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Research Article

Pricing of Proactive Hedging European Option with Dynamic Discrete Position Strategy

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Proactive hedging European option is an exotic option for hedgers in the options market proposed recently by Wang et al. It extends the classical European option by requiring option holders to continuously trade in underlying assets according to a predesigned trading strategy, to proactively hedge part of the potential risk from underlying asset price changes. To generalize this option design for practical application, in this study, a proactive hedging option with discrete trading strategy is developed and its pricing formula is deducted assuming the underlying asset price follows Geometric Fractional Brownian Motion. Simulation studies show that proactive hedging option with discrete trading strategy still enjoys strong price advantage compared to the classical European option for majority of parameter space. The observed price advantage is stronger when the underlying asset has more volatility or when the asset price follows closer to Geometric Brownian Motion. Additionally, we found that a higher frequency trading strategy has stronger price advantage if there is no trading cost. The findings in this research strongly facilitate the practical application of the proactive hedging option, making this lower-cost trading tool more feasible.

1. Introduction

Exotic options, such as Asian, lookback, barrier, and passport options, have been a key focus of mathematical finance research since the late 1980s and early 1990s [1–9]. In this paper, we focus on an exotic option that is a proactive hedging strategy bundled into the classical European option, called *proactive hedging European option*. This exotic option has a built-in condition that requires option holders to trade the underlying asset and linearly adjust the holding position according to its price fluctuation within the option period. The potential loss of the underlying asset covered by such proactive actions is no longer the responsibility of the option writer. Therefore, compared to classical European options, this proactive hedging European option can significantly reduce the risk taken by the option writer, thus making it a theoretically less expensive option. This type of exotic option is particularly suitable to hedgers who seek to cover their risk of exposure at a minimum cost. Although very promising in theory, the continuous linear position strategy makes this option not very practical for trading purposes. In this paper, we try to improve the feasibility of this option to make it more adaptable to a real market scenario by making the continuous linear position strategy discrete. We then deduce the theoretical pricing formula for guiding trading practice in the market.

Proactive hedging European option with continuous linear position was first introduced by Wang et al. [10]. With the addition of a mandatory condition to the classical European option, option holders need to buy in (sell out) the underlying asset when its price goes up (goes down). Specifically, for a call option in which the prediction is that the future value of the underlying stock will increase, the option holder holds a certain amount of capital A at the beginning of the option period. When the price of the underlying stock goes up to $X_e + \delta(\delta \geq 0)$, the option holder spends $\beta_0 \bullet A$ to buy in the stock. The parameter β_0 is called *the initial capital utilization coefficient* and is a constant between 0 and 1. The option holder linearly and continuously adjusts the capital utilization to increase the holding if the price continues to increase, until the price reaches $(1+\alpha)(X_e+\delta)(\alpha > 0, \delta \geq 0)$ and the total capital spending reaches $\beta \bullet A$, where α is a

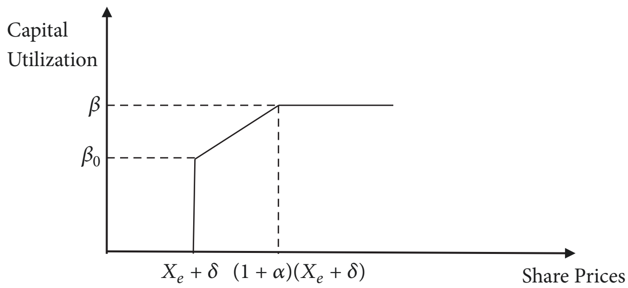

Figure 1: Capital utilization coefficient function with varying stock price under a linear position strategy.

positive number and referred to as the *investment strategy index* and β is the *maximum capital utilization coefficient*. Figure 1 describes this process. The expected loss resulting from the asset price increasing from $X_e+\delta$ to $(1+\alpha)(X_e+\delta)$, which is supposed to borne by the option writer, is partly retrieved by the dynamic strategy. The dynamic hedging option works in a similar fashion for a put option in which the prediction is that the future value of the underlying stock will decrease.

Wang et al. [11, 12] obtained the pricing formula for this exotic option by extending the Black-Scholes model under the assumption that the asset price movement follows Geometric Brownian Motion (GBM). Recently, numerous studies indicate that some features of asset prices such as fat-tails [13, 14] are more compatible with Geometric Fractional Brownian Motion (GFBM)[15–17] rather than GBM. Li et al. [18], following the work of Wang et al. [11, 12], derived the theoretical pricing formulas of this exotic option and some of its simplified special forms under the GFBM assumption by using risk-neutral evaluation principle. They compared the price of this exotic option with that of the classical European option using simulations and found that this exotic option almost always has a lower price than the classical European option. This price advantage can be as large as 65% under some parameter settings and may be greater if the asset price distributes closer to the standard GBM.

Although this exotic option enjoys a significant price advantage, it currently remains an unrealistic option choice for hedgers since proactive hedging actions must be taken *continuously* along a linear function. In this paper, we build on the work of Li et al. [18] by making its proactive hedging strategy discrete to increase its feasibility for practical use and derive its pricing formulas under the GFBM assumption. Even though making the proactive hedging strategy discrete would likely sacrifice some price advantage, simulations indicate that this discrete strategy still enjoys a strong price advantage compared to the classical European option. This advantage will be stronger when the underlying asset has more uncertainty, or when the dynamic hedging strategy is more frequent.

The rest of the paper is organized as follows. In Section 2, we describe this exotic option with discrete proactive hedging actions in detail and derive its value function. Section 3 gives the theoretical pricing formula under the GFBM assumption and the simplified formula for application to some special cases. In Section 4, we use simulations to evaluate the price premium of this exotic option with discrete proactive hedging actions compared to the exotic option with continuous linear proactive hedging actions and the classical European option. Since the pricing formula derivations are very similar for call and put options, in this paper, we only present results for call options.

