $\beta = 1$, by (9),

$$\mathbb{E}\left[\tilde{\eta}_n(x)\right] \le \mathbb{E}\left[\left|\mathbb{E}_{\mathcal{F}}\left[\tilde{\eta}_n(x)\right]\right|\right] \le c_{2,8}\mathbb{E}\left[\xi_n(x)\right] \tag{24}$$

for all $x \in \mathbb{Z}$; and, if $\beta \in (1,2)$, by (8) and (19),

$$\begin{aligned} &\mathbb{E}_{\mathcal{F}}\left[\tilde{\eta}_n(x)\right] \\ &\quad \le \mathbb{E}_{\mathcal{F}}\left[\xi_n(x)\,|\sigma(x)|\, I\left(|\sigma(x)| \ge t_n V_{\beta,n}\xi_n^{-1}(x)\right)\right] \\ &\quad = \xi_n(x)\int_{t_n V_{\beta,n}\xi_n^{-1}(x)}^{\infty} \mathbb{P}\left(|\sigma(x)| > u\right) du \\ &\quad \le c_{2,9}\xi_n(x)\int_{t_n V_{\beta,n}\xi_n^{-1}(x)}^{\infty} u^{-\beta} du \\ &\quad \le c_{2,10}t_n^{1-\beta}V_{\beta,n}^{1-\beta}\xi_n^{\beta}(x) \end{aligned} \tag{25}$$

for all $x \in \mathbb{Z}$. Hence, by (23)-(25) and making use of the fact that $\mathbb{E}[\tilde{\eta}_n(x)] = \mathbb{E}[\mathbb{E}_{\mathcal{F}}[\tilde{\eta}_n(x)]]$, we have

$$\left(t_n V_{\beta,n}\right)^{-1}\sum_{x\in\mathbb{Z}}\left|\mathbb{E}\left[\tilde{\eta}_n(x)\right]\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{26}$$

Thus, by (22) and (26),

$$\begin{aligned}
&\mathbb{P}\left(\sum_{x\in\mathbb{Z}}\widetilde{\eta}_n(x)\geq t_nV_{\beta,n}\right)\\
&\quad\leq \mathbb{P}\left(\sum_{x\in\mathbb{Z}}\left(\widetilde{\eta}_n(x)-\mathbb{E}\left[\widetilde{\eta}_n(x)\right]\right)\geq\frac{t_nV_{\beta,n}}{2}\right)\\
&\quad\leq 4\left(t_nV_{\beta,n}\right)^{-2}\sum_{x\in\mathbb{Z}}\mathbb{E}\left[\left(\widetilde{\eta}_n(x)-\mathbb{E}\left[\widetilde{\eta}_n(x)\right]\right)^2\right]\\
&\quad\leq 4\left(t_nV_{\beta,n}\right)^{-2}\sum_{x\in\mathbb{Z}}\mathbb{E}\left[\left(\widetilde{\eta}_n(x)\right)^2\right]\leq c_{2,11}t_n^{-\beta}.
\end{aligned}\tag{27}$$

Noting that we can rewrite K_n as

$$K_n=\sum_{x\in\mathbb{Z}}\xi_n(x)\sigma(x),\tag{28}$$

we have that

$$\begin{aligned}
\mathbb{P}\left(K_n\geq t_nV_{\beta,n}\right)\leq&\sum_{x\in\mathbb{Z}}\mathbb{P}\left(\left|\sigma(x)\right|\geq t_nV_{\beta,n}\xi_n^{-1}(x)\right)\\
&+\mathbb{P}\left(\sum_{x\in\mathbb{Z}}\widetilde{\eta}_n(x)\geq t_nV_{\beta,n}\right).
\end{aligned}\tag{29}$$

It follows from (20), (27), and (29) that

$$\mathbb{P}\left(K_n\geq t_nV_{\beta,n}\right)\leq c_{2,12}t_n^{-\beta}.\tag{30}$$

By replacing $\sigma(x)$ with $-\sigma(x)$, we have

$$\mathbb{P}\left(-K_n\geq t_nV_{\beta,n}\right)\leq c_{2,13}t_n^{-\beta}.\tag{31}$$

This, together with (30), yields

$$\mathbb{P}\left(\left|K_n\right|\geq t_nV_{\beta,n}\right)\leq c_{2,14}t_n^{-\beta}.\tag{32}$$

It yields the right-hand side of (17).

