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*Research Article*

# Optimal Control Strategies Depending on Interest Level for the Spread of Rumor

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Many media channels such as broadcast, newspaper, and social networks diffuse a variety of information which can cause spread of many rumors. There are social damage and economic damage due to the spread of rumors. Thus one needs to establish strategies for controlling the rumors. We first propose rumor model with three control strategies for preventing the spread of rumor, (1) announcing the truth before ignorant receives rumor, (2) punishing spreaders, and (3) deleting information of the rumor in media, and consider optimal control problems to minimize the number of spreaders while minimizing the cost of three control strategies for preventing the spread of rumors. The analysis of optimal control problems is conducted as Pontryagin's Maximum Principle. Furthermore, adapted optimal control is performed to investigate the effect of three controls under isoperimetric constraints. By using numerical simulations, we compare the number of spreaders before and after applying the three controls and confirm when and how each control should be applied with respect to the interest level of rumor. The lower the interest level of rumor is, the greater the number of spreaders drops after the three controls are applied. In terms of timing of three controls, control (1) should be applied in the early stage of rumor spreading and control (2) is required when the rumors spread the most. After the rumors spread the most, control (3) is needed. Commonly the higher the interest level is, the more controls (1) and (2) are required. On the other hand, control (3) is needed a lot when the interest level is low.

## 1. Introduction

All kinds of rumors are overflowing in newspapers, TV, Internet, and other mass media. There are too many speculative articles that we doubt whether we should believe. The development of the Internet, social networking sites, and blogs makes ordinary people as well as celebrities such as politicians and entertainers be swept to the rumors. Rumors especially spread so fast in social networks [1] that the influence of rumors is beyond our imagination [2]. Once rumors begin to spread, they are topics for the gossips and are being accepted as true even if their certainty has not been confirmed. So anyone can become a victim of the rumor or a perpetrator. Sometimes it happened that some people could not endure the malicious rumors about them and ended their lives, and so it stirred up society. For example, recently, a team of Mèdecins Sans Fróntiers (MSF) had to temporarily stop work at the isolation ward in Macenta in Guinea, because of the rumors that MSF had brought the Ebola virus to Guinea. As the rumor becomes more socially serious, it attracts the attention of many scholars and many attempts have been made to express and interpret the spread of rumor as a mathematical model.

Daley-Kendall's model [3] and Maki-Thompson's model [4], the classical mathematical models of rumor, described the spread of rumor by considering three categories (ignorants, spreaders, and stiflers). Kawachi suggested age-independent rumor transmission models which are extensions of Daley-Kendall's model and age-structured rumor transmission model. By using basic reproduction number $R_0$, he investigated sufficient conditions for the local stability of equilibrium point and for the uniform strong persistence [5]. For examining what influences the spread of rumor from various angles, Kawachi et al. [6] suggested a rumor transmission model with various contact interactions. Huo et al. [7] analyzed the rumor model with incubation by using $R_0$. Huo et al. [8] proposed the rumor transmission model with a nonmonotone and nonlinear incidence rate that describes psychological effect with rumor transmission in the emergency.

Zhao and Wang [9] have included a quantity of rumor in the medium state in Daley-Kendall's model [3] because the development of media has a great influence on the propagation of rumors in modern society. They investigated the influence of medium on the spread of the rumor by using numerical simulation and found that to control the rumor spreading, it is needed to control not only the rate of change of the spreader subclass but also the change of the information about a rumor in the medium. After that, Zhao and Wang [10] proposed a mathematical model which was based on the previous model [9] and included the measures of government which are to issue the actual message through the medium and to punish the spreaders. Based on the extended model, they investigated more effective strategies for controlling the spread of rumor by using sensitivity analysis.

However, static control measures of government conducted in Zhao and Wang [10] have difficulties in analyzing the prevention for the rumor problem. When a government takes measures for a social phenomenon, it must consider when and how they should be applied. The phenomenon such as rumor spreading is no exception. Therefore, optimal control is one of the methods to analyze this phenomenon. To authors' best knowledge, no mathematical model has been formulated to represent optimal control for rumor diffusion problem. In addition, the types of rumors vary, and the rate and extent of rumor spreading depend on the personality, age, and occupation of the individual. So the government should control the rumor problem according to the type or interest of rumor. But the paper of Zhao and Wang [10] lacks this point.

In this paper, we consider the rumor model with three control strategies depending on interest level of rumor for preventing the spread of rumor, unlike previous papers. Three strategies are composed of announcing the truth before ignorants receive rumor, punishing spreaders, and deleting messages in media [2]. We investigate optimal strategies to prevent the spread of rumor from three perspectives. First, we focus on the interest level of rumor and confirm the change of spreaders and controls depending on the level. The degree of interest in rumors varies according to sex, occupation, and age. A high interest level of rumor will lead to a rapid spread, while at a low level it will be slow. So when controlling a rumor, we need to consider the interest level of rumor. We define the interest level as the parameter "$\theta$" in our paper.

Second, to prevent the spread of rumor effectively, we investigate when and how controls should be applied. This perspective is important since each control has a different characteristic and an optimal time point of application. To see these contents, we consider the optimal control problem and analyze it via Pontryagin's Maximum Principle. By using numerical simulation, we find the optimal strategy to prevent the spread of rumor.

Finally, when the amount of controls is limited, we check how three controls are applied, respectively. Since time, costs and so on must be taken into account for implementing the policy or strategy, the perspective is required. Therefore, the adapted optimal control is used to show the effect of controls under isoperimetric constraints.

The rest of this paper is organized as follows. Section 2 describes the mathematical model for the spread of rumor with medium state and three control terms. The necessary conditions for an optimal control and the corresponding states are derived using Pontryagin's Maximum Principle. In Section 3, we consider the optimal control of limited strategy for reality. Section 4 discusses the numerical simulation results. Finally, discussion and conclusions are given in Section 5.

## 2. Rumor Model with Three Strategies

In this section, we now introduce the rumor model including three controls for prevention of rumors (e.g., telling ignorants the truth, punishing spreaders, and deleting messages in media). The model is given as system (1) below, and the model flow diagram with three controls is shown in Figure 1.

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t) - u_1(t)I(t),\\
\frac{dS(t)}{dt} &= \theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N}\\
&\quad - mS(t) - \frac{\xi_2 S(t)R(t)}{N} - u_2(t)S(t),\\
\frac{dR(t)}{dt} &= (1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t)\\
&\quad + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N}\\
&\quad + u_1(t)I(t) + u_2(t)S(t),\\
\frac{dW(t)}{dt} &= \lambda S(t) - k\left(1 + u_3(t)\right)W(t).
\end{aligned} \tag{1}$$

$I(t)$ is the population of ignorants, $S(t)$ is the population of spreaders, $R(t)$ is the population of stiflers, and $W(t)$ is quantity of rumor in medium, such as website news, mobile phone, Facebook, Instagram, and twitter. They supposed that the ignorant population turns to the spreader population at direct transmission rate $\theta\beta(I(t)S(t)/N)$ and at indirect transmission rate $\theta\alpha I(t)W(t)$ and turns to the stifler population at direct transmission rate $(1-\theta)\beta(I(t)S(t)/N)$ and at indirect transmission rate $(1-\theta)\alpha I(t)W(t)$. The spreader population becomes stifler population at rates $2(\xi_1 S^2(t)/N)$, $mS(t)$, and $\xi_2 S(t)R(t)/N$ which represent the fact that the spreaders are bored and lose interest in rumors. The control factor $u_1(t)$ represents the rate that ignorant becomes stifler by knowing the truth of rumor. It is defined by the control factor $u_2(t)$ that spreader can become stifler via the punishment. The messages in media are reduced by control factor $u_3(t)$. If there are no three controls, the model is identical to Zhao and Wang's model [10].