2. Description of Proactive Hedging European Option with Dynamic Discrete Position Strategy

2.1. Constraints for the Proactive Hedging European Option. The proactive hedging European option is proposed based on the following assumptions:

(1) A call option holder holds one piece of contract and an initial capital of amount of $A = Q \times X_e$ at the beginning of the option period, where Q is the number of stock units for the piece of option contract and X_e is the exercising price according to the contract.

(2) The call option holder should buy in the underlying stock according to the price changes subject to the dynamic discrete position strategy attached to the option contract, which will be presented in more detail in Section 2.2.

(3) The potential loss from the underlying asset covered by such proactive actions is no longer the responsibility of the option writer.

(4) There are no transaction costs for buying or selling the stock or the option.

2.2. Dynamic Discrete Position Strategy. The option holder holds a certain amount of capital A at the beginning of the option period. The dynamic discrete position strategy will be activated when the underlying asset price reaches X_e. The option holder will only buy in when the asset price hits a series of equally spaced points $S_n, \{S_n : S_n = X_e + n \bullet \Delta,\ n = 1, \ldots, N\}$, where Δ is a positive constant representing the price distance for two consecutive trading actions and N is the total number of trades of the stock in the entire option period. Similar to previous studies, β is the *maximum capital utilization coefficient,* so the maximum amount of capital tradable or available is $\beta \bullet A$. The strategy also assumes the option holder will evenly distribute the capital over the N trades, such that each buy-in trade will spend a capital of $\beta A/N$ for $\beta A/(N \bullet S_n)$ pieces of stock. Please refer to Figure 2 for an illustration of the discrete position strategy.

2.3. The Value Function for Proactive Hedging European Option with Dynamic Discrete Position Strategy. For the classical European option, the option holder will suffer an expected loss L as

$$L = Q\left(S - X_e\right) = \frac{A}{X_e}\left(S - X_e\right) \tag{1}$$

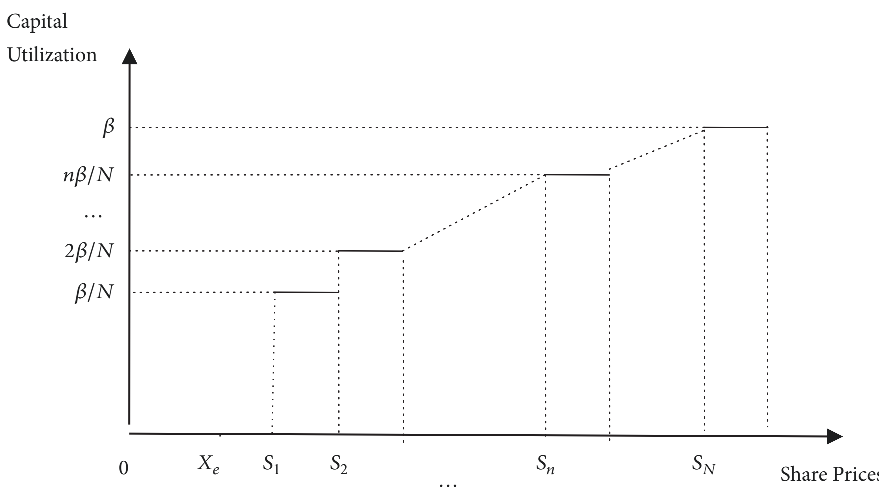

FIGURE 2: Graphical illustration of the discrete linear proactive hedging strategy.

for each piece of the option contract as the stock price rises from X_e to S, for $S > X_e$. In an exotic option with proactive hedging strategy, the option holder is required to actively buy in the underlying stock to hedge the risk from the fluctuations of the underlying asset. Assume the option holder trades with the discrete position strategy described in Section 2.2 and buys in $(\beta \bullet A)/(N \bullet S_n)$ pieces of stock when the stock price is S_n. Then, when the stock price increases from S_n to S, with $S_n < S < S_{n+1} \leq S_N$, the option holder will make a return R of

$$R = \frac{\beta A}{N S_n}(S - S_n) \tag{2}$$

by holding the stocks. Since the discrete position requires the option holder to buy in the underlying stock at every stock price of $\{S_1, S_2, \ldots, S_N\}$, each with a capital of $\beta A/N$, when the stock price reaches S with $S_m \leq S < S_{m+1} \leq S_N$, the option holder will make a cumulative return $R(S)$ of

$$R(S) = \sum_{n=1}^{m} \frac{\beta A}{N S_n}(S - S_n). \tag{3}$$

Thus, when $S_m \leq S < S_{m+1} \leq S_N$, the expected loss taken by the option writer, $L(S)$, should be the expected total loss L in (1) minus the cumulative return $R(S)$ in (3), that is,

$$\begin{aligned} L(S) &= \frac{A}{X_e}(S - X_e) - \sum_{n=1}^{m} \frac{\beta A}{N S_n}(S - S_n) \\ &= \frac{A}{X_e}(S - X_e) - \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{S_n} + \frac{m \beta A}{N} \\ &= \frac{AS}{X_e} - \frac{\beta A S}{N} \sum_{n=1}^{m} \frac{1}{X_e + n\Delta} + \left(\frac{m\beta}{N} - 1\right) A. \end{aligned} \tag{4}$$

When $S \geq S_N$, the expected loss taken by the option writer is

$$L(S) = \frac{AS}{X_e} - \frac{\beta A S}{N} \sum_{n=1}^{N} \frac{1}{X_e + n\Delta} + (\beta - 1) A. \tag{5}$$