To verify the left-hand side of (17), we denote by A_x and B_x the events $\{|\xi_n(x)\sigma(x)|\geq(1+\varepsilon)t_nV_{\beta,n}\}$ and $\{|\sum_{y\in\mathbb{Z},y\neq x}\xi_n(y)\sigma(y)|<\varepsilon t_nV_{\beta,n}\}$, $\varepsilon>0, x\in\mathbb{Z}$, respectively. By (19) and some conditional argument, we have

$$\sum_{x\in\mathbb{Z}}\mathbb{P}\left(A_x\right)\asymp t_n^{-\beta}\longrightarrow 0.\tag{33}$$

On the other hand, by (28),

$$\begin{aligned}
\mathbb{P}\left(B_x\right)\geq&\mathbb{P}\left(\left|K_n\right|\leq\left(\frac{\varepsilon}{2}\right)t_nV_{\beta,n}\right)\\
&-\mathbb{P}\left(\left|\xi_n(x)\sigma(x)\right|\geq\left(\frac{\varepsilon}{2}\right)t_nV_{\beta,n}\right).
\end{aligned}\tag{34}$$

This, together with (20) and (32), yields that

$$\mathbb{P}\left(B_x\right)\longrightarrow 1\quad\forall x\in\mathbb{Z}.\tag{35}$$

Note that

$$\begin{aligned}
\mathbb{P}\left(\left|K_n\right|>t_nV_{\beta,n}\right)&\geq\mathbb{E}\left[\mathbb{P}\left(\bigcup_{x\in\mathbb{Z}}\left(A_x\cap B_x\right)\mid\mathscr{F}\right)\right]\\
&=\mathbb{E}\left[\lim_{M\longrightarrow\infty}\mathbb{P}\left(\bigcup_{x=-M}^{M}\left(A_x\cap B_x\right)\mid\mathscr{F}\right)\right]\\
&\geq\mathbb{E}\left[\lim_{M\longrightarrow\infty}\left\{\sum_{x=-M}^{M}\mathbb{P}\left(A_x\cap B_x\mid\mathscr{F}\right)\right.\right.\\
&\qquad\left.\left.-\sum_{-M\leq x<y\leq M}\mathbb{P}\left(\left(A_x\cap B_x\right)\cap\left(A_y\cap B_y\right)\mid\mathscr{F}\right)\right\}\right]\\
&\geq\sum_{x\in\mathbb{Z}}\mathbb{P}\left(A_xB_x\right)-\sum_{-\infty\leq x<y\leq\infty}\mathbb{P}\left(A_xA_y\right)\\
&=\sum_{x\in\mathbb{Z}}\mathbb{P}\left(A_x\right)\mathbb{P}\left(B_x\right)-\left(\sum_{x\in\mathbb{Z}}\mathbb{P}\left(A_x\right)\right)^2.
\end{aligned}\tag{36}$$

Thus, by (33)-(36),

$$\mathbb{P}\left(\left|K_n\right|>t_nV_{\beta,n}\right)\geq c_{2,15}\sum_{x\in\mathbb{Z}}\mathbb{P}\left(A_x\right)\geq c_{2,16}t_n^{-\beta}.\tag{37}$$

It yields the left-hand side of (17). The proof of Lemma 6 is completed. □

We will also need the following two technical results.

Lemma 7. *Let $\{X_i, i\geq 1\}$ be a sequence of i.i.d. nonnegative random variables with a common distribution F. Assume that F is absolutely continuous and $1-F(x)\sim c_{2,17}x^{-\alpha}$, $0<\alpha<2$. Then, for all $r>0$,*

$$S_n\geq n^{1/\alpha}\left(\log n\right)^{-r}\quad\text{a.s.}\tag{38}$$

Proof. Let $M_n=\max\{X_1,\ldots,X_n\}$, $\overline{F}(x)=1-F(x)=P(X_1>x)$, and $\overline{F}^*$ be the inverse of $\overline{F}$. Let $U,U_1,U_2,\ldots,U_n$ be i.i.d. random variables with the distribution of U uniform over $(0,1)$ and $M_n^*=\max\{U_1,U_2,\ldots,U_n\}$. Let $\theta>1$ be a constant which will be chosen later on and $n_k=\lfloor\theta^k\rfloor$, $k\geq 1$. By making use of the fact that $F(X_n)$ is a uniform $(0,1)$ random variable, we have $M_n^*\overset{\mathrm{d}}{=}F(M_n)$, $n\geq 1$. On the other hand, $\overline{F}^*(y)\sim c_{2,18}y^{-1/\alpha}$, $0<y\leq 1$. Thus, since X_i's are nonnegative, $\overline{F}$ and $\overline{F}^*$ are nonincreasing:

$$\begin{aligned}
P\left(S_{n_k}\leq n_k^{1/\alpha}\left(\log n_k\right)^{-r}\right)&\leq P\left(M_{n_k}\leq n_k^{1/\alpha}\left(\log n_k\right)^{-r}\right)\\
&\leq P\left(\overline{F}^*\left(\overline{F}\left(M_{n_k}\right)\leq\overline{F}^*\left(c_{2,19}n_k^{-1}\left(\log n_k\right)^{\alpha r}\right)\right)\right)\\
&=P\left(\overline{F}\left(M_{n_k}\right)\geq c_{2,19}n_k^{-1}\left(\log n_k\right)^{\alpha r}\right)\\
&=P\left(1-M_{n_k}^*\geq c_{2,19}n_k^{-1}\left(\log n_k\right)^{\alpha r}\right)\\
&=P\left(M_{n_k}^*\leq 1-c_{2,19}n_k^{-1}\left(\log n_k\right)^{\alpha r}\right)
\end{aligned}$$

$$= \left(P\left(U \le 1 - c_{2,19} n_k^{-1} (\log n_k)^{\alpha r}\right)\right)^{n_k}$$
$$\le \exp\left(-c_{2,19} (\log n_k)^{\alpha r}\right). \tag{39}$$

By making use of Borel-Cantelli lemma,

$$\liminf_{k\longrightarrow\infty} n_k^{-1/\alpha} (\log n_k)^r S_{n_k} \ge 1 \quad \text{a.s.} \tag{40}$$

To each n, there exists an integer k such that $n_k \le n \le n_{k+1}$. Thus, by (40),

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty} n^{-1/\alpha} (\log n)^r S_n \\
&\quad \ge \liminf_{k\longrightarrow\infty} \min_{n_k\le n\le n_{k+1}} n^{-1/\alpha} (\log n)^r S_n \\
&\quad \ge \liminf_{k\longrightarrow\infty} \left(n_k^{1/\alpha} n_{k+1}^{-1/\alpha}\right) n_k^{-1/\alpha} (\log n_k)^r S_{n_k} \ge \theta^{-1/\alpha} \\
&\qquad \text{a.s.}
\end{aligned} \tag{41}$$

Letting $\theta \downarrow 1$, (38) is proved. The proof of Lemma 7 is completed. □

Lemma 8. *Let $\{\sigma(x), x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables satisfying (5) and (9), and $\{X_i, i \ge 1\}$ be a sequence of arbitrary random variables and independent of σ's. Let $\{a_n, n \ge 1\}$ be a nondecreasing sequences of positive integers such that $a_n \longrightarrow \infty$, $W_n = \sum_{x=-a_n}^{a_n} \xi_n(x)\sigma(x)$ and $\gamma_n = (\sum_{x=-a_n}^{a_n} \mathbb{E}[\xi_n^\beta(x)])^{1/\beta}$. Then*

$$\limsup_{n\longrightarrow\infty} \left|\gamma_n^{-1} W_n\right|^{1/\log\log n} \le e^{1/\beta} \quad \text{a.s.} \tag{42}$$

Proof. Let $n_0 = 0$ and $n_k = \inf\{n : \gamma_n \ge 2^k\}$ $(k \ge 1)$. Since γ_n is increasing and $\gamma_n \longrightarrow \infty$, we have that $n_k \longrightarrow \infty$ and $2^k \le \gamma_{n_k} < 2^{k+1}$. Noting

$$\gamma_n \le \left(\sum_{x\in\mathbb{Z}} \xi_n^2(x)\right)^{1/\beta} \le \left(\sum_{x\in\mathbb{Z}} \xi_n(x)\right)^{2/\beta} = n^{2/\beta}, \tag{43}$$

we have

$$n_k \ge \gamma_{n_k}^{\beta/2} \ge 2^{\beta k/2}. \tag{44}$$

For the sake of convenience, we denote $u_k = \gamma_{n_k} (\log n_{k-1})^{(1+\varepsilon)/\beta}$, $\mathcal{S}_n = \{x \in \mathbb{Z} : x \in [-a_n, a_n]\}$,

