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Research Article

On the Blended Solutions of Polynomial-Like Iterative Equation with Multivalued Functions

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The *blended* solutions are more general than the *unblended* solutions for polynomial-like iterative equation with multivalued functions. In this paper we study the *blended* solutions of polynomial-like iterative equation with multivalued functions.

1. Introduction

The polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in X, \quad (1)$$

is an interesting form of functional equations, where X is a topological vector space, $F : X \longrightarrow X$ is a given function, and f^i, $i = 1, 2, 3, \cdots, n$, stands for the i–th iteration of the unknown function $f : X \longrightarrow X$, i.e., $f^i(x) = f(f^{i-1}(x))$ and $f^0(x) = x$. Since iteration is an important problem in many mathematical subjects, such as dynamical systems and numerical computation, and many fields of natural science, in recent years some attentions have been paid to (1) and its generalizations [1–7]. On the other hand, multivalued function is an important class of mappings, which has been extensively employed in control theory [8], stochastics [9], artificial intelligence [10], and economics [11]. Many nice results [12–16] were obtained on functional equations with multivalued functions. It is important to investigate (1) with multivalued functions, i.e., the equation

$$\lambda_1 g(x) + \lambda_2 g^2(x) + \cdots + \lambda_n g^n(x) = G(x), \quad x \in I := [a, b], \quad (2)$$

where $n \geq 2$ is an integer, $\lambda_1, \lambda_2, \cdots \lambda_n$ are real constants, $G : I \longrightarrow cc(I)$ is a given multivalued function, $cc(I)$ denotes the family of all nonempty convex compact subsets of I, and $g : I \longrightarrow cc(I)$ is an unknown multivalued function; the i–th iteration g^i of the multivalued function g is defined inductively as

$$g^i(x) := \cup \left\{ g(y) : y \in g^{i-1}(x) \right\} \quad (3)$$

and $g^0(x) :\equiv x$ for all $x \in I$. In 2004, Nikodem and Zhang [2] first studied (2) for $n = 2$ with an increasing upper semicontinuous (*USC* for short) multivalued function G on $I = [a, b]$. The result on the existence and uniqueness of *USC* solutions is given. As indicated in [2], the upper semicontinuity for multivalued functions is much weaker than the continuity for functions; the method used for continuous solutions [6, 7] is improved substantially in order to obtain *USC* solutions. In 2011, Xu, Nikodem, and Zhang [4] considered the general case $n \geq 3$ of the equation

$$\lambda_1 F(x) = G(x) - \lambda_2 F^2(x) - \cdots - \lambda_n F^n(x), \quad \forall x \in I, \quad (4)$$

which is a modified version of (2), for the *piecewise Lipschitzian* multivalued functions defined in [4]. *USC unblended* multivalued solutions of this equation were given in the inclusion sense. As defined in [4], a multivalued function F is said to be *unblended* if it satisfies $\zeta_{k+1} \in F(\zeta_k)$ on a sequence $S := \{\zeta_k\}_{k \geq 0}^{\infty}$ in I; that is, $\zeta_{k+1} \in$

$[\min F(\zeta_k), \max F(\zeta_k)]$ for each k; otherwise, F is said to be blended on S. Clearly, the blended requirement for multivalued functions is much weaker than the unblended requirement for multivalued functions. It is an interesting object to study the blended multivalued solutions of (2) in the inclusion sense. So far, we find no results on the *USC blended* multivalued solutions of (2) in the inclusion sense. In this paper, we investigate the existence of the *USC blended* multivalued solutions of (2) in the inclusion sense.

2. Preliminaries

The family $cc(I)$ endowed with the Hausdorff distance is defined by

$$h(A;B) = \max\{\sup\{d(a;B) : a \in A\}; \sup\{d(b;A) : b \in B\}\}; \tag{5}$$

where $d(a;B) = \inf\{|a-b| : b \in B\}$ is a complete metric space (cf., e.g., [17], Cor. 4.3.12).