Therefore, the expected loss taken by the option writer is a stepwise function of S, specifically,

$$L(S) = \begin{cases} 0, & S < X_e \\ \dfrac{A}{X_e}(S - X_e), & X_e \leq S < S_1 \\ \dfrac{AS}{X_e} - \dfrac{\beta A S}{N} \displaystyle\sum_{n=1}^{m} \dfrac{1}{X_e + n\Delta} + \left(\dfrac{\beta m}{N} - 1\right) A, & S_m \leq S < S_{m+1} \leq S_N \\ \dfrac{AS}{X_e} - \dfrac{\beta A S}{N} \displaystyle\sum_{n=1}^{N} \dfrac{1}{X_e + n\Delta} + (\beta - 1) A. & S_N \leq S \end{cases} \tag{6}$$

Since the option holder can buy A/X_e units of underlying stocks at price of X_e with all the initial capital, therefore, the expected loss for each unit of stock is

$$L_u(S) = \frac{L(S)}{A/X_e} = \begin{cases} 0, & S < X_e \\ S - X_e, & X_e \le S < S_1 \\ S - \dfrac{\beta S X_e}{N}\displaystyle\sum_{n=1}^{m}\frac{1}{X_e + n\Delta} + \left(\dfrac{\beta m}{N} - 1\right)X_e, & S_m \le S < S_{m+1} \le S_N \\ S - \dfrac{\beta S X_e}{N}\displaystyle\sum_{n=1}^{N}\frac{1}{X_e + n\Delta} + (\beta - 1)X_e, & S_N \le S \end{cases} \quad (7)$$

The pricing of an option is proportional to the units of underlying stocks specified in the contract; therefore, without loss of generality, in this study we set the intrinsic value function, $f(S)$, of the option as the loss function *per unit (per share)*, $L_u(S)$, that is,

$$f(S) = L_u(S), \quad (8)$$

where $L_u(S)$ is defined as in (7). The intrinsic value function $f(S)$ will include the same four terms as $L_u(S)$ in (7), and these four terms will be further denoted as f_1, f_2, f_3, and f_4.

3. Pricing of Proactive Hedging European Option Based on GFBM

3.1. Asset Price Behavior Based on GFBM. Let $S(t)$ be the asset price at time t. If $S(t)$ follows GFBM, then it satisfies the following equation:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB_H(t) \quad (9)$$

where $S(0)$, the draft μ, and the volatility σ of the asset price are all positive constants for $t \geq 0$. The Hurst parameter H is a measure of the long-range dependence in the stochastic process of GFBM. If $H = 1/2$, $B_H(t)$ reduces to an uncorrelated Brownian motion series $B(t)$. A time series with H value larger than 0.5 has long-range positive dependence, and a larger H value indicates stronger positive dependence. A time series with H value below 0.5 has long-range negative dependence, and a smaller H value indicates stronger negative dependence. As described in Section 1, many previous researchers have shown long-range positive dependence of the price changes of financial assets, and therefore we only present the case of $H \geq 0.5$ here.

We assume the stochastic process $S(t)$ follows GFBM. By applying the fractional Wick-Itô formula, Hu and Øksendal proved that (9) can be rewritten as follows [19]:

$$S(t) = S(0)\exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right). \quad (10)$$

For any two time points t_1 and t_2, where $0 \le t_1 \le t_2 \le T$, the relationship between $S(t_1)$ and $S(t_2)$ can be obtained by applying (10):

$$S(t_2) = S(t_1)\exp\left[\mu(t_2 - t_1) - \frac{1}{2}\sigma^2\left(t_2^{2H} - t_1^{2H}\right) + \sigma\left(B_H(t_2) - B_H(t_1)\right)\right]. \quad (11)$$

3.2. Fractional European Option Pricing Formula. Black and Scholes (1973) derived the famous B-S partial differential function for the theoretical price of a classical European option by applying the Ito Lemma [20]:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf, \quad (12)$$

where f is the option price, t is time, r is the risk-free return rate, σ is the volatility of the stock return, and S is the stock price. The analytical solution to the B-S formula (12) under GBM is as follows:

$$V(S(t), t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} f\left(S(t)\,e^{\sigma\sqrt{T-t}\bullet Z + (T-t)(r-\sigma^2/2)} \bullet e^{-Z^2/2}\right)dZ, \quad (13)$$

where T is the option period and $f(\bullet)$ is the intrinsic value function of the option. The intrinsic value of an option is the value of the option at maturity date $t = T$.

Equation (13) is the value function of a classical European option under the GBM assumption. However, as discussed earlier in Section 1, the GFBM assumption is more applicable for practical purpose than the GBM assumption. Therefore, we need to derive a function of form (13) that meets the GFBM assumption with the intrinsic value function $f(\bullet)$ from Section 2.3. The work of Li et al. (2018) suggests a path for similar derivation work. By applying the risk-neutral evaluation principle, Li et al. (2018) obtained an analytical solution of pricing formula when stock prices follow GFBM and then validated this pricing formula through simulations. However, one limitation is that their results require the assumption of continuous hedging actions, an impractical restriction. Thus, similar to the steps taken by Li et al. (2018)

and applying the risk-neutral evaluation principle, here we develop an analytical solution of pricing formula that allows for dynamic discrete hedging actions.

The basic idea for pricing of exotic options is based on the risk-neutral evaluation principle: at the maturity date T, the value of the classical European option, V_T, is equal to its intrinsic value, which can be written as follows:

$$V_T(S(T)) = (S(T) - X_e)^+, \quad (14)$$

where X_e, the exercising price, is a given constant and $S(T)$, the asset price at maturity date T, is a random variable. The option pricing function at any time point t, $0 \le t \le T$, is then equivalent to the solution of the following equation:

$$V_t = V(S(t), t), \quad (15)$$

and is also equal to the solution of (14) when $t = T$.