The objective of our work is to minimize the number of spreaders and efforts of three control strategies which

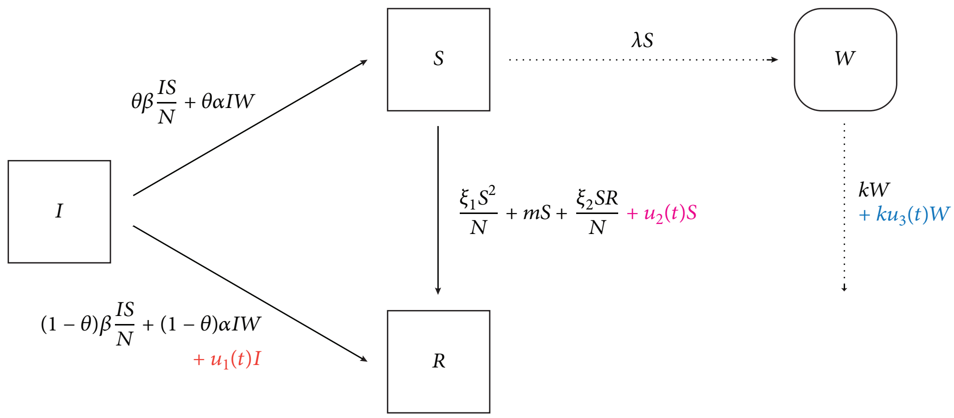


Figure 1: Schematic diagram of the rumor model with three controls.

are presented above. Thus, an optimal control problem with objective functional is given by

$$\mathcal{F}(u_1,u_2,u_3)=\int_0^{t_f} AS(t)+Bu_1(t)^2+Cu_2(t)^2+Du_3(t)^2\,dt. \tag{2}$$

The quantities $A$, $B$, $C$, and $D$ represent the weight constants of spreader, $u_1$, $u_2$, and $u_3$, in the objective cost functional, respectively. The optimal control problem is to seek optimal control functions $(u_1^*(t), u_2^*(t), u_3^*(t))$ such that

$$\mathcal{F}(u_1^*(t),u_2^*(t),u_3^*(t))=\min\{\mathcal{F}(u_1,u_2,u_3),(u_1,u_2,u_3)\in U\}, \tag{3}$$

subject to system (1) and appropriate initial conditions given at $t=0$, where the control set is defined as

$$U=\{u=(u_1,u_2,u_3)\mid u_i(t) \text{ is Lebesque measurable}, 0\le u_i(t)\le 1,\ t\in[0,T] \text{ for } i=1,2,3\}. \tag{4}$$

First we prove the existence of an optimal control for problem (3) and then derive the optimality system.

**Theorem 1.** *Given the cost functional* $\mathcal{F}(u_1,u_2,u_3)=\int_0^{t_f} AS(t)+Bu_1(t)^2+Cu_2(t)^2+Du_3(t)^2dt$ *and the control set* $U$ *(4), there exists an optimal control* $u^*=(u_1^*,u_2^*,u_3^*)$ *such that* $\mathcal{F}(u_1^*,u_2^*,u_3^*)=\min\{\mathcal{F}(u_1,u_2,u_3),(u_1,u_2,u_3)\in U\}$.

*Proof.* To prove the existence of an optimal control, we use the result in [11]. Note that the control and the state variable are nonnegative values. The set of all the control variables $(u_1,u_2,u_3)\in U$ is convex and closed by definition. In this minimizing problem, the convexity of the objective functional in $u_1$, $u_2$, and $u_3$ is satisfied. The optimal system is bounded which determines the compactness that is necessary for the existence of the optimal control. Moreover, the integrand in functional (2), $AS(t)+Bu_1(t)^2+Cu_2(t)^2+Du_3(t)^2$, is convex on the control set $U$. Also we can easily check that there exist a constant $\delta>1$ and numbers $\phi_1,\phi_2$ such that

$$\mathcal{F}(u_1,u_2,u_3)\ge\phi_1\left(u_1^2+u_2^2+u_3^2\right)^{\delta/2}-\phi_2, \tag{5}$$

because the state variables are bounded, which completes the existence of an optimal control. □

In order to find an optimal solution of the system, first we should find the Lagrangian for the optimal control problem (1)-(2) [12, 13]. The Lagrangian of the control problem is given by

$$L=AS(t)+Bu_1(t)^2+Cu_2(t)^2+Du_3(t)^2. \tag{6}$$

To seek for the minimal value of the Lagrangian, we define the Hamiltonian function $H$ for the system, where $\lambda_i$, $i=1,2,3,4$, are the adjoint variables:

$$\begin{aligned}H={}&AS+Bu_1^2+Cu_2^2+Du_3^2+\lambda_1\left[-\beta\frac{I(t)S(t)}{N}-\alpha I(t)W(t)-u_1(t)I(t)\right]+\lambda_2\left[\theta\beta\frac{I(t)S(t)}{N}\right.\\&\left.+\theta\alpha I(t)W(t)-\frac{\xi_1S^2(t)}{N}-mS(t)-\frac{\xi_2S(t)R(t)}{N}-u_2(t)S(t)\right]+\lambda_3\left[(1-\theta)\beta\frac{I(t)S(t)}{N}\right.\\&\left.+(1-\theta)\alpha I(t)W(t)+\frac{\xi_1S^2(t)}{N}+mS(t)+\frac{\xi_2S(t)R(t)}{N}+u_1(t)I(t)+u_2(t)S(t)\right]\\&+\lambda_4\left[\lambda S(t)-k(1+u_3(t))W(t)\right].\end{aligned} \tag{7}$$

In order to derive the necessary conditions, we use Pontryagin's Maximum Principle [14] as follows. If $(x,u)$ is an optimal solution of an optimal control problem, then there exists a nontrivial vector function $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfying the following inequalities:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H(t, x, u, \lambda)}{\partial \lambda}, \\ 0 &= \frac{\partial H(t, x, u, \lambda)}{\partial u}, \\ \frac{d\lambda}{dt} &= -\frac{\partial H(t, x, u, \lambda)}{\partial x}. \end{aligned} \tag{8}$$

We now drive the necessary conditions that optimal control functions and corresponding states must satisfy. In the following theorem, we present the adjoint system and control characterization.

**Theorem 2.** *Given an optimal control $u^* = (u_1^*, u_2^*, u_3^*)$ and a solution $y^* = (I^*, S^*, R^*, W^*)$ of the corresponding state system (1)-(2), there exist adjoint variables $\lambda_i$, $i = 1, 2, 3, 4$, satisfying*

$$\begin{aligned} \lambda_1'(t) &= -\frac{\partial H}{\partial I} \\ &= -\lambda_1\left[-\frac{\beta S}{N} - \alpha W - u_1\right] - \lambda_2\left[\theta\frac{\beta S}{N} + \theta\alpha W\right] \\ &\quad - \lambda_3\left[(1-\theta)\frac{\beta S}{N} + (1-\theta)\alpha W + u_1\right] \\ \lambda_2'(t) &= -\frac{\partial H}{\partial S} \\ &= -A - \lambda_1\left[-\frac{\beta I}{N}\right] \\ &\quad - \lambda_2\left[\theta\frac{\beta I}{N} - 2\frac{\xi_1 S}{N} - m - \frac{\xi_2 R}{N} - u_2\right] \\ &\quad - \lambda_3\left[(1-\theta)\frac{\beta I}{N} + 2\frac{\xi_1 S}{N} + m + \frac{\xi_2 R}{N} + u_2\right] \\ &\quad - \lambda_4[\lambda] \\ \lambda_3'(t) &= -\frac{\partial H}{\partial R} = -\lambda_2\left[-\frac{\xi_2 S}{N}\right] - \lambda_3\left[\frac{\xi_2 S}{N}\right] \\ \lambda_4'(t) &= -\frac{\partial H}{\partial W} \\ &= -\lambda_1[-\alpha I] + \lambda_2[\theta\alpha I] + \lambda_3[(1-\theta)\alpha I] \\ &\quad + \lambda_4[-k(1+u_3)] \end{aligned} \tag{9}$$

*with transversality conditions*

$$\lambda_i(t_{\text{end}}) = 0, \quad i = 1, 2, 3, 4. \tag{10}$$

*Furthermore, the control functions $u_1^*$, $u_2^*$, and $u_3^*$ are given by*

$$\begin{aligned} u_1^* &= \min\{1, \max\{0, R_1\}\} \quad \textit{where } R_1 = \frac{(\lambda_1 - \lambda_3) I}{2B}, \\ u_2^* &= \min\{1, \max\{0, R_2\}\} \quad \textit{where } R_2 = \frac{(\lambda_2 - \lambda_3) S}{2C}, \\ u_3^* &= \min\{1, \max\{0, R_3\}\} \quad \textit{where } R_3 = \frac{k\lambda_4 W}{2D}. \end{aligned} \tag{11}$$

*Proof.* To determine the adjoint equations and the transversality conditions we use the Hamiltonian (7). The adjoint system results from Pontryagin's Maximum Principle [14]:

$$\begin{aligned} \lambda_1'(t) &= -\frac{\partial H}{\partial I}, \\ \lambda_2'(t) &= -\frac{\partial H}{\partial S}, \\ \lambda_3'(t) &= -\frac{\partial H}{\partial R}, \\ \lambda_4'(t) &= -\frac{\partial H}{\partial W}, \end{aligned} \tag{12}$$

with $\lambda_i(t_f) = 0$.