$$\begin{aligned}
\eta_n(x) &= \eta_n(k, x) \\
&= \begin{cases} \xi_n(x)\sigma(x) I\left(\sigma(x) \le u_k \xi_{n_k}^{-1}(x)\right) & \text{if } 0 < \beta \le 1, \\ \xi_n(x) \min\left\{\sigma(x), u_k \xi_{n_k}^{-1}(x)\right\} & \text{if } 1 < \beta < 2, \end{cases}
\end{aligned} \tag{45}$$

and $\widetilde{W}_n = \sum_{x\in\mathcal{S}_n} \eta_n(x)$ for $\varepsilon > 0$, $x \in \mathbb{Z}$ and $n_{k-1} < n \le n_k$.

By (19) and (44),

$$\mathbb{P}\left(\sigma(x) > u_k \xi_{n_k}^{-1}(x) \mid \mathcal{F}\right) \le c_{2,20} k^{-(1+\varepsilon)} \gamma_{n_k}^{-\beta} \xi_{n_k}^{\beta}(x). \tag{46}$$

It follows that

$$\sum_{x=-a_{n_k}}^{a_{n_k}} \mathbb{P}\left(\sigma(x) > u_k \xi_{n_k}^{-1}(x)\right) \le c_{2,21} k^{-(1+\varepsilon)}. \tag{47}$$

Since

$$\begin{aligned}
&\min\left\{\sigma(x), u_k \xi_{n_k}^{-1}(x)\right\} \\
&\quad = \sigma(x) I\left(\sigma(x) \le u_k \xi_{n_k}^{-1}(x)\right) \\
&\qquad + u_k \xi_{n_k}^{-1}(x) I\left(\sigma(x) > u_k \xi_{n_k}^{-1}(x)\right),
\end{aligned} \tag{48}$$

following the same argument as the proof of (29), we have

$$\begin{aligned}
&u_k^{-1} \max_{n_{k-1}<n\le n_k} \left|\mathbb{E}\left[\widetilde{W}_n \mid \mathcal{F}\right]\right| \\
&\quad \le u_k^{-1} \max_{n_{k-1}<n\le n_k} \sum_{x=-a_n}^{a_n} \left|\mathbb{E}\left[\eta_n(x) \mid \mathcal{F}\right]\right| \longrightarrow 0
\end{aligned} \tag{49}$$

as $k \longrightarrow \infty$ for $0 < \beta < 2$. On the other hand, by (22) and noting

$$\begin{aligned}
&\mathbb{E}\left[\eta_n^2(x) \mid \mathcal{F}\right] \\
&\quad \le 2\mathbb{E}\left[\xi_{n_k}^2(x)\sigma^2(x) I\left(\sigma(x) \le u_k \xi_{n_k}^{-1}(x)\right) \mid \mathcal{F}\right] \\
&\qquad + 2u_k^2 \mathbb{P}\left(\sigma(x) \ge u_k \xi_{n_k}^{-1}(x) \mid \mathcal{F}\right),
\end{aligned} \tag{50}$$

we have for $n_{k-1} < n \le n_k$ and $0 < \beta < 2$,

$$\mathbb{E}\left[\left(\eta_n(x)\right)^2 \mid \mathcal{F}\right] \le c_{2,22} u_k^{2-\beta} \xi_{n_k}^{\beta}(x). \tag{51}$$

It follows that

$$\sum_{x\in\mathcal{S}_{n_k}} \mathbb{E}\left[\left(\eta_{n_k}(x)\right)^2 \mid \mathcal{F}\right] \le c_{2,23} u_k^{2-\beta} \gamma_{n_k}^{\beta}. \tag{52}$$

From Newman and Wright [28], we call a finite collection of random variables Z_i, $1 \le i \le m$, which is associated if any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^m$ such that $F_i = f_i(Z_1, \ldots, Z_m)$ have finite variance for $i = 1, 2$, $\operatorname{cov}(F_1, F_2) \ge 0$; an infinite collection is associated if every finite subcollection is associated. It is not difficult to demonstrate that independent variables are always associated. Moreover, given $\mathcal{F}$, $\eta_n(x) - \mathbb{E}[\eta_n(x) \mid \mathcal{F}]$ are nonincreasing functions on σ and are also associated variables by Esary et al. [29]. Consequently, by Theorem 2 of Newman and Wright [28] and (52),