According to the risk-neutral evaluation principle, option pricing V_t before the maturity date should be equal to the discounted value of the option value V_T at the maturity date, and the discount rate should be the risk-free interest rate r. The pricing formula can then be obtained by solving (16):

$$V_t = \mathrm{e}^{-r(T-t)} \widehat{E}\left[V_T(S(T))\right]. \quad (16)$$

where $S(T)$ is a function of $S(t)$ and t.

To obtain the analytical solution of the option pricing formula of (16) under the GFBM assumption, Li et al. (2018) applied a result by Necula [21], which is quoted here as Theorem 1.

Theorem 1. *Let f be a function such that $E[f(B_H(T))] < \infty$. Then for every $0 \le t \le T$,*

$$\widehat{E}_t\left[f(B_H(T))\right] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(T^{2H} - t^{2H})}} \exp\left(-\frac{(x - B_H(t)^2)}{2(T^{2H} - t^{2H})}\right) \cdot f(x)\,dx. \quad (17)$$

Combining Theorem 1 with (11), Li et al.(2018) also obtained analytical solution to (16), and it is restated here as Lemma 2 below. Lemma 2 will serve as the starting point for derivations in this study.

Lemma 2. *Assume the price of the underlying asset at time point t, $S(t)$, satisfies (9). Then the valuation of the option at time point t is*

$$V(S(t), t) = \frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} f\left(S(t)\,\mathrm{e}^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})}\right) \bullet \mathrm{e}^{-(Z^2/2)}dZ. \quad (18)$$

where $f(\bullet)$ is the intrinsic value function of the European option at maturity T.

3.3. Pricing Formula of Proactive Hedging Option with Discrete Position Based on GFBM. This section discusses the pricing formula for the proactive hedging option when using the dynamic discrete position strategy that was introduced in Section 2.2. The pricing formula is obtained by solving the integral expression after combining the intrinsic value function in (7) and the pricing formula of the fractional European option in 3.9). The intrinsic value function in (7) is a stepwise function, thus the pricing formula consists of four parts:

$$V(S(t), t) = V_1(S(t), t) + V_2(S(t), t) + V_3(S(t), t) + V_4(S(t), t), \quad (19)$$

where

$$\begin{aligned}
V_1((S(t), t)) &= \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{S<X_e} f_1(\bullet)\,\mathrm{e}^{-Z^2/2}dz = 0,\\
V_2((S(t), t)) &= \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{X_e\le S<S_1} f_2(\bullet)\,\mathrm{e}^{-Z^2/2}dz,\\
V_3((S(t), t)) &= \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{S_m\le S<S_{m+1}\le S_N} f_3(\bullet)\,\mathrm{e}^{-Z^2/2}dz,\\
V_4((S(t), t)) &= \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{S_N\le S} f_4(\bullet)\,\mathrm{e}^{-Z^2/2}dz.
\end{aligned} \quad (20)$$

We then derive the analytical forms for the four value functions.

Since $f_1(\bullet) = 0$, it is easy to see that $V_1((S(t), t)) = 0$. For $V_2(S(t), t)$, let $U = S(t)e^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})}$. When $X_e \le U < S_1$, we have

$$\frac{\ln(X_e/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}} \le Z < \frac{\ln(S_1/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}. \quad (21)$$

By letting $Z_0 = (\ln(X_e/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H} - t^{2H}))/\sigma\sqrt{T^{2H} - t^{2H}}$ and $Z_1 = (\ln(S_1/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H} - t^{2H}))/\sigma\sqrt{T^{2H} - t^{2H}}$, we have

$$V_2((S(t), t)) = \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \cdot \int_{Z_0}^{Z_1} \left(S(t)\,\mathrm{e}^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})} - X_e\right) \cdot \mathrm{e}^{-Z^2/2}dZ = \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{Z_0}^{Z_1} S(t) \cdot \mathrm{e}^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})} \bullet \mathrm{e}^{-Z^2/2}dZ$$

$$-\frac{X_e \mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}} \int_{Z_0}^{Z_1} \mathrm{e}^{-Z^2/2} dZ = \frac{S(t)}{\sqrt{2\pi}} \cdot \int_{Z_0-\sigma\sqrt{T^{2H}-t^{2H}}}^{Z_1-\sigma\sqrt{T^{2H}-t^{2H}}} \mathrm{e}^{-(1/2)(Z-\sigma\sqrt{T^{2H}-t^{2H}})^2} d\left(Z - \sigma\sqrt{T^{2H}-t^{2H}}\right) - X_e \mathrm{e}^{-r(T-r)} \left(N(Z_1) - N(Z_0)\right) = S(t)\left[N\left(Z_1 - \sigma\sqrt{T^{2H}-t^{2H}}\right) - N\left(Z_0 - \sigma\sqrt{T^{2H}-t^{2H}}\right)\right] - X_e e^{-r(T-r)} \left(N(Z_1) - N(Z_0)\right). \quad (22)$$

The condition $S_m \le U < S_{m+1} \le S_N$ is equivalent to $S_m \le S(t)e^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})} < S_{m+1}$. Thus,

$$\frac{\ln(S_m/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}} \le Z < \frac{\ln(S_{m+1}/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H}-t^{2H})}{\sigma\sqrt{T^{2H}-t^{2H}}}. \quad (23)$$