To get the characterization of the optimal control given by (11), by solving the equations,

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= 0, \\ \frac{\partial H}{\partial u_2} &= 0, \\ \frac{\partial H}{\partial u_3} &= 0, \end{aligned} \tag{13}$$

on the interior of the control set and using the property of the control space $U$, we can derive the desired characterization (11). □

## 3. Optimal Control of Limited Strategy

For control strategies $u_1, u_2$, and $u_3$ mentioned above, the budget and range of strategies in country are limited. Therefore to identify an effective strategy more actually, we limit the quantity of control and compare it according to time and effect of optimal control. This is represented mathematically through the addition of an integral constraint on the control and may be included in the above model by introducing a new state variable $G(t)$ such that

$$G_i(t) = \int_0^t u_i(s)\,ds, \quad \text{for } i = 1, 2, 3 \tag{14}$$

with $G_i(t_f) = M_i$, where $M_i$ is a constant for $i = 1, 2, 3$. This type of constraint is known as an isoperimetric constraint.

Firstly, we consider the following problem about the control $u_1$ which represents that an ignorant becomes a stifler by knowing the truth of rumors. The optimal control problem in which $u_1$ is considered only is as follows:

$$\min_{u_1} \int_0^{t_f} AS(t) + Bu_1(t)^2\, dt \tag{15}$$

subject to

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t) - u_1(t)I(t),\\
\frac{dS(t)}{dt} &= \theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} - mS(t) - \frac{\xi_2 S(t)R(t)}{N},\\
\frac{dR(t)}{dt} &= (1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t) + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N} + u_1(t)I(t),\\
\frac{dW(t)}{dt} &= \lambda S(t) - kW(t),\\
\frac{dG_1(t)}{dt} &= u_1(t),
\end{aligned} \tag{16}$$

with $0 \leqslant u_1 \leqslant 1$, $\int_0^{t_f} u_1(t)dt = M_1$. We begin by forming the Hamiltonian $H_1$:

$$\begin{aligned}
H_1 = {} & AS(t) + Bu_1(t)^2 + \lambda_1\left[-\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t) - u_1(t)I(t)\right] + \lambda_2\left[\theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} - mS(t) - \frac{\xi_2 S(t)R(t)}{N}\right]\\
& + \lambda_3\left[(1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t) + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N} + u_1(t)I(t)\right]\\
& + \lambda_4[\lambda S(t) - kW(t)] + \lambda_5[u_1(t)].
\end{aligned} \tag{17}$$

The adjoint equations are given by

$$\begin{aligned}
\lambda_1'(t) &= -\frac{\partial H}{\partial I} = -\lambda_1\left[-\frac{\beta S}{N} - \alpha W - u_1\right] - \lambda_2\left[\theta\frac{\beta S}{N} + \theta\alpha W\right] - \lambda_3\left[(1-\theta)\frac{\beta S}{N} + (1-\theta)\alpha W + u_1\right]\\
\lambda_2'(t) &= -\frac{\partial H}{\partial S} = -A - \lambda_1\left[-\frac{\beta I}{N}\right] - \lambda_2\left[\theta\frac{\beta I}{N} - 2\frac{\xi_1 S}{N} - m - \frac{\xi_2 R}{N} - u_2\right] - \lambda_3\left[(1-\theta)\frac{\beta I}{N} + 2\frac{\xi_1 S}{N} + m + \frac{\xi_2 R}{N}\right] - \lambda_4[\lambda]\\
\lambda_3'(t) &= -\frac{\partial H}{\partial R} = -\lambda_2\left[-\frac{\xi_2 S}{N}\right] - \lambda_3\left[\frac{\xi_2 S}{N}\right]\\
\lambda_4'(t) &= -\frac{\partial H}{\partial W} = -\lambda_1[-\alpha I] + \lambda_2[\theta\alpha I] + \lambda_3[(1-\theta)\alpha I] + \lambda_4[-k]\\
\lambda_5'(t) &= -\frac{\partial H}{\partial G_1} = 0\\
\lambda_i(t_{\mathrm{end}}) &= 0, \quad i = 1,2,3,4,5.
\end{aligned} \tag{18}$$

The optimality condition is

$$\frac{\partial H}{\partial u_1} = 2Bu_1(t) + (\lambda_3 - \lambda_1)I(t) + \lambda_5 = 0. \tag{19}$$

By using similar method, the optimality system for cases of $u_2, u_3$ can be derived. (See Appendix for details.)

## 4. Numerical Simulations

In this section, the system with three controls was solved numerically by using the forward-backward sweep method. In our simulation, we fixed the total population by $I(t) + S(t) + R(t) = N(t) = 5000000$ and compared effects of two scenarios in compliance with the initial spreader population. The first scenario is when the rumors spread less at the initial stage ($S(0) = 10$). In the second scenario the rumors already spread a lot ($S(0) = 100000$). The parameter values of our simulation were as those of Zhao and Wang [9] (Table 1). However, the value of $\theta$ was not fixed because each interest level for the rumor is different depending on age, sex, and job, and so on. Therefore, we examined each quantity of three controls and proportions of spreader by changing the value of $\theta$ at constant scope.

Figure 2 shows the proportion of spreader to total population in the case including and excluding three controls

Table 1: Parameter values in the model.

| Parameter | Description | Value | Reference |
|---|---|---|---|
| $\beta$ | $S(t)$-to-$I(t)$ transmission rate | 10 | [9] |
| $\alpha$ | $W(t)$-to-$I(t)$ transmission rate | 0.00001 | [9] |
| $\theta$ | The rate of being $S(t)$ after transmission | 0.5 | [9] |
| $m$ | The automatical transformation rate of $S(t)$ into $R(t)$ | 0.1 | [9] |
| $k$ | The submerged rate of message | 2 | [9] |
| $\lambda$ | The discharge quantity of message into medium | 5 | [9] |
| $\xi_1$ | $S(t)$-to-$S(t)$ transmission rate | 30 | [9] |
| $\xi_2$ | $R(t)$-to-$I(t)$ transmission rate | 15 | [9] |