A multivalued function $F : I \longrightarrow cc(I)$ is increasing (resp., strictly increasing) if for every $x, y \in I$ with $x < y$, we have $\max F(x) \le \min F(y)$ (resp., $\max F(x) < \min F(y)$) (cf. [18], Def. 3.5.1). A multivalued function $F : I \longrightarrow cc(I)$ is *USC* at a point $x_0 \in I$ if for every open set $V \subset R$ with $F(x_0) \subset V$ there exists a neighborhood U_{x_0} of x_0 such that $F(x) \subset V$ for every $x \in U_{x_0}$. F is *USC* on I if it is *USC* at every point in I. For convenience, let

$$USI(I) := \{F \in \mathscr{F}(I) : F \text{ is } USC \text{ and strictly increasing}\}, \tag{6}$$

where $\mathscr{F}(I)$ is the set of all multivalued functions $F : I \longrightarrow cc(I)$. Some useful properties are summarized in the following Lemma (cf. [4, 15, 19]).

Lemma 1. *For $A, B, C, D \in cc(I)$ and for an arbitrary real λ, the following properties hold:*

(i) $h(A + C, B + C) = h(A, B)$,

(ii) $h(\lambda A, \lambda B) = |\lambda| h(A, B)$,

(iii) $h(A + C, B + D) \le h(A, B) + h(C, D)$.

Lemma 2. *If $F_1, F_2 \in USI(I)$, then $F_1 \circ F_2 \in USI(I)$.*

As indicated in [4], if a function $F \in USI(I)$ is not single-valued, there exists at least a point $\xi \in I$ such that the cardinal of the set $F(\xi)$ is more than 1; i.e., card $F(\xi) \ge 2$. Actually, $F(\xi)$ is a nontrivial interval because $F(\xi) \in cc(I)$. Since F is strictly increasing, there exist two small open intervals $V_\delta^- := \{x \in I \mid \xi - \delta < x < \xi\}$ and $V_\delta^+ := \{x \in I \mid \xi < x < \xi + \delta\}$ such that F is single-valued in both of them and satisfies $\min F(\xi) > F(x)$, $\forall x \in V_\delta^-$, and $\max F(\xi) < F(x), \forall x \in V_\delta^+$. We call ξ a jump-point of F or a jump simply. For every $F \in USI(I)$, let $J(F)$ denote the set of all jumps of F. We easily see that each $F \in USI(I)$ has at most countably infinite many jumps, i.e., the cardinal card $J(F) \le \aleph$. In fact, for each $\xi \in J(F)$, the set $F(\xi)$ is a nontrivial compact subinterval of I. By the strict monotonicity, $\{F(\xi) : \xi \in J(F)\}$ is a set of disjoint nonempty compact subintervals of I. Choose a rational number $r(\xi) \in F(\xi)$ for each $\xi \in J(F)$. Then card $J(F) \le$ card $\mathbb{Q} = \aleph$. According to above the argument, $F \in USI(I)$ was divided into two cases in [4]: one is unblended multivalued functions; the other is blended ones.

Since functions in $USI(I)$ are strictly increasing, it suffices to discuss multivalued functions F in $USI(I)$ which satisfy either $\min F(x) > x$ for all $x \in \operatorname{int} I$ or $\max F(x) < x$ for all $x \in \operatorname{int} I$. Let $USI^*(I)$ and $USI_*(I)$ denote the two classes of multivalued functions, respectively.