$$\begin{aligned}
&\mathbb{E}\left[\max_{n_{k-1}<n\le n_k} \left(\widetilde{W}_n - \mathbb{E}\left[\widetilde{W}_n \mid \mathcal{F}\right]\right)^2 \mid \mathcal{F}\right] \\
&\quad \le \mathbb{E}\left[\left(\widetilde{W}_{n_k} - \mathbb{E}\left[\widetilde{W}_{n_k} \mid \mathcal{F}\right]\right)^2 \mid \mathcal{F}\right] \\
&\quad \le \sum_{x\in\mathcal{S}_{n_k}} \mathbb{E}\left[\left(\eta_{n_k}(x)\right)^2 \mid \mathcal{F}\right] \le c_{2,23} u_k^{2-\beta} \gamma_{n_k}^{\beta}.
\end{aligned} \tag{53}$$

Hence

$$u_k^{-2}\mathbb{E}\left[\max_{n_{k-1}<n\le n_k}\left(\widetilde{W}_n-\mathbb{E}\left[\widetilde{W}_n\mid\mathcal{F}\right]\right)^2\mid\mathcal{F}\right]\le c_{2,24}k^{-(1+\varepsilon)}. \tag{54}$$

Note that

$$\begin{aligned}\mathbb{P}\left(\max_{n_{k-1}<n\le n_k}W_n\ge u_k\right)&\le\sum_{x=-a_{n_k}}^{a_{n_k}}\mathbb{P}\left(\sigma(x)\xi_{n_k}(x)>u_k\right)\\&+\mathbb{P}\left(\max_{n_{k-1}<n\le n_k}\widetilde{W}_n\ge u_k\right)\le\sum_{x=-a_{n_k}}^{a_{n_k}}\mathbb{P}\left(\sigma(x)\right.\\&\left.>u_k\xi_{n_k}^{-1}(x)\right)+\mathbb{E}\left[\mathbb{P}\left(\max_{n_{k-1}<n\le n_k}\left(\widetilde{W}_n-\mathbb{E}\left[\widetilde{W}_n\mid\mathcal{F}\right]\right)\right.\right.\\&\left.\left.\ge 2^{-1}u_k\mid\mathcal{F}\right)\right]\le\sum_{x=-a_{n_k}}^{a_{n_k}}\mathbb{P}\left(\sigma(x)>u_k\xi_{n_k}^{-1}(x)\right)\\&+\mathbb{E}\left[u_k^{-2}\mathbb{E}\left[\max_{n_{k-1}<n\le n_k}\left(\widetilde{W}_n-\mathbb{E}\left[\widetilde{W}_n\mid\mathcal{F}\right]\right)^2\mid\mathcal{F}\right]\right].\end{aligned} \tag{55}$$

Thus, by (47), (54), (55) and making use of Borel-Cantelli lemma,

$$\limsup_{k\longrightarrow\infty}\max_{n_{k-1}<n\le n_k}W_n\gamma_{n_k}^{-1}\left(\log n_{k-1}\right)^{-(1+\varepsilon)/\beta}\le c_{2,25}\quad\text{a.s.} \tag{56}$$

By replacing $\sigma(x)$ with $-\sigma(x)$, following the same argument, we have that (56) also holds if W_n is replaced with $-W_n$. It yields

$$\limsup_{k\longrightarrow\infty}\max_{n_{k-1}<n\le n_k}\left|W_n\right|\gamma_{n_k}^{-1}\left(\log n_{k-1}\right)^{-(1+\varepsilon)/\beta}\le c_{2,26}\quad\text{a.s.} \tag{57}$$

Therefore, by (57),

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left|W_n\right|\gamma_n^{-1}\left(\log n\right)^{-(1+\varepsilon)/\beta}\\&\le\limsup_{k\longrightarrow\infty}\max_{n_{k-1}<n\le n_k}\left|W_n\right|\gamma_n^{-1}\left(\log n\right)^{-(1+\varepsilon)/\beta}\\&\le c_{2,27}\limsup_{k\longrightarrow\infty}\max_{n_{k-1}<n\le n_k}\left|W_n\right|\gamma_{n_k}^{-1}\left(\log n_{k-1}\right)^{-(1+\varepsilon)/\beta}\\&\le c_{2,28}\quad\text{a.s.}\end{aligned} \tag{58}$$