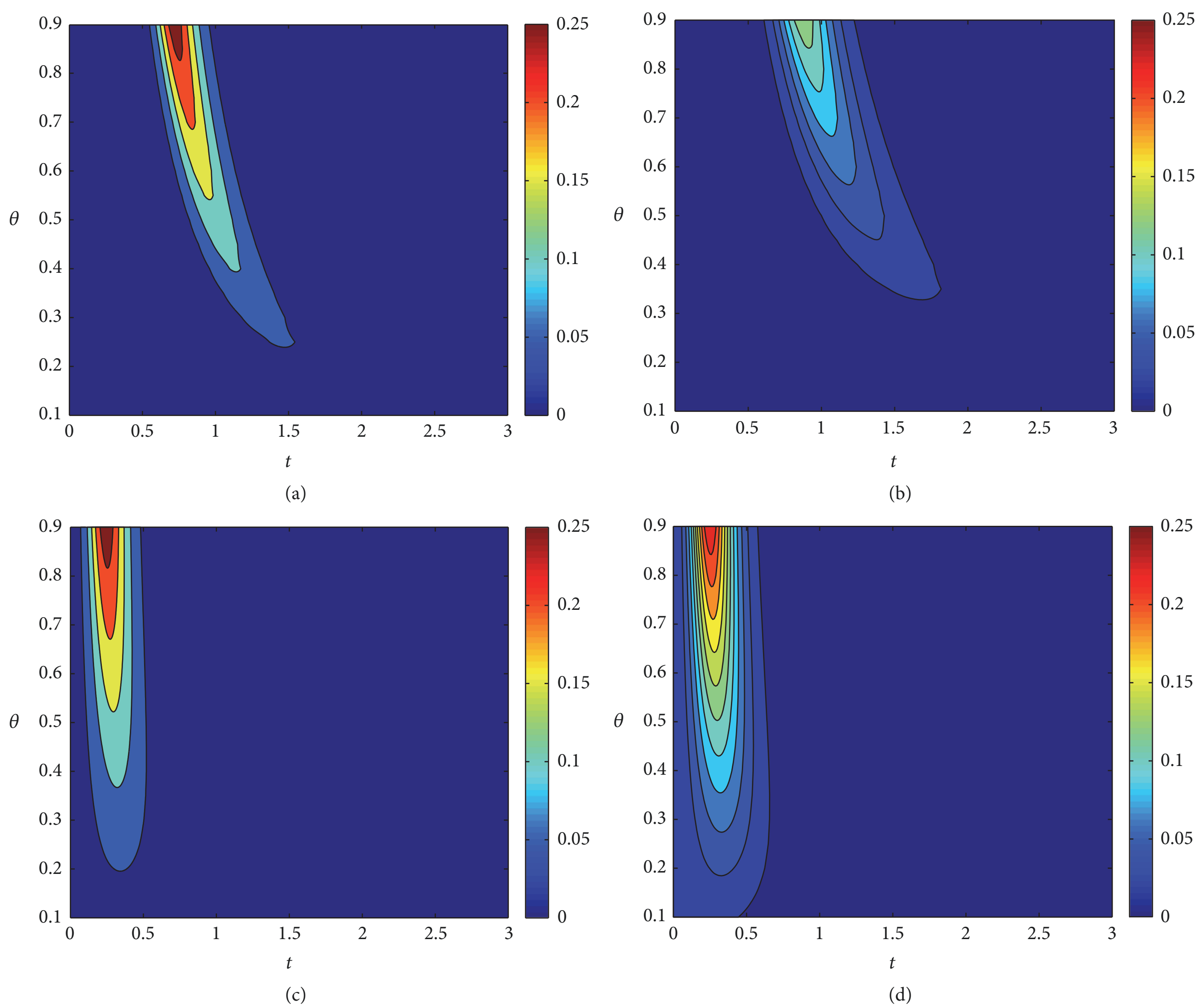


Figure 2: (a), (b) show the proportion of spreader to total population depending on $\theta$ in the first scenario without control and with control, respectively, and (c), (d) show the proportion of spreader to total population in the second scenario without control and with control, respectively. Other parameter values are given in Table 1.

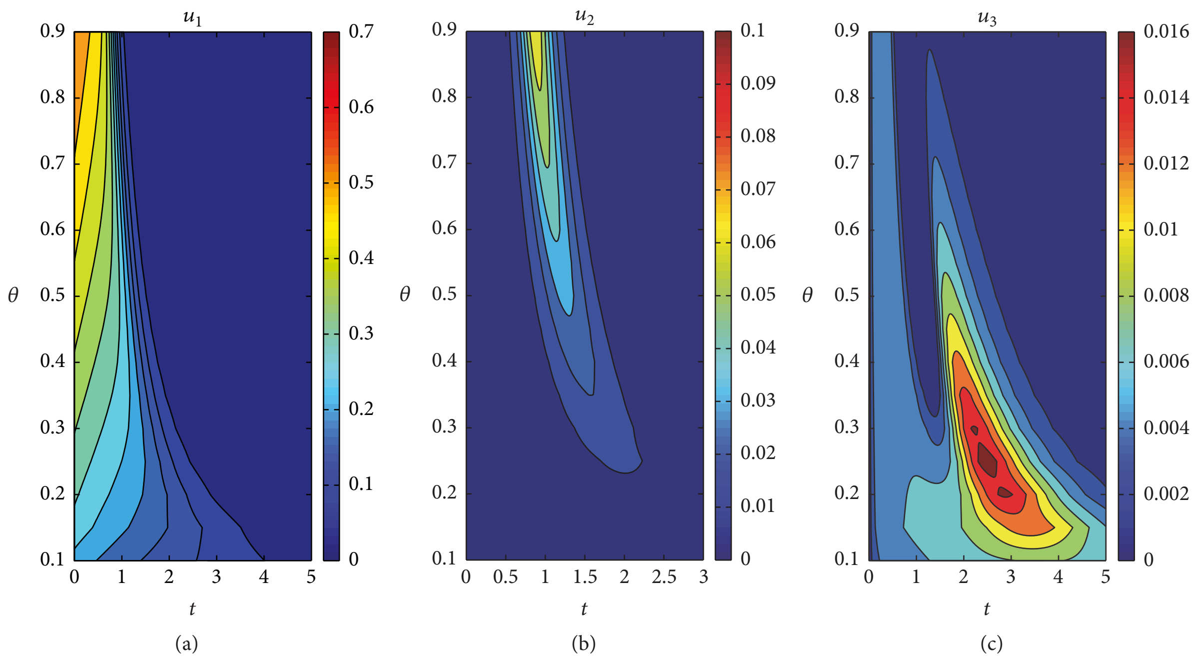


Figure 3: Distribution of controls (a) $u_1$, (b) $u_2$, and (c) $u_3$ for various values of $\theta$ in the first scenario, respectively. Other parameter values are given in Table 1.

while changing the value of $\theta$ in the range 0.1–0.9. Firstly, we considered one scenario in Figures 2(a) and 2(b). Under the absence of three controls, spreaders were distributed from $t = 0.5$ to $t = 1.5$ if $\theta$ was more than 0.2. Also, the maximum proportion of spreader was 0.25. But, in the case with three controls, spreaders existed between $t = 0.5$ and $t = 2$ when $\theta$ was more than 0.3 and the maximum proportion was 0.12.

From Figures 2(c) and 2(d), the other scenario was considered. If three controls were not applied, the spreaders appeared from $t = 0.1$ to $t = 0.5$ when $\theta$ was more than 0.2 and the maximum proportion of spreaders was still 0.25. With controls, the spreaders were in the time range 0–0.6 when $\theta$ was more than 0.1. The maximum proportion was 0.23 and so there was no big difference from noncontrol.

And we found that the controls are needed in order of $u_1$, $u_2$, and $u_3$ by comparing each required quantity of controls, respectively, at the first scenario, which can be seen in Figure 3. For control $u_1$, it was distributed from $t = 0$. Moreover the bigger $\theta$ was, the more it was needed. The control $u_2$ was applied from $t = 0.5$ to $t = 2.5$ when $\theta$ was more than 0.2. If $\theta$ was less than 0.5, the control $u_3$ was mostly distributed between $t = 2$ and $t = 3$. In particular, it was required most widely when $\theta$ was in the range 0.2–0.3.

In Figure 4, three controls were compared at the second scenario. The control $u_1$ was distributed from $t = 0$ to $t = 0.6$, but mostly existed around $t = 0$. For the control $u_2$, it was most applied between $t = 0.2$ and $t = 0.4$. In the case of control $u_3$, it was distributed between $t = 0.5$ and $t = 2.3$. The bigger $\theta$ was, the more $u_1$ and $u_2$ were needed. But the case of control $u_3$ was an opposite case. Therefore when comparing Figures 3 and 4, we recognized that the quantity of control at the second scenario had to be concentrated in the initial stage during a shorter time than at the first scenario. Also for the maximum of each required control, the cases of $u_1, u_2$ in $S(0) = 100000$ (Figure 4) were higher than $S(0) = 10$ (Figure 3), but the case of $u_3$ was contrary. In other words, the more the initial spreaders were, the more $u_1$ and $u_2$ have to be applied, but the less $u_3$ has to be applied.

Furthermore, we examined the effect of three controls at both scenarios by limiting the total quantity of controls to 1, 2, and 3, respectively, in Figures 5 and 6. For this numerical simulation, the adapted forward-backward sweep method [15] was used. First, the control $u_1$ was only considered. The optimality system for cases of $u_2$ and $u_3$ can be also derived using the similar method. The algorithm is as follows. At both endpoints, all stages are free. For adjoint equations $\lambda_1, \ldots, \lambda_5$, $\lambda_j(N+1) = 1$, $j = 1, 2, 3, 4$; but $\lambda_5(N+1)$ is unknown. Suppose that we give a guess $\lambda_5(N+1) = \sigma$. Then, by using forward-backward sweep method with Runge-Kutta 4, we can solve the optimal control problem. Let $Z$ be a real-valued function defined by

$$Z(\sigma) = \sum_{n=0}^{N} \int_{n}^{n+1} u_1(t, \sigma)\, dt - M_1, \quad (20)$$

where $M_1$ is the total quantity of the control $u_1$. Since $Z$ must be zero for the guessed $\sigma$ value, by assuming $Z = 0$ and using the Secant method, we have to determine $\sigma$. A rough algorithm is given by Algorithm 1.