Let $Z_m = (\ln(S_m/S(t)) - (T-t)r + (\sigma^2/2)(T^{2H} - t^{2H}))/\sigma\sqrt{T^{2H}-t^{2H}}$ and $\sigma\sqrt{T^{2H}-t^{2H}}Z + (T-t)r - (\sigma^2/2)(T^{2H}-t^{2H}) = \Sigma$. Then we have

$$V_3((S(t),t)) = \frac{\mathrm{e}^{-r(T-r)}}{\sqrt{2\pi}} \sum_{m=1}^{N-1} \int_{Z_m}^{Z_{m+1}} S(t)\mathrm{e}^{\Sigma}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e+n\Delta}\right)\mathrm{e}^{-Z^2/2}dZ + \frac{\mathrm{e}^{-r(T-r)}}{\sqrt{2\pi}} \cdot \sum_{m=1}^{N-1}\int_{Z_m}^{Z_{m+1}}\left(\frac{\beta m}{N}-1\right)X_e \mathrm{e}^{-Z^2/2}dZ = \frac{S(t)}{\sqrt{2\pi}} \cdot \sum_{m=1}^{N-1}\int_{Z_m}^{Z_{m+1}}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e+n\Delta}\right) \cdot \mathrm{e}^{\sigma\sqrt{T^{2H}-t^{2H}}Z-(\sigma^2/2)(T^{2H}-t^{2H})-Z^2/2}dZ + \mathrm{e}^{-r(T-r)}X_e\sum_{m=1}^{N-1}\left(\frac{\beta m}{N}-1\right)\frac{1}{\sqrt{2\pi}}\int_{Z_m}^{Z_{m+1}}\mathrm{e}^{-Z^2/2}dZ = S(t)\sum_{m=1}^{N-1}\frac{1}{\sqrt{2\pi}}\int_{Z_m-\sigma\sqrt{T^{2H}-t^{2H}}}^{Z_{m+1}-\sigma\sqrt{T^{2H}-t^{2H}}}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e+n\Delta}\right)\mathrm{e}^{-(1/2)(Z-\sigma\sqrt{T^{2H}-t^{2H}})}dZ - \sigma\sqrt{T^{2H}-t^{2H}} + \mathrm{e}^{-r(T-r)}X_e\sum_{m=1}^{N-1}\left(\frac{\beta m}{N}-1\right) \cdot \left(N(Z_{m+1}) - N(Z_m)\right) = S(t)\sum_{m=1}^{N-1}\left(1 - \frac{\beta X_e}{N} \cdot \sum_{n=1}^{m}\frac{1}{X_e+n\Delta}\right)\left(N\left(Z_{m+1}-\sigma\sqrt{T^{2H}-t^{2H}}\right) - N\left(Z_m - \sigma\sqrt{T^{2H}-t^{2H}}\right)\right) + \mathrm{e}^{-r(T-t)}X_e\sum_{m=1}^{N-1}\left(\frac{\beta m}{N} - 1\right)\left(N(Z_{m+1}) - N(Z_m)\right). \quad (24)$$

For $V_4(S(t),t)$, $U \ge S_N$ means that $S(t)e^{\sigma\sqrt{T^{2H}-t^{2H}}Z+(T-t)r-(\sigma^2/2)(T^{2H}-t^{2H})} \ge S_N$. Therefore,

$$V_4(S(t),t) = \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}}\int_{Z_N}^{+\infty}\left[S(t)\mathrm{e}^{\Sigma} - \frac{\beta S(t)\mathrm{e}^{\Sigma}X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e+n\Delta} + (\beta-1)X_e\right]\mathrm{e}^{-Z^2/2}dZ = \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}}\int_{Z_N}^{+\infty}S(t)\mathrm{e}^{\Sigma}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e+n\Delta}\right) \cdot \mathrm{e}^{-Z^2/2}dZ + \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}}\int_{Z_N}^{+\infty}(\beta-1)X_e \bullet \mathrm{e}^{-Z^2/2}dZ = S(t)\int_{Z_N-\sigma\sqrt{T^{2H}-t^{2H}}}^{+\infty}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e+n\Delta}\right) \cdot \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(Z-\sigma\sqrt{T^{2H}-t^{2H}})^2/2}dZ - \sigma\sqrt{T^{2H}-t^{2H}} + (\beta-1)X_e\int_{Z_N}^{+\infty}\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi}}\mathrm{e}^{-Z^2/2}dZ = S(t)\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e+n\Delta}\right)\left(1 - N\left(Z_N - \sigma\sqrt{T^{2H}-t^{2H}}\right)\right) + (\beta-1)X_e \bullet \mathrm{e}^{-r(T-t)}\left(N(-Z_N)\right). \quad (25)$$

By adding the derived analytical parts from (22), (24), and (25), we obtain the pricing formula of this exotic option as in (26).

$$V(S(t),t) = V_1(S(t),t) + V_2(S(t),t) + V_3(S(t),t) + V_4(S(t),t) = S(t)\left[N\left(Z_1 - \sigma\sqrt{T^{2H}-t^{2H}}\right) - N\left(Z_0 - \sigma\sqrt{T^{2H}-t^{2H}}\right)\right] - X_e\mathrm{e}^{-r(T-r)}\left(N(Z_1)\right.$$

$$- N(Z_0)) + S(t)\sum_{m=1}^{N-1}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e + n\Delta}\right) \cdot \left(N\left(Z_{m+1} - \sigma\sqrt{T^{2H} - t^{2H}}\right) - N\left(Z_m - \sigma\sqrt{T^{2H} - t^{2H}}\right)\right) + e^{-r(T-t)} X_e \sum_{m=1}^{N-1}\left(\frac{\beta m}{N} - 1\right)(N(Z_{m+1}) - N(Z_m)) + (\beta - 1) X_e \bullet e^{-r(T-t)}(N(-Z_N)) + S(t)\left(1 - \frac{\beta X_e}{N} \cdot \sum_{n=1}^{N}\frac{1}{X_e + n\Delta}\right)\left(1 - N\left(Z_N - \sigma\sqrt{T^{2H} - t^{2H}}\right)\right) \tag{26}$$

3.4. Special Cases. In this section, we discuss a few special cases of the exotic option. Specifically, by manipulating parameter H and the intensity of proactive hedging option actions, we can obtain several simplified versions of the pricing formula in Section 3.3.