It follows that

$$\log\left(\left|W_n\right|\gamma_n^{-1}\right)-\left(\frac{1+2\varepsilon}{\beta}\right)\log\log n\le 0\quad\text{a.s.} \tag{59}$$

Letting $\varepsilon\downarrow 0$, we obtain (49). The proof of Lemma 8 is completed. □

3. Proofs

Proof of Theorem 1 . Let $\varepsilon\in(0,1)$ and $\tau\in(0,1/\alpha)$ be two arbitrary constants. Let $\mathcal{S}_n$, W_n, and γ_n be defined as in Lemma 8 with $a_n=n^{1/\alpha}(\log n)^{1/\alpha+\tau}$. By Chover's law of the iterated logarithm (see Chover [26] and Qi and Cheng [30]) we have $S_n=o(a_n)$ a.s. Thus, for any sample point ω for which it holds, there exists $N_0=N_0(\omega)$ such that for all $n\ge N_0$ and $|x|\ge\operatorname{Card}(\mathcal{S}_n)$, where $\operatorname{Card}(\mathcal{S}_n)$ is the number of integers belonging to $\mathcal{S}_n$, $\xi_n(x)(\omega)=\sum_{i=0}^n I[S_i=x](\omega)=0$. It follows that, for all $n\ge N_0$,

$$\begin{aligned}\sum_{x\notin\mathcal{S}_n}\sigma(x)\xi_n(x)&=0\quad\text{a.s.},\\\sum_{x\notin\mathcal{S}_n}\xi_n^\beta(x)&=0\quad\text{a.s.}\end{aligned} \tag{60}$$

By (28) and (60), we have that $K_n=W_n$ a.s. and $V_{\beta,n}=\gamma_n$ a.s. Hence, to prove (10), by (28), it suffices to prove that, for all $0<\varepsilon<1$,

$$\limsup_{n\longrightarrow\infty}\left|W_n\right|\gamma_n^{-1}\left(\log n\right)^{-(1+\varepsilon)/\beta}\le c_{3,1}\quad\text{a.s.} \tag{61}$$

and

$$\limsup_{n\longrightarrow\infty}\left|W_n\right|\gamma_n^{-1}\left(\log n\right)^{-(1-\varepsilon)/\beta}\ge c_{3,2}\quad\text{a.s.} \tag{62}$$

By (42), (61) holds. Thus, it remains to prove (62). Let $\mathcal{T}_n=\{x\in\mathbb{Z}:x\in[0,a_n]\}$. Let $\theta>1$ be a constant. For $k\ge 1$, let $m_k=e^{k^\theta}$ and $n_k=\inf\{n:n\in\{m_1,m_2,\ldots\},\gamma_n\ge e^{k^\theta}\}$. Since γ_n is increasing and $\gamma_n\longrightarrow\infty$, we have that n_k's are well-defined, $n_k\longrightarrow\infty$ and $e^{k^\theta}\le\gamma_{n_k}<e^{(k+1)^\theta}$. Noting $n^\beta\sim(\sum_{x\in\mathcal{T}_n}\xi_n(x))^\beta\le\gamma_n^\beta$ if $0<\beta\le 1$ and $n\sim\sum_{x\in\mathcal{T}_n}\xi_n(x)\le\gamma_n^\beta$ if $1<\beta<2$, we have, for $0<\beta<2$,

$$n_k\le\gamma_{n_k}^{(\beta\vee 1)}\le e^{(\beta\vee 1)(k+1)^\theta}. \tag{63}$$

For $k\ge 1$, we denote $\lambda_{1,k}:=\sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}}\mathbb{E}[\xi_{n_{2k+1}}^\beta(x)]$, $\lambda_{2,k}:=\sum_{x\in\mathcal{T}_{n_{2k-1}}}\mathbb{E}[\xi_{n_{2k+1}}^\beta(x)]$, and $\nu_k:=\lambda_{1,k}^{1/\beta}(\log n_{2k+1})^{(1-\varepsilon)/\beta}$. Denote by C_x, D_x, and E_x the events $C_x=\{|\sigma(x)\xi_{n_{2k+1}}(x)|\ge(1+\varepsilon)\nu_k\}$, $D_x=\{|\sum_{y\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}},y\ne x}\sigma(y)\xi_{n_{2k+1}}(y)|<\varepsilon\nu_k\}$, and $E_k=\{|\sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}}\sigma(x)\xi_{n_{2k+1}}(x)|\ge\nu_k\}$. Let $\mathcal{G}_k$ be the σ-field generated by $\{\sigma(x),x\in\mathcal{T}_{n_{2k+1}}\}\cup\mathcal{F}$. Then, $E_k\in\mathcal{G}_k$. By (19) and (63),

$$\begin{aligned}&\sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}}\mathbb{P}\left(C_x\mid\mathcal{G}_{k-1}\right)\ge c_{3,3}\left(\log n_{2k+1}\right)^{-1+\varepsilon}\\&\ge c_{3,4}k^{-(1-\varepsilon)\theta}.\end{aligned} \tag{64}$$