The effect of three controls at the first scenario is described in Figure 5. The controls $u_2$ and $u_3$ appeared in a similar aspect as requiring fixed quantity. On the other hand, the control $u_1$ was most required in the initial stages and gradually paralleled as the limited total quantity of control

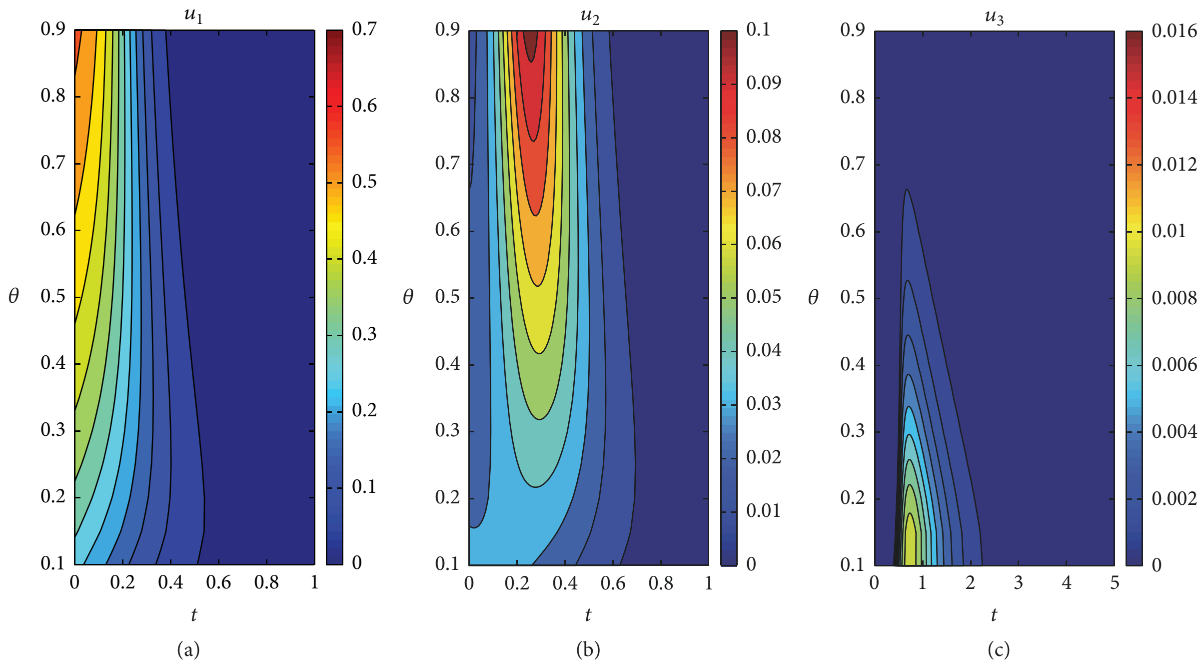


FIGURE 4: Distribution of controls (a) $u_1$, (b) $u_2$, and (c) $u_3$ for various values of $\theta$ in the second scenario, respectively. Other parameter values are given in Table 1.

increases. Looking at the number of spreaders, in the case of $u_1$, spreaders decreased considerably in peak point in which the number of spreaders reaches its maximum. In addition, the peak point was getting ahead when the total quantity of $u_1$ increased. The end point in which the number of spreaders is less than 1 was also more delayed than in noncontrol. When $u_2$ and $u_3$ were applied, respectively, the peak point was put forward in both cases, but the end point in the case of $u_2$ was longer than the one in the case of $u_3$. The more the total quantity of $u_3$ was, the more the end point was advanced. For the number of spreaders, it was certainly decreased after the peak point compared with noncontrol in both cases. However, in spite of increasing the total quantity of $u_2$ and $u_3$, respectively, there was no significant change of total spreader population.

Figure 6 represents the case of the other scenario. We noticed that there was little change of spreaders compared with the first scenario even if the total quantity of controls was increased. Also, all controls failed to advance the end point.

Lastly, we compared the cumulative spreader population in two scenarios depending on the interest level of rumors from Figure 7. When three controls were applied in the first scenario, the spreader population diminished more than noncontrol as the value of $\theta$ was less. On the other hand, regardless of $\theta$, three controls could not decrease the spreader population a lot in the second scenario.

## 5. Discussion and Conclusions

Modern people have been able to obtain information easily and quickly by developing media such as the Internet and SNS, but the rumors that are not so clear or true spread more quickly than ever before. Therefore, the study of mathematical modeling of rumors that have been carried out in the past could be extended considering the domain of media. However, although we could express the proliferation of rumors more specifically, the research on effective strategies to prevent rumors from spreading is lacking. In this paper, we considered the optimal control problem of rumor model. Three types of control functions in connection with telling the truth, punishing the spreaders, and deleting messages in media are considered. The aim of optimal control is to minimize the spreaders and messages in media by three control functions above. For the existence of an optimal control, we used Pontryagin's Maximum Principle in order to determine the necessary conditions. Furthermore, we investigated the system by numerical simulation to find the optimal strategies for reducing spreaders and preventing rumors from spreading under two scenarios mentioned in Section 4.

First, we demonstrated the effects of preventing the spread of rumors before and after applying the controls. In the first scenario, if applying the controls, the rumors spread less no matter how high the interest level of a rumor is. Furthermore, the spread of rumors was slower than before applying the controls. On the other hand, the controls had no effective result in the second scenario except that the spreaders were slightly reduced. Thus, the spreaders were reduced by the controls in both scenarios, but the controls could prevent the spread of rumors more effectively in the first scenario (see Figure 2).

Also, we compared the effects of three controls, respectively, in two scenarios above. Looking at the case of $u_1$, there were differences in two scenarios. In the aspect of quantity,