3.4.1. Pricing of Proactive Hedging European Option Based on GBM. By letting $H = 1/2$ while keeping the same values for the other parameters, the pricing formula (26) can be simplified as a pricing formula under the GBM assumption, as specified here in (27).

$$V(S(t), t) = V_1(S(t), t) + V_2(S(t), t) + V_3(S(t), t) + V_4(S(t), t) = S(t)\left[N\left(Z_1 - \sigma\sqrt{T-t}\right) - N\left(Z_0 - \sigma\sqrt{T-t}\right)\right] - X_e e^{-r(T-r)}(N(Z_1) - N(Z_0)) + S(t)\sum_{m=1}^{N-1}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e + n\Delta}\right) \cdot \left(N\left(Z_{m+1} - \sigma\sqrt{T-t}\right) - N\left(Z_m - \sigma\sqrt{T-t}\right)\right) + e^{-r(T-t)} X_e \sum_{m=1}^{N-1}\left(\frac{\beta m}{N} - 1\right)(N(Z_{m+1}) - N(Z_m)) + S(t)\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e + n\Delta}\right)\left(1 - N\left(Z_N - \sigma\sqrt{T-t}\right)\right) + (\beta - 1) X_e \bullet e^{-r(T-t)}(N(-Z_N)) \tag{27}$$

It can be verified easily that the same result can be obtained by including the value function (7) in the B-S formula (13).

3.4.2. Pricing of Classical European Option Based on GFBM. Classical European option can be taken as a special case of this exotic option without proactive hedging strategy. By letting $\beta = 0$, pricing formula (26) can be simplified into that of the classical European option as shown in (28).

$$V(S(t), t) = V_1(S(t), t) + V_2(S(t), t) + V_3(S(t), t) + V_4(S(t), t) = S(t)\left[N\left(Z_1 - \sigma\sqrt{T^{2H} - t^{2H}}\right) - N\left(Z_0 - \sigma\sqrt{T^{2H} - t^{2H}}\right)\right] - X_e e^{-r(T-r)}(N(Z_1) - N(Z_0)) + S(t)\left(N\left(Z_N - \sigma\sqrt{T^{2H} - t^{2H}}\right) - N\left(Z_1 - \sigma\sqrt{T^{2H} - t^{2H}}\right)\right) - e^{-r(T-t)} X_e (N(Z_N) - N(Z_1)) + S(t)\left(1 - N\left(Z_N - \sigma\sqrt{T^{2H} - t^{2H}}\right)\right) - X_e \bullet e^{-r(T-t)}(N(-Z_N)). \tag{28}$$

3.4.3. Pricing of Classical European Option Based on GBM. The assumption of GBM corresponds to the cases of $\beta = 0$ and $H = 1/2$ in (26). A model with these assumptions is equivalent to the classic option pricing B-S model. We now derive the pricing formula for this special case, and we will also confirm that the reduced pricing model is the same as the classic B-S model.

First, by letting $H = 1/2$, we can further simplify some terms as

$$Z_0 = \frac{\ln(X_e/S(t)) + (T-t)(\sigma^2/2 - r)}{\sigma\sqrt{T-t}}, \quad Z_m = \frac{\ln(S_m/S(t)) + (T-t)(\sigma^2/2 - r)}{\sigma\sqrt{T-t}}, \quad \text{and } Z_N = \frac{\ln(S_N/S(t)) + (T-t)(\sigma^2/2 - r)}{\sigma\sqrt{T-t}}. \tag{29}$$

$V_1(S(t), t) = 0$, as it was previously. The other parts of (19) can be simplified as follows:

$$V_2(S(t), t) = S(t)\left[N\left(Z_1 - \sigma\sqrt{T-t}\right) - N\left(Z_0 - \sigma\sqrt{T-t}\right)\right] - X_e e^{-r(T-t)}[N(Z_1) - N(Z_0)]$$

$$V_3(S(t), t) = S(t)\sum_{m=1}^{N-1}\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{m}\frac{1}{X_e + n\Delta}\right) \cdot \left(N\left(Z_{m+1} - \sigma\sqrt{T-t}\right) - N\left(Z_m - \sigma\sqrt{T-t}\right)\right) + e^{-r(T-t)} X_e \sum_{m=1}^{N-1}\left(\frac{\beta m}{N} - 1\right)(N(Z_{m+1}) - N(Z_m)) \tag{30}$$

$$V_4(S(t), t) = S(t)\left(1 - \frac{\beta X_e}{N}\sum_{n=1}^{N}\frac{1}{X_e + n\Delta}\right) \cdot N\left(\sigma\sqrt{T-t} - Z_N\right) + e^{-r(T-t)}(\beta - 1) X_e.$$