Similar to (36), we have

$$\begin{aligned}
&\mathbb{P}\left(E_k \mid \mathcal{G}_{k-1}\right) \\
&\quad \geq \mathbb{P}\left(\bigcup_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \left(C_x \cap D_x\right) \mid \mathcal{G}_{k-1}\right) \\
&\quad \geq \sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \mathbb{P}\left(C_x \mid \mathcal{G}_{k-1}\right) \left\{\mathbb{P}\left(D_x \mid \mathcal{G}_{k-1}\right)\right. \\
&\quad \left. - \sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \mathbb{P}\left(C_x \mid \mathcal{G}_{k-1}\right)\right\}.
\end{aligned} \tag{65}$$

Similar to (33) and (35), we have $\sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \mathbb{P}(C_x \mid \mathcal{G}_{k-1}) \longrightarrow 0$ and $\mathbb{P}(D_x \mid \mathcal{G}_{k-1}) \longrightarrow 1$ for all $x \in \mathcal{T}_{n_{2k+1}} \setminus \mathcal{T}_{n_{2k-1}}$, respectively. Thus, by (64) and (65), we have

$$\begin{aligned}
\mathbb{P}\left(E_k \mid \mathcal{G}_{k-1}\right) &\geq c_{3,5} \sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \mathbb{P}\left(C_x \mid \mathcal{G}_{k-1}\right) \\
&\geq c_{3,6} k^{-(1-\varepsilon)\theta}.
\end{aligned} \tag{66}$$

By choosing $\theta > 1$ small enough such that $(1-\varepsilon)\theta < 1$, and making use of Lemma 5,

$$\mathbb{P}\left(E_k \text{ i.o.}\right) = 1. \tag{67}$$

By the definitions of m_k and n_k, we have that

$$\frac{\gamma_{n_{2k}}}{\gamma_{n_{2k+1}}} \leq \frac{e^{(2k)^\theta}}{e^{(2k+1)^\theta}} \leq e^{-(2k)^{\theta-1}} \longrightarrow 0, \tag{68}$$

and that there exist integers $j < l$ such that $n_{2k-1} = m_j$ and $n_{2k} = m_l$. It follows that

$$\frac{n_{2k-1}\left(\log n_{2k-1}\right)^{2\alpha\tau}}{n_{2k}} = \frac{j^{2\alpha\tau\theta} e^{j^\theta}}{e^{l^\theta}} \longrightarrow 0. \tag{69}$$

By Lemma 7, we have almost surely $S_n > n^{1/\alpha}(\log n)^{-\tau}$ for all large n. This, together with (60), (68), and (69), yields

$$\begin{aligned}
\lambda_{2,k} &= \sum_{x\in\mathcal{T}_{n_{2k-1}}} \mathbb{E}\left[\xi^\beta_{n_{2k-1}(\log n_{2k-1})^{2\alpha\tau}}(x)\right] \leq \gamma^\beta_{n_{2k}} \\
&= o\left(\gamma^\beta_{n_{2k+1}}\right).
\end{aligned} \tag{70}$$

Since $\gamma^\beta_{n_{2k+1}} = \lambda_{1,k} + \lambda_{2,k}$, by (70), we have $\gamma^\beta_{n_{2k+1}} \sim \lambda_{1,k}$. Thus, (67) remains true when $\lambda_{1,k}$ is replaced with $\gamma^\beta_{n_{2k+1}}$. Hence, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x)\right|\right. \\
&\qquad \left. \geq \gamma_{n_{2k+1}}\left(\log n_{2k+1}\right)^{(1-\varepsilon)/\beta} \text{ i.o.}\right) = 1.
\end{aligned} \tag{71}$$