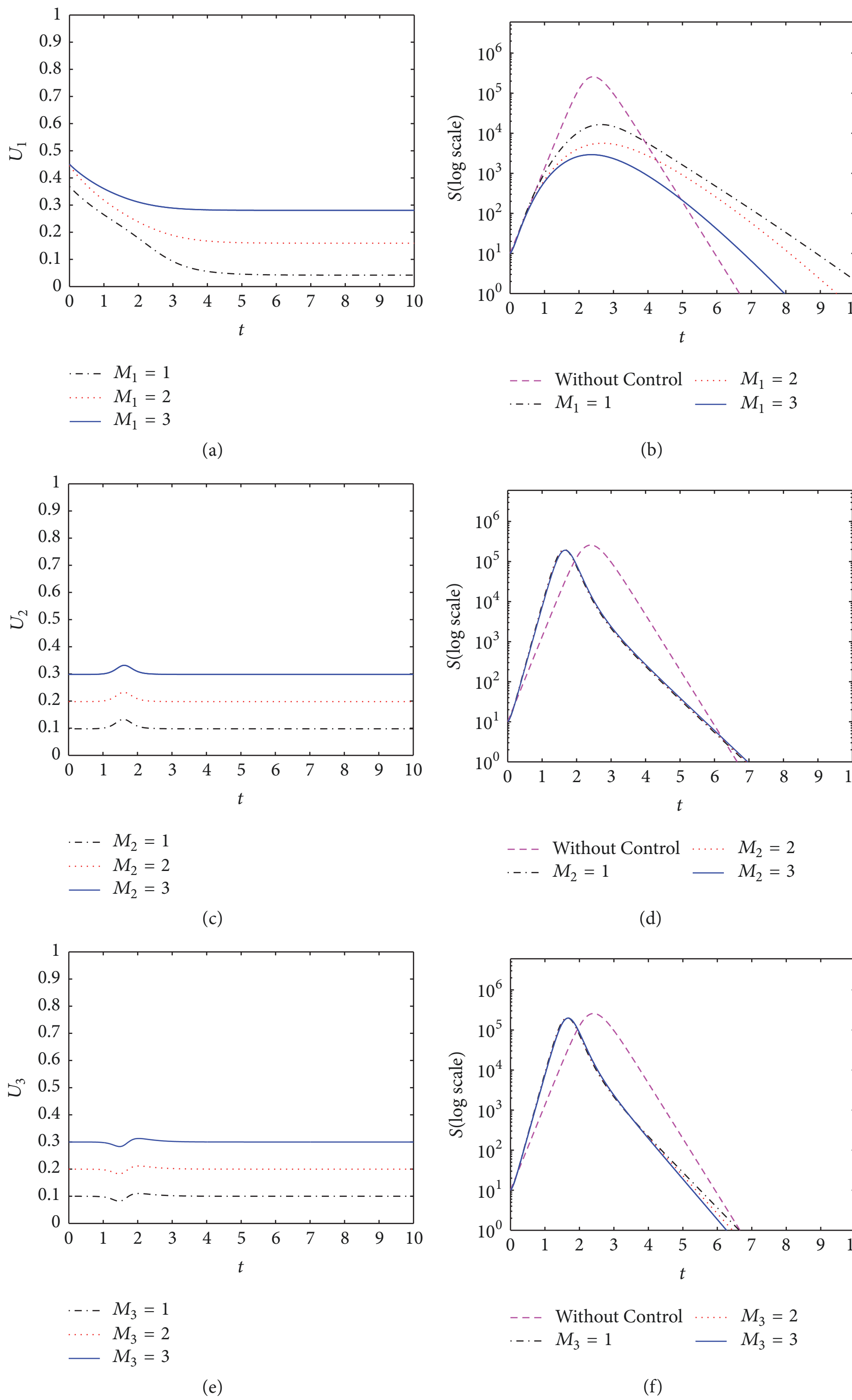


FIGURE 5: (a) The distribution of control $u_1$ and (b) the change of spreader population with the variation of the total amount of $u_1$ when there are no $u_2$ and $u_3$ in the first scenario. (c) The distribution of control $u_2$ and (d) the change of spreader population with the variation of the total amount of $u_2$ when there are no $u_1$ and $u_3$ in the first scenario. (e) The distribution of control $u_3$ and (f) the change of spreader population with the variation of the total amount of $u_3$ when there are no $u_1$ and $u_2$ in the first scenario. Other parameters are given in Table 1.

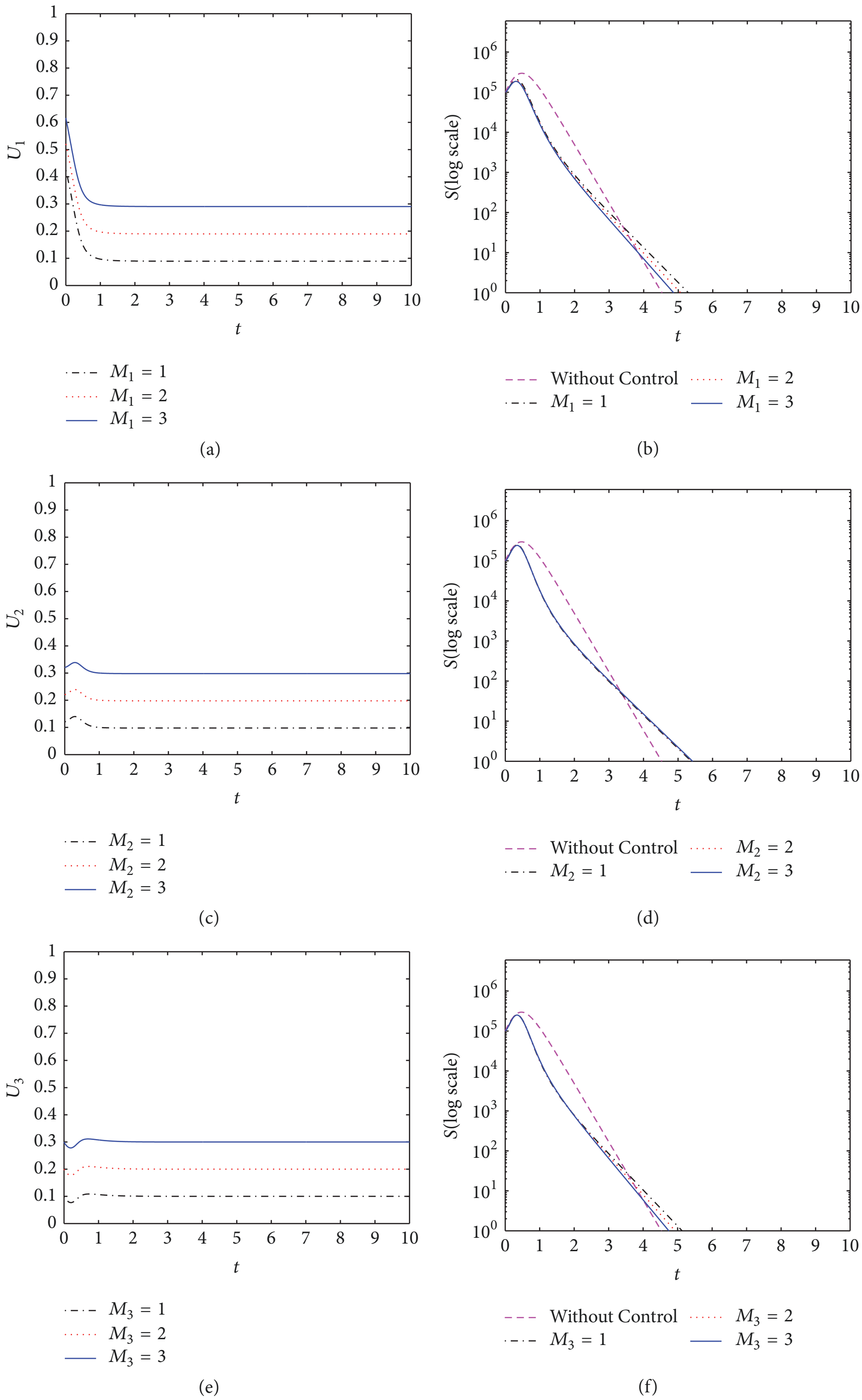


Figure 6: (a) The distribution of control $u_1$ and (b) the change of spreader population with the variation of the total amount of $u_1$ when there are no $u_2$ and $u_3$ in the second scenario. (c) The distribution of control $u_2$ and (d) the change of spreader population with the variation of the total amount of $u_2$ when there are no $u_1$ and $u_3$ in the second scenario. (e) The distribution of control $u_3$ and (f) the change of spreader population with the variation of the total amount of $u_3$ when there are no $u_1$ and $u_2$ in the second scenario. Other parameters are given in Table 1.

(1) Suppose the transversality condition $\lambda_5(t_f)$ takes $\sigma$ and give an initial guess for $u_1$.

$$\lambda_5(t_f) = \sigma, \ u_1(0) = 0$$

(2) Solve the system forward in time by using Runge-Kutta 4 method

$$X_i(t+h) = X_i(t) + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}), \ i = 1, 2, 3, 4, 5$$

with the initial conditions:

$$X_1(0) = 5000000 - X_2(0), \ X_2(0) = 10 \text{ or } 100000, \ X_3(0) = 0, \ X_4(0) = 0, \ X_5(0) = 0,$$

where $X_1(t) = I(t), \ X_2(t) = S(t), \ X_3(t) = R(t), \ X_4(t) = W(t), \ X_5(t) = G_1(t)$.

(3) Solve the system backward in time by using Runge-Kutta 4 method

$$\lambda_i(t-h) = \lambda_i(t) - \frac{h}{6}(l_{1i} + 2l_{2i} + 2l_{3i} + l_{4i}), \ i = 1, 2, 3, 4$$

with the transverality conditions:

$$\lambda_1(t_f) = 0, \ \lambda_2(t_f) = 0, \ \lambda_3(t_f) = 0, \ \lambda_4(t_f) = 0.$$

(4) Update $u_1$ by entering the new state and adjoint values into the characterization of the optimal control.

(5) Output the current value as a solution if value of a variable in this iteration and the last iteration are negligibly close. Otherwise, return to Step (2).

$$\frac{|u_1 - \text{old}u_1|}{|u_1|} \le \delta,$$

where $\delta$ is the accepted tolerance.