Lemma 3. *Suppose that $F \in USI^*(I)$ (resp., $\in USI_*(I)$) is unblended on the strictly increasing (resp., decreasing) sequence $S = \{\zeta_k\}_{k\ge0}^\infty$. If $S \in J(F)$ and satisfies that $\zeta_0 = a$ and $\lim_{k\longrightarrow\infty}\zeta_k = b$ (resp., $\zeta_0 = b$ and $\lim_{k\longrightarrow\infty}\zeta_k = a$), then for each integer $i \ge 1$, (i) $J(F^i) \subset S$, and (ii) $F^i((\zeta_k, \xi_{k+1})) \subset (\zeta_{k+i}, \zeta_{k+1+i})$ (resp., $F^i((\zeta_{k+1}, \zeta_k)) \subset (\zeta_{k+1+i}, \zeta_{k+i})$), $\forall k \ge 0$.*

A function $F \in USI(I)$ is said to be *piecewiseLipschitzian* on I with the sequence Λ and constants $M > m > 0$ if for each $k \ge 0$,

(C1) $J(F) \subset \Lambda$,

(C2) $m(x_2 - x_1) \le F(x_2) - F(x_1) \le M(x_2 - x_1)$ $\forall x_1, x_2 \in (\eta_k, \eta_{k+1})$ with $x_1 < x_2$,

(C3) $\max F(\eta_{k+1}) - \min F(\eta_k) \le M(\xi_{k+1} - \eta_k)$ $\forall \eta_k, \eta_{k+1} \in \Lambda$,

where Λ is a strictly monotonic sequence in I such that $\operatorname{int}(I)$ Λ is a union of disjoint open intervals; i.e.,

$$\operatorname{int}(I) \setminus \Lambda = \bigcup_{k\ge0} (\eta_k, \eta_{k+1}), \tag{7}$$

where each η_k is either an element of Λ or an endpoint of I. For a strictly increasing sequence $S = \{\zeta_k\}_{k\ge0}^\infty$ in I such that $\zeta_0 = a$ and $\lim_{k\longrightarrow\infty}\zeta_k = b$, define

$$USI_u^* := \{F \in USI^*(I) : F \text{ is unblended on } S \text{ and } J(F) \subset S\}. \tag{8}$$

Similarly, for a strictly decreasing sequence $S = \{\zeta_k\}_{k\ge0}^\infty$ in I such that $\zeta_0 = b$ and $\lim_{k\longrightarrow\infty}\zeta_k = a$ define

$$USI_{u*} := \{F \in USI_*(I) : F \text{ is unblended on } S \text{ and } J(F) \subset S\}. \tag{9}$$

Lemma 4. *$USI(I, \Lambda, m, M)$ is a complete metric space equipped with the distance D, defined by $D(F_1, F_2) := \sup\{h(F_1(x), F_2(x)) : x \in I\}$, $\forall F_1, F_2 \in \Phi(I)$.*

Define

$$\begin{aligned} USI^*(I,S,m,M) &:= USI^*(I) \bigcap USI(I,S,m,M), \\ USI_*(I,S,m,M) &:= USI_*(I) \bigcap USI(I,S,m,M), \\ USI_u^*(I,S,m,M) &:= USI_u^*(I,S) \bigcap USI(I,S,m,M), \\ USI_{u*}(I,S,m,M) &:= USI_{u*}(I,S) \bigcap USI(I,S,m,M). \end{aligned} \tag{10}$$

Clearly, $USI^*(I,S,m,M)$, $USI_*(I,S,m,M)$, $USI_u^*(I,S,m,M)$, and $USI_{u*}(I,S,m,M)$ are all closed subsets of $USI(I,S,m,M)$.

Lemma 5. *$F^i \in USI^*(I,S,m^i,M^i)$ (resp., $USI_*(I,S,m^i,M^i)$) if $F \in USI_u^*(I,S,m,M)$ (resp., $USI_{u*}(I,S,m,M)$).*

Lemma 6. *If either $F_1,F_2 \in USI_u^*(I,S,m,M)$ or $F_1,F_2 \in USI_{u*}(I,S,m,M)$, then*

$$D\left(F_1^i,F_2^i\right) \le \left(\sum_{j=0}^{i-1} M^j\right) D\left(F_1,F_2\right). \tag{11}$$