If we let $d_0 = Z_0 - \sigma\sqrt{T-t} = (\ln(X_e/S(t)) + (T-t)(\sigma^2/2 - r))/\sigma\sqrt{T-t} - \sigma\sqrt{T-t} = (\ln(X_e/S(t)) - (T-t)(\sigma^2/2 + r))/\sigma\sqrt{T-t}$, and $d_m = Z_m - \sigma\sqrt{T-t} = (\ln(S_m/S(t)) + (T-t)(\sigma^2/2 - r))/\sigma\sqrt{T-t} - \sigma\sqrt{T-t} = (\ln(S_m/S(t)) - (T-t)(\sigma^2/2+r))/\sigma\sqrt{T-t}$, then we can rewrite the parts of value function (28) as

$$\begin{aligned} V_2(S(t),t) &= S(t)\left[N(d_1) - N(d_0)\right] \\ &\quad - X_e e^{-r(T-t)}\left[N(Z_1) - N(Z_0)\right] \\ V_3(S(t),t) &= S(t)\left[N(d_N) - N(d_{N-1}) + N(d_{N-1})\right. \\ &\quad \left.- N(d_{N-2}) + \cdots + N(d_2) - N(d_1)\right] \\ &\quad - X_e e^{-r(T-t)}\left[N(Z_N) - N(Z_{N-1}) + N(Z_{N-1})\right. \\ &\quad \left.- N(Z_{N-2}) + \cdots + N(Z_2) - N(Z_1)\right] \\ V_4(S(t),t) &= S(t)\left[1 - N(d_N)\right] - X_e e^{-r(T-t)}(1 \\ &\quad - N(Z_N)). \end{aligned} \tag{31}$$

By substituting the parts in the reduced form in (31) into (28), we can obtain the pricing formula for this special case as

$$\begin{aligned} V(S(t),t) &= V_1(S(t),t) + V_2(S(t),t) + V_3(S(t),t) \\ &\quad + V_4(S(t),t) \\ &= S(t)\left[1 - N(d_0)\right] \\ &\quad - X_e e^{-r(T-t)}\left[1 - N(Z_0)\right] \\ &= S(t)N(-d_0) + X_e e^{-r(T-t)}N(-Z_0) \end{aligned} \tag{32}$$

which is exactly the same as the above classic pricing formula for the B-S model.

4. Simulation Studies

4.1. Comparison of the Option Prices between the Exotic Option with Discrete Position Strategy versus the Classic European Option. In this section, we compare the pricing of the exotic option allowing dynamic discrete position strategy and the classic European option. The option price with dynamic discrete position, P_{dis}, is calculated based on (26), and the option price of the classic European option, P_{class}, is calculated based on the B-S model. We preset some parameters as $t = 0$, $T = 0.5$ (half of a year), $X_e = S(0) =$\$20 per share, $\beta_0 = 0$, and $r = 6\%$. For the dynamic discrete position, we allow the price step Δ to be \$1, \$2, \$5, and \$10, corresponding to numbers of steps N of 10, 5, 2, and 1, respectively, for a total price rise of \$10 for all simulations. We report the price ratio of these two options as

$$ratio_A = \frac{P_{dis}}{P_{class}} \tag{33}$$

Table 1: Numerical values of the price $ratio_A$ and $ratio_B$ for different parameter settings.

β	H	Δ	σ	$ratio_A$	$ratio_B$
0.8	0.5	2	0.1	0.9760	0.9386
0.8	0.5	2	0.4	0.7727	0.9119
0.8	0.5	5	0.1	0.9995	0.9166
0.8	0.5	5	0.4	0.8537	0.8253
0.8	0.7	2	0.1	0.9831	0.9414
0.8	0.7	2	0.4	0.8055	0.9138
0.8	0.7	5	0.1	0.9999	0.9256
0.8	0.7	5	0.4	0.8852	0.8315
1	0.5	2	0.1	0.9700	0.9228
1	0.5	2	0.4	0.7158	0.8811
1	0.5	5	0.1	0.9993	0.8957
1	0.5	5	0.4	0.8172	0.7718
1	0.7	2	0.1	0.9789	0.9264
1	0.7	2	0.4	0.7569	0.8853
1	0.7	5	0.1	0.9998	0.9070
1	0.7	5	0.4	0.8566	0.7823

for different values of Δ, σ, β, and H. This price ratio represents the extent to which the proactive hedging option with dynamic discrete position strategy reduces the option price relative to the classic European option under the same parameters. Please note that, in the classic option model, the parameter β is always 0, but the β in dynamic discrete position can be 0.8 or 1. The $ratio_A$ values are presented in a set of plots in Figure 3.

For example, when $H = 0.5$ and $\beta = 0.8$, the theoretical price of the classic European option, P_{class}, is 2.5239, and the price of the exotic option allowing discrete proactive hedging will change as the value of Δ changes. When $\Delta = 2$, the exotic option price P_{dis} is 1.9502, and the price ratio $ratio_A$ is 77.27%; that is, the dynamic discrete position strategy enjoys a price advantage of about 22.73% compared to the classic European option. When $\Delta = 5$, the exotic option price P_{dis} is 2.1548, and the price ratio $ratio_A$ is 85.37%, indicating that the dynamic discrete position strategy enjoys a price advantage of 14.63% compared to the classical European option (these price ratios can be read in Table 1). As we can see from Figure 3, in general, if the discrete strategy allows a smaller step in trading price (a smaller Δ value) and if the market has more fluctuation (a larger σ value), then this proactive hedging option has a stronger price advantage compared to the classic European option. Additionally, when other parameters are constant, this new exotic option has the maximum price advantage when $H = 0.5$ (when the underlying asset prices follow GBM). More price comparison ratios can be read from Table 1.

4.2. Comparison of the Option Prices between Exotic Options with Dynamic Discrete Position Strategy versus Continuous Linear Position. In this section, we compare the pricing of two kinds of exotic options with a proactive hedging strategy, one with a continuous linear position and the other with a discrete position strategy.