(6) Calculate and check the $Z$ values. If $Z$ is negligibly close to 0, i.e. $|Z(\sigma)| < \epsilon$ for tolerance $\epsilon$, output the value of $\sigma$ as solution. Otherwise, reassume the value of $\sigma$, not equal to the previous value and then return to Step (3). However, after this step is repeated $n$ times, if $Z$ is not negligibly close to 0, i.e. $|Z(\sigma_{n+1})| \ge \epsilon$ for $n = 1, 2, \ldots$, update the new value of $\sigma_{n+2}$ by using the Secant method and return to Step (2). The Secant method is as follows:

$$\sigma_{n+2} = \sigma_{n+1} - \frac{Z(\sigma_{n+1})}{Z'(\sigma_{n+1})}, \text{ where } Z'(\sigma_{n+1}) = \frac{Z(\sigma_{n+1}) - Z(\sigma_n)}{\sigma_{n+1} - \sigma_n}.$$

Algorithm 1

$u_1$ was applied more in the second scenario than in the first scenario under the rumor of same interest level; that is, time, cost, and so on were used a lot to apply this control to the second scenario. This means that it is hard to prevent the rumors when the rumors already spread a lot. There was also a difference in the aspect of time. In the first scenario, $u_1$ was applied for a long time until recently. On the other hand, in the second scenario, $u_1$ was concentrated most at the beginning. This is because ignorants appear a lot later when the number of spreaders in the initial stage is low. Since we assumed that the total population was fixed and few spreaders were in the first scenario, ignorants rarely moved into spreaders or stiflers at the beginning and so still remained a lot afterward. Namely, there were a lot of ignorants who could not get rumors. Thus, the control $u_1$ was applied for a long time until late stage because it is the control for ignorants. In the case of $u_2$, two scenarios had something in common; that is, $u_2$ was required after the rumors somewhat spread, unlike $u_1$. This is because $u_1$ is telling the truth for ignorants before the rumors spread a lot, whereas $u_2$ is punishing spreaders after the rumors spread a lot. However, there were two differences in two scenarios. One of them is that $u_2$ was almost unnecessary for uninteresting rumors in the first scenario in contrast with the second scenario because not only do not the rumors of low interest level spread well, but also the number of spreaders at the initial stage is low. The other difference is that, in the second scenario, $u_2$ was focused on the initial stage, but it was applied afterwards in the first scenario. This difference happened due to a reason similar to the case of $u_1$. When the rumors spread less at the beginning, spreaders appeared a lot late in fixed total population. Therefore, $u_2$ is required afterwards in the first scenario. Lastly, in the case of $u_3$, there were interesting results unlike the cases of $u_1$ and $u_2$. The higher the interest level of a rumor was, the more the spreaders appeared in both scenarios commonly; that is, the number of ignorants decreased a lot in the fixed total population. Then $u_3$ was not much needed since it prevents ignorants from moving to spreaders by deleting the messages in media. In contrast to this, if the interest level was lower, $u_3$ was more applied in both scenarios. In addition, there was a distinct difference between two scenarios. Despite few spreaders in the initial stage, $u_3$ was more required in the first scenario than in the second scenario. Generally, the controls are much applied to prevent the spread of rumors when the rumors already spread a lot. However, $u_3$ was an opposite case because spreaders who appeared late posted information about rumors on media in the first scenario. This means that, in the first scenario, many ignorants remained until late and so plenty of rumors on media could change them into spreaders. Therefore, $u_3$ was more required in the first scenario than in the second scenario even if spreaders were few at the beginning (see Figures 3 and 4).

And we showed the effects of each control when restricting the total amount of controls, respectively, which means that time, cost, and so on are limited if the government implements policies or strategies. Thus, we analyzed the results by increasing the total amount of each control within

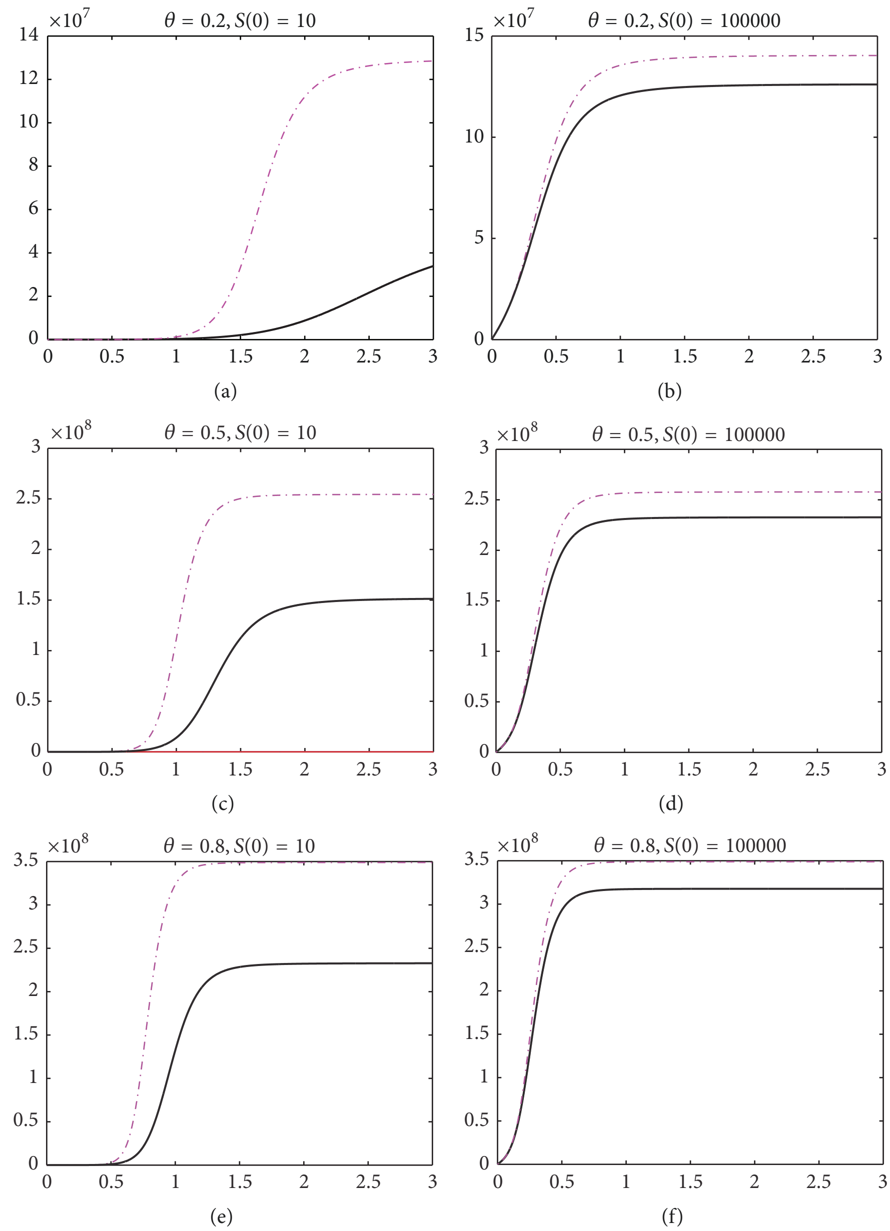


Figure 7: Cumulative spreader population for various values of $\theta$ and $S(0)$. Other parameters are given in Table 1.

the certain range. In both scenarios, the distributions of $u_2$ and $u_3$ were applied almost constantly during the period of control even if the total amount of each control increased. The case of $u_1$ was different. The control $u_1$ was still concentrated on the initial stage when the total amount of $u_1$ increased. This is because most of the ignorants existed at the beginning and $u_1$ is the control for ignorants. We could also confirm the change of the number of spreaders depending on each control. In both scenarios, the number of spreaders was most declined via $u_1$ among the three controls even if $u_1$ reduced the spreaders a little in the second scenario. It follows that $u_1$ is the optimal strategy to decrease the number of spreaders most. Furthermore, $u_1$ could advance the end point mentioned in Section 4 when the total amount of $u_1$ increased. Namely, if $u_1$ is applied a lot, we can reduce the spreaders and so quash the rumors quickly. The control $u_3$ could also put the end point forward, but was different from the case of $u_1$. The control $u_3$ advanced the end point more than noncontrol even if $u_3$ could not diminish spreaders much. This implies that the role of media is important to dispel the rumors. If the total amount of $u_3$ increases abundantly in both scenarios, the rumors in media are wiped out and so ignorants can not receive the rumors in media. Then, since ignorants do not move into spreaders, the spread of rumors slows down and eventually ends sooner. Thus, $u_3$ is the optimal strategy to quash the rumors quickly (see Figures 5 and 6).

Consequently, our results in this paper can be summarized as follows:

(i) The three controls are more effective at controlling the spread of rumors when the rumors spread less at the initial stage than when the rumors are already widespread.

(ii) The control of announcing the truth before ignorants receive rumor ($u_1$) must be applied in the early stage of rumor spreading. When the rumors spread the most, the control of punishing spreaders ($u_2$) is required. Finally the control of deleting information about the rumor in media ($u_3$) should be applied after the rumors spread the most.

(iii) The higher the interest level is, the more the controls of announcing the truth before ignorants receive rumor ($u_1$) and punishing spreaders ($u_2$) are required. On the other hand, the control of deleting information about the rumor in media ($u_3$) is needed a lot at a suitable point of low interest level.