3. Main Results

Theorem 7. *Suppose that $\lambda_1 > 0$, $\lambda_i \le 0$ $(i = 2,..,n)$, and $\sum_{i=1}^n \lambda_i = 1$ and that sequences $S = \{\zeta_k\}_{k\ge0}^\infty$ and $S^* = \{\zeta_k^*\}_{k\ge0}^\infty$ are strictly increasing sequences in I such that $S \subset S^*$, $\zeta_0 = a$, $\lim_{k\longrightarrow\infty}\zeta_k^* = b$, and card of $(\zeta_k,\zeta_{k+1})\bigcap(S^*\setminus S) \le 1$ and that $G \in USI^*(I,S,m_0,M_0)$, where $M_0 > m_0 > 0$, such that $\sum_{i=1}^n \lambda_i\zeta_{k+i}^* \in G(\zeta_k^*)$, $\forall k \ge 0$; then, for arbitrary constants $M > m > 0$ such that*

$$\begin{gathered} m \le \frac{m_0 + \sum_{i=2}^n |\lambda_i| m^i}{\lambda_1}, \\ M \le \frac{m_0 + \sum_{i=2}^n |\lambda_i| M^i}{\lambda_1}, \\ \rho := \frac{1}{\lambda_1}\sum_{i=2}^n |\lambda_i| \sum_{j=0}^{i-1} M^i < 1, \end{gathered} \tag{12}$$

(4) has a unique solution $F \in USI_u^*(I,S^*,m,M)$, which is a solution of (2) in the inclusion sense and is blended in S.

Proof. Sets $USI^*(I,S,m_0,M_0)$ and $USI_u^*(I,S^*,m,M)$ are well defined since the sequences $S = \{\zeta_k\}_{k\ge0}^\infty$ and $S^* = \{\zeta_k^*\}_{k\ge0}^\infty$ are strictly increasing sequences in I such that $S \subset S^*$, $\zeta_0 = a$, $\lim_{k\longrightarrow\infty}\zeta_k^* = b$, and card of $(\zeta_k,\zeta_{k+1})\bigcap(S^*\setminus S) \le 1$. Define operator $\mathscr{L} : USI_u^*(I,S^*,m,M) \longrightarrow \mathscr{F}(I)$ by

$$\mathscr{L}(F)(x) = \frac{1}{\lambda_1}\left(G(x) - \sum_{i=2}^n F^i(x)\right), \quad \forall x \in I, \tag{13}$$

where $F \in USI_u^*(I,S^*,m,M)$. We can deduce $\mathscr{L}(F) \in USI^*(I)$ from hypotheses $\lambda_1 > 0$, $\lambda_i \le 0$ $(i = 2,..,n)$, $\sum_{i=1}^n \lambda_i = 1$, $G \in USI^*(I,S,m_0,M_0)$, and $F \in USI_u^*(I,S^*,m,M)$. Since $F \in USI_u^*(I,S^*,m,M)$ and $\sum_{i=1}^n \lambda_i\zeta_{k+i}^* \in G(\zeta_k^*)$, $\forall k \ge 0$, we have

$$\begin{aligned} \zeta_{k+1}^* &\in \frac{1}{\lambda_1}\left(G(\zeta_k^*) - \sum_{i=2}^n \lambda_i\zeta_{k+i}^*\right) \\ &\subset \frac{1}{\lambda_1}\left(G(\zeta_k^*) - \sum_{i=2}^n \lambda_i F(\zeta_{k+i}^*)\right) = \mathscr{L}(F)(\zeta_k^*), \\ &\qquad \forall k \ge 0, \end{aligned} \tag{14}$$