By (68), (70) and following the same argument as the proof of (32),

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{x\in\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x)\right|\right. \\
&\qquad \left. \geq c_{3,7}\gamma_{n_{2k+1}}\left(\log n_{2k+1}\right)^{(1-\varepsilon)/\beta}\right) \\
&\quad = \mathbb{P}\left(\left|\sum_{x\in\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x)\right|\right. \\
&\qquad \left. \geq c_{3,8}\lambda_{2,k}\frac{\gamma_{n_{2k+1}}}{\lambda_{2,k}}\left(\log n_{2k+1}\right)^{(1-\varepsilon)/\beta}\right) \\
&\quad \leq \mathbb{P}\left(\left|\sum_{x\in\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x)\right| \geq c_{3,9}e^{(2k)^{\theta-1}}\lambda_{2,k}\right) \\
&\quad \leq c_{3,10}e^{-\beta(2k)^{\theta-1}}.
\end{aligned} \tag{72}$$

Thus, by making use of Borel-Cantelli lemma,

$$\begin{aligned}
&\mathbb{P}\left(\left|\sum_{x\in\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x)\right|\right. \\
&\qquad \left. \geq c_{3,7}\gamma_{n_{2k+1}}\left(\log n_{2k+1}\right)^{(1-\varepsilon)/\beta} \text{ i.o.}\right) = 0.
\end{aligned} \tag{73}$$

Noting

$$\begin{aligned}
W\left(n_{2k+1}\right) &= \sum_{x\in\mathcal{T}_{n_{2k+1}}\setminus\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x) \\
&\quad + \sum_{x\in\mathcal{T}_{n_{2k-1}}} \sigma(x)\,\xi_{n_{2k+1}}(x),
\end{aligned} \tag{74}$$

by (71) and (73),

$$\begin{aligned}
&\mathbb{P}\left(\left|W\left(n_{2k+1}\right)\right| \geq c_{3,11}\gamma_{n_{2k+1}}\left(\log n_{2k+1}\right)^{(1-\varepsilon)/\beta} \text{ i.o.}\right) \\
&\quad = 1.
\end{aligned} \tag{75}$$

This yields (62). The proof of Theorem 1 is completed. □

Proof of Theorem 4 . Fix $0 < \lambda \leq 1/\beta$. Let $\nu = 1/\beta\lambda$ and $n_k = e^{k^\nu}$ $(k \geq 1)$. It is enough to prove that

$$\limsup_{k\longrightarrow\infty}\left(V^{-1}_{n_k}\left|K_{n_k}\right|\right)^{1/\log\log n_k} = e^\lambda \quad \text{a.s.} \tag{76}$$

To prove (76), it suffices to prove that, for all $\varepsilon > 0$, with probability one,

$$V^{-1}_{n_k}\left|K_{n_k}\right| > \left(\log n_k\right)^{(1+\varepsilon)\lambda} \quad \text{for at most finitely many } k \tag{77}$$

and

$$V_{n_k}^{-1}\left|K_{n_k}\right| > \left(\log n_k\right)^{(1-\varepsilon)\lambda} \quad \text{for infinitely many } k. \tag{78}$$

By Lemma 6,

$$\begin{aligned}\mathbb{P}\left(V_{n_k}^{-1}\left|K_{n_k}\right| > \left(\log n_k\right)^{(1+\varepsilon)\lambda}\right) &\le c_{3,12}\left(\log n_k\right)^{-(1+\varepsilon)\beta\lambda} \\ &\le c_{3,13}k^{-(1+\varepsilon)}.\end{aligned} \tag{79}$$

By making use of Borel-Cantelli lemma, we obtain (77).

It remains to prove (78). For the case $\lambda = 1/\beta$, following the same lines as the proof of (62), we have that there exists a subsequence of the subsequence $\{e^k, k \ge 1\}$ such that (78) holds. For the case $0 < \lambda < 1/\beta$, we have $\nu > 1$. For $k \ge 1$, let $m_k = \inf\{n : n \in \{n_1, n_2, \ldots\}, V_n \ge e^{k^\nu}\}$. Following the same lines as the proof of (62), we have, with probability one,

$$V_{m_k}^{-1}\left|K_{m_k}\right| > \left(\log m_k\right)^{(1-\varepsilon)\lambda} \quad \text{for infinitely many } k. \tag{80}$$

On the other hand, $\{m_k\}$ is a subsequence of the subsequence $\{n_k\}$. Thus, we obtain (78) again. The proof of Theorem 4 is completed. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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