(iv) The control of announcing the truth before ignorants receive rumor ($u_1$) is a direct and effective way to reduce the spreaders significantly. Also, The control of deleting information about the rumor in media ($u_3$) is an indirect and effective way to finish the rumor a bit sooner.

(v) Therefore, if we apply different custom strategies based on interest level and an initial value of spreader, the rumors are controlled effectively.

There is also a weak point in this study. It is that the results of the optimal control may vary depending on the parameter or the initial value or the objective functional. However, we could compare the three controls and determine when each control has to be applied, depending on the extent of rumor spreading. And since this study was proceeded without the actual data, there can be reasonable results if we use the actual data. We leave the analysis through the data as a future work.

## Appendix

## Optimal Control with an Isoperimetric Constraint for $u_2$ and $u_3$

We consider the following problem about the control $u_2$ which represents that a spreader becomes a stifler by punishment of the spreader. The optimal control problem where $u_2$ is considered only is as follows:

$$\min_{u_2} \quad \int_0^{t_f} AS(t) + Cu_2(t)^2\,dt \tag{A.1}$$

subject to

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t),\\
\frac{dS(t)}{dt} &= \theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} - mS(t) - \frac{\xi_2 S(t)R(t)}{N} - u_2(t)S(t),\\
\frac{dR(t)}{dt} &= (1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t) + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N} + u_2(t)S(t),\\
\frac{dW(t)}{dt} &= \lambda S(t) - kW(t),\\
\frac{dG_2(t)}{dt} &= u_2,
\end{aligned} \tag{A.2}$$

with $0 \leqslant u_2 \leqslant 1$, $\int_0^{t_f} u_2(t)dt = M_2$.

We begin by forming the Hamiltonian $H_2$.

$$\begin{aligned}
H_2 = {}& AS(t) + Cu_2(t)^2 + \lambda_1\left[-\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t)\right] + \lambda_2\left[\theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} - mS(t) - \frac{\xi_2 S(t)R(t)}{N} - u_2(t)S(t)\right]\\
&+ \lambda_3\left[(1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t) + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N} + u_2(t)S(t)\right] + \lambda_4\left[\lambda S(t) - kW(t)\right] + \lambda_5\left[u_2(t)\right].
\end{aligned} \tag{A.3}$$

The adjoint equations are given by

$$\begin{aligned}
\lambda_1'(t) &= -\frac{\partial H}{\partial I}\\
&= -\lambda_1\left[-\frac{\beta S}{N} - \alpha W\right] - \lambda_2\left[\theta\frac{\beta S}{N} + \theta\alpha W\right] - \lambda_3\left[(1-\theta)\frac{\beta S}{N} + (1-\theta)\alpha W\right]
\end{aligned}$$

$$\begin{aligned}
\lambda_2'(t) &= -\frac{\partial H}{\partial S} \\
&= -A - \lambda_1\left[-\frac{\beta I}{N}\right] \\
&\quad - \lambda_2\left[\theta\frac{\beta I}{N} - 2\frac{\xi_1 S}{N} - m - \frac{\xi_2 R}{N} - u_2\right] \\
&\quad - \lambda_3\left[(1-\theta)\frac{\beta I}{N} + 2\frac{\xi_1 S}{N} + m + \frac{\xi_2 R}{N} + u_2\right] \\
&\quad - \lambda_4[\lambda] \\
\lambda_3'(t) &= -\frac{\partial H}{\partial R} = -\lambda_2\left[-\frac{\xi_2 S}{N}\right] - \lambda_3\left[\frac{\xi_2 S}{N}\right] \\
\lambda_4'(t) &= -\frac{\partial H}{\partial W} \\
&= -\lambda_1[-\alpha I] - \lambda_2[\theta\alpha I] - \lambda_3[(1-\theta)\alpha I] \\
&\quad + \lambda_4[k] \\
\lambda_5'(t) &= -\frac{\partial H}{\partial G_2} = 0.
\end{aligned} \tag{A.4}$$

The optimality condition is

$$\frac{\partial H}{\partial u_2} = 2Cu_2(t) - (\lambda_2 - \lambda_3)S(t) + \lambda_5 = 0. \tag{A.5}$$

We consider following problem about the control $u_3$ which represents deleting messages in media. The optimal control problem where $u_3$ is considered only is as follows:

$$\min_{u_3} \int_0^{t_f} AS(t) + Du_3(t)^2\,dt \tag{A.6}$$

subject to

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\beta\frac{I(t)S(t)}{N} - \alpha I(t)W(t), \\
\frac{dS(t)}{dt} & \\
&= \theta\beta\frac{I(t)S(t)}{N} + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} \\
&\quad - mS(t) - \frac{\xi_2 S(t)R(t)}{N}, \\
\frac{dR(t)}{dt} & \\
&= (1-\theta)\beta\frac{I(t)S(t)}{N} + (1-\theta)\alpha I(t)W(t) \\
&\quad + \frac{\xi_1 S^2(t)}{N} + mS(t) + \frac{\xi_2 S(t)R(t)}{N}, \\
\frac{dW(t)}{dt} &= \lambda S(t) - k(1+u_3(t))W(t), \\
\frac{dG_3(t)}{dt} &= u_3,
\end{aligned} \tag{A.7}$$

with $0 \leqslant u_3 \leqslant 1$, $\int_0^{t_f} u_3(t)dt = M_3$.

We begin by forming the Hamiltonian $H_3$.

$$\begin{aligned}
H_3 &= AS(t) + Du_3(t)^2 + \lambda_1\left[-\beta\frac{I(t)S(t)}{N}\right. \\
&\quad \left.- \alpha I(t)W(t)\right] + \lambda_2\left[\theta\beta\frac{I(t)S(t)}{N}\right. \\
&\quad + \theta\alpha I(t)W(t) - \frac{\xi_1 S^2(t)}{N} - mS(t) \\
&\quad \left.- \frac{\xi_2 S(t)R(t)}{N}\right] + \lambda_3\left[(1-\theta)\beta\frac{I(t)S(t)}{N}\right. \\
&\quad + (1-\theta)\alpha I(t)W(t) + \frac{\xi_1 S^2(t)}{N} + mS(t) \\
&\quad \left.+ \frac{\xi_2 S(t)R(t)}{N}\right] + \lambda_4[\lambda S(t) \\
&\quad - k(1+u_3(t))W(t)] + \lambda_5[u_3(t)].
\end{aligned} \tag{A.8}$$

The adjoint equations are given by

$$\begin{aligned}
\lambda_1'(t) &= -\frac{\partial H}{\partial I} \\
&= -\lambda_1\left[-\frac{\beta S}{N} - \alpha W\right] - \lambda_2\left[\theta\frac{\beta S}{N} + \theta\alpha W\right] \\
&\quad - \lambda_3\left[(1-\theta)\frac{\beta S}{N} + (1-\theta)\alpha W\right] \\
\lambda_2'(t) &= -\frac{\partial H}{\partial S} \\
&= -A - \lambda_1\left[-\frac{\beta I}{N}\right] \\
&\quad - \lambda_2\left[\theta\frac{\beta I}{N} - 2\frac{\xi_1 S}{N} - m - \frac{\xi_2 R}{N}\right] \\
&\quad - \lambda_3\left[(1-\theta)\frac{\beta I}{N} + 2\frac{\xi_1 S}{N} + m + \frac{\xi_2 R}{N} + u_2\right] \\
&\quad - \lambda_4[\lambda] \\
\lambda_3'(t) &= -\frac{\partial H}{\partial R} = -\lambda_2\left[-\frac{\xi_2 S}{N}\right] - \lambda_3\left[\frac{\xi_2 S}{N}\right]
\end{aligned}$$

$$\begin{aligned}\lambda_4'(t) &= -\frac{\partial H}{\partial W} \\ &= -\lambda_1[-\alpha I] - \lambda_2[\theta\alpha I] - \lambda_3[(1-\theta)\alpha I] \\ &\quad + \lambda_4[k(1+u_3)] \\ \lambda_5'(t) &= -\frac{\partial H}{\partial G_3} = 0.\end{aligned} \tag{A.9}$$

The optimality condition is

$$\frac{\partial H}{\partial u_3} = 2Du_3(t) - \lambda_4 kW(t) + \lambda_5 = 0. \tag{A.10}$$

## Disclosure

The authors would like to note that partial results of their work were presented by a poster at the 2017 Society for Mathematical Biology Annual meeting.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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