which implies

$$\mathscr{L}(F) \in USI_u^*(I,S^*). \tag{15}$$

By G in $USI^*(I,S,m_0,M_0)$ and card of $(\zeta_k,\zeta_{k+1})\bigcap(S^*\setminus S) \le 1$, we can obtain that $G \in USI^*(I,S^*,m_0,M_0)$. In fact, Since $G \in USI^*(I,S,m_0,M_0)$, $S \subset S^*$, and card of $(\zeta_k,\zeta_{k+1})\bigcap(S^*\setminus S) \le 1, k \ge 0$, then $J(G) \subset S \subset S^*$, $m(x_2 - x_1) \le G(x_2) - G(x_1) \le M(x_2 - x_1)$, $\forall x_1,x_2 \in (\zeta_k^*,\zeta_{k+1}^*)$ with $x_1 < x_2$, and $\max G(\zeta_{k+1}^*) - \min G(\zeta_k^*) \le M(\zeta_{k+1}^* - \zeta_k^*)$ $\forall \zeta_k^*,\zeta_{k+1}^* \in S^*$; i.e., $G \in USI^*(I,S^*,m_0,M_0)$. From G in $USI^*(I,S^*,m_0,M_0)$, Lemma 5, and (13), we can infer that

$$\begin{aligned} \mathscr{L}(F) \in USI\Bigg(I,S^*,&\frac{1}{\lambda_1}\left(m_0 + \sum_{i=2}^n |\lambda_i| m^i\right), \\ &\frac{1}{\lambda_1}\left(M_0 + \sum_{i=2}^n |\lambda_i| M^i\right)\Bigg), \end{aligned} \tag{16}$$

which together with (15) yields

$$\begin{aligned} \mathscr{L}(F) \in USI_u^*\Bigg(I,S^*,&\frac{1}{\lambda_1}\left(m_0 + \sum_{i=2}^n |\lambda_i| m^i\right), \\ &\frac{1}{\lambda_1}\left(M_0 + \sum_{i=2}^n |\lambda_i| M^i\right)\Bigg). \end{aligned} \tag{17}$$

Furthermore, for F_1, $F_2 \in USI_u^*(I,S^*,m_0,M_0)$, by (13) and Lemma 6 we have

$$\begin{aligned} D(\mathscr{L}F_1,\mathscr{L}F_2) = \sup_{x\in I} h\Bigg(&\frac{1}{\lambda_1}\left(G(x) - \sum_{i=2}^n \lambda_i F_1^i(x)\right), \\ &\frac{1}{\lambda_1}\left(G(x) - \sum_{i=2}^n \lambda_i F_2^i(x)\right)\Bigg) \le \frac{1}{\lambda_1}\sum_{i=2}^n |\lambda_i| \\ &\cdot \sup_{x\in I} h\left(F_1^i(x),F_2^i(x)\right). \end{aligned} \tag{18}$$

In view of Lemma 6,

$$\begin{aligned} D(\mathscr{L}F_1,\mathscr{L}F_2) &\le \frac{1}{\lambda_1}\sum_{i=2}^n |\lambda_i| D\left(F_1^i,F_2^i\right) \\ &\le \frac{1}{\lambda_1}\sum_{i=2}^n |\lambda_i| \sum_{j=0}^{i-1} M^j D(F_1,F_2) \\ &\le \rho D(F_1,F_2). \end{aligned} \tag{19}$$

We know that operator $\mathscr{L}$ is contract since $\rho < 1$. Making use of *Banach's* fixed point Theorem, $\mathscr{L}$ has a unique fixed point F in $USI_u^*(I,S^*,m,M)$ such that

$$F(x) = \frac{1}{\lambda_1}\left(G(x) - \sum_{i=2}^n F^i(x)\right), \quad \forall x \in I, \tag{20}$$

and since $F(x)$, $G(x)$, and $\sum_{i=2}^n F^i(x)$ are three sets, according to sets operation relations we can get

$$\sum_{i=1}^n \lambda_i F^i(x) \supset G(x), \quad \forall x \in I, \tag{21}$$

which implies F is a unique solution of (2) in the inclusion sense. In the following, we shall prove that F is *blended* in S. By reduction to absurd, we suppose F is *unblended* in S. Since F is *unblended* in S^*, for $\zeta \in S^* \setminus S$, there exists some interval$(\zeta_{k_0}, \zeta_{k_0+1})$ such that $\zeta \in (\zeta_{k_0}, \zeta_{k_0+1})$, which implies $\zeta_{k_0+1} \in F(\zeta)$. $\zeta_{k_0+1} \in F(\zeta_{k_0})$ because F is *unblended*. Therefore, $F(\zeta) \bigcap F(\zeta_{k_0}) \neq \emptyset$, which contradicts to the condition that F is strictly increasing on S^*; thus F is *blended* in S. □

Theorem 8. *Suppose that $\lambda_1 > 0$, $\lambda_i \leq 0$ $(i = 2, .., n)$, and $\sum_{i=1}^{n}$, $\lambda_i = 1$ and that sequences $S = \{\zeta_k\}_{k\geq 0}^{\infty}$ and $S^* = \{\zeta_k^*\}_{k\geq 0}^{\infty}$ are strictly decreasing sequences in I such that $S \subset S^*$, $\zeta_0 = b$, $\lim_{k\longrightarrow\infty}\zeta_k^* = a$, and card of $(\zeta_{k+1}, \zeta_k) \bigcap (S^* \setminus S) \leq 1$ and that $G \in USI_*(I, S, m_0, M_0)$, where $M_0 > m_0 > 0$, such that $\sum_{i=1}^{n} \lambda_i \zeta_{k+i}^* \in G(\zeta_k^*)$, $\forall k \geq 0$; then, for arbitrary constants $M > m > 0$ satisfying (12), (4) has a unique solution $F \in USI_{u*}(I, S^*, m, M)$, which is a solution of (2) in the inclusion sense and is blended in S.*

The proof is similar to Theorem 7, so we omit it.

Remark 9. Although we strengthen the condition of Theorem 4.1 in [4] in Theorem 7, more general solutions of (2) are given.

4. Example

In order to show the rationality conditions in Theorems, we consider the equation

$$\lambda_1 F(x) + \lambda_2 F^2(x) = G(x), \quad \forall x \in I := [0, 1], \tag{22}$$

where $\lambda_1 = 3/2$, $\lambda_2 = -1/2$,

$$G(x) := \begin{cases} \left[\frac{1}{8}, \frac{3}{16}\right], & x = 0, \\ \frac{3}{4}x + \frac{3}{16}, & x \in \left(0, \frac{1}{2}\right), \\ \left[\frac{9}{16}, \frac{11}{16}\right], & x = \frac{1}{2}, \\ \frac{5}{8}x + \frac{3}{8}, & x \in \left(\frac{1}{2}, 1\right), \end{cases} \tag{23}$$

and, obviously, $G \in USI^*(I, S, m_0, M_0)$, where

$$\begin{aligned} m_0 &= \min\left\{\frac{3}{4}, \frac{5}{8}\right\} = \frac{5}{8}, \\ M_0 &= \left\{\frac{3}{4}, \frac{5}{8}, \frac{11/16 - 1/8}{1/2}\right\} = \frac{9}{8}, \end{aligned} \tag{24}$$

$S = \{\zeta_k\}_{k=0}^{\infty}$ with $\zeta_k = 1 - 2^{-k}$, $k \geq 0$, for $S^* = \{\zeta_k^*\}_{k=0}^{\infty}$ with $\zeta_0^* = 0$, $\zeta_1^* = 1/4$, $\zeta_2^* = 1/2$, $\zeta_k^* = 1 - 2^{-k+1}$, $k \geq 2$; one can check

$$\sum_{i=1}^{2} \lambda_i \zeta_{k+i}^* \in G(\zeta_k^*) \quad \forall k \geq 0. \tag{25}$$

Taking $m = 1/2$, $M = 3/2$, we can see that conditions (12) hold. Thus, (2) has a solution $F \in USI_u^*(I, S^*, m, M)$ by Theorem (12).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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