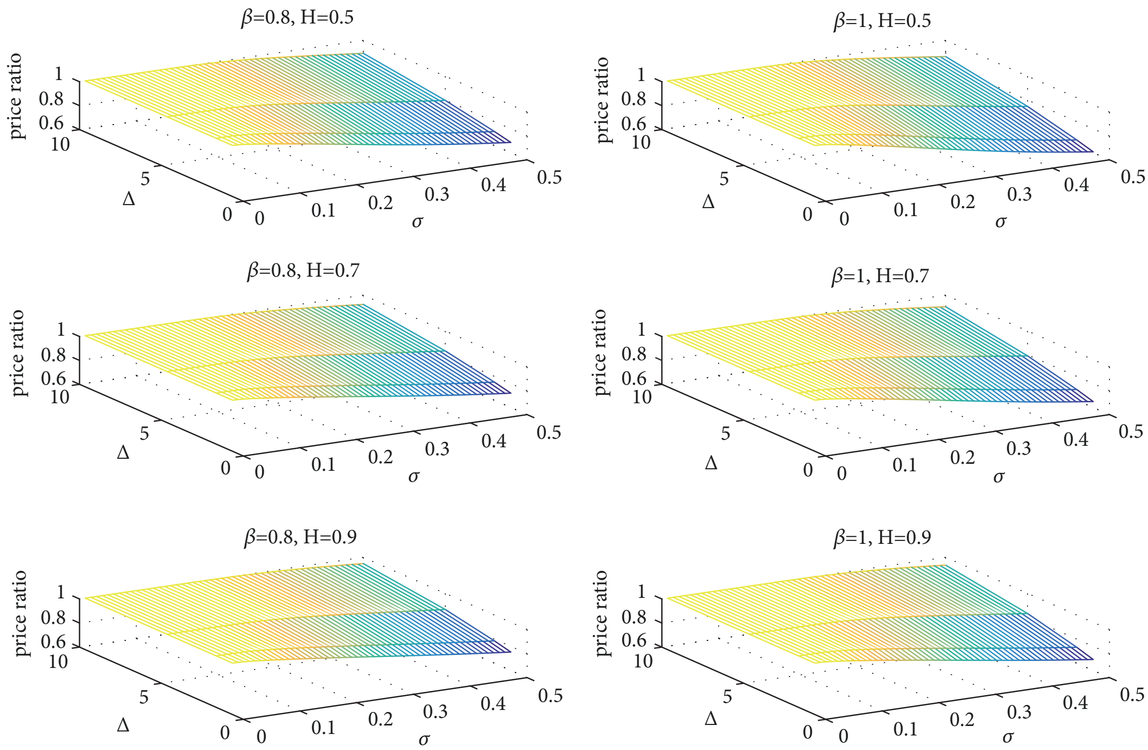

Figure 3: The price ratios between the proactive hedging European option and the classic European option for different parameter settings of Δ, σ, β, and H.

The option price with discrete position strategy, P_{dis}, is calculated based on (25), and the option price with continuous linear position, P_{con}, is calculated based on (18), as in Li et al. (2018). We preset some parameters as $t = 0$, $T = 0.5$ (half a year), $X_e = S_0$ =\$20 per share, and $\beta_0 = 0$, $r = 6\%$. Specific parameters were set as $\Delta = 0$ and $\alpha = 0.5$ in P_{con}. Since $\alpha = 0.5$ corresponds to a total price change of \$10 compared to the exercising price X_e in the continuous linear position, for the discrete position strategy we set the price step Δ as \$1, \$2, \$5, and \$10, corresponding to the numbers of steps N of 10, 5, 2, and 1, respectively. We compute the price ratio as

$$ratio_B = \frac{P_{con}}{P_{dis}} \tag{34}$$

for different values of Δ, σ, β, and H. This $ratio_B$ represents the price sacrifice resulting from making the continuous linear hedging strategy discrete. The ratios are presented as a panel of plots as in Figure 4.

For example, when $H = 0.5$ and $\beta = 0.8$, the theoretical price of the exotic option allowing continuous proactive hedging, P_{con}, is 1.7783, and the price of the exotic option allowing discrete proactive hedging will change with Δ value. When $\Delta = 2$, the exotic option price P_{dis} is 1.9502, and the price ratio $ratio_B$ is 91.19%, indicating that by transforming the continuous linear hedging strategy into a discrete linear strategy, allowing trade-in for every price appreciation of 2 dollars, it will decrease the price advantage by 8.81%. When $\Delta = 5$, the exotic option price P_{dis} is 2.1548, and the price ratio $ratio_B$ is 82.53%, indicating that the change to a discrete position strategy, allowing trade-in for every price appreciation of 5 dollars, will decrease the price advantage by 17.47% (these price ratios can be read from Table 1). As shown in Figure 4, given the market condition (by fixing the σ, β, and H values), a smaller-step (a smaller Δ) discrete strategy will better maintain the price advantage of this proactive hedging option. Given the discrete strategy step size (by fixing Δ), if the market shows less fluctuation (smaller σ), the discrete strategy can better maintain the price advantage of the proactive hedging option, and a smaller step size can generally help resist the influence of market uncertainty. Similarly, we find that when the underlying asset price follows closely to GBM, the exotic option has the maximum advantage. More price comparison ratios can be read from Table 1.

5. Conclusion

In this paper, we further extended the proactive hedging option by making the continuous linear position strategy discrete, so that it is more feasible for practical trading purposes. By applying results from extension of Li et al. (2018) under the risk-neutral evaluation principle, we derived the analytical form of pricing formula for this exotic option and compared its pricing advantage to the classic European option and the proactive hedging option with continuous linear position strategy. When the option allows a small-step-position strategy and when there is greater uncertainty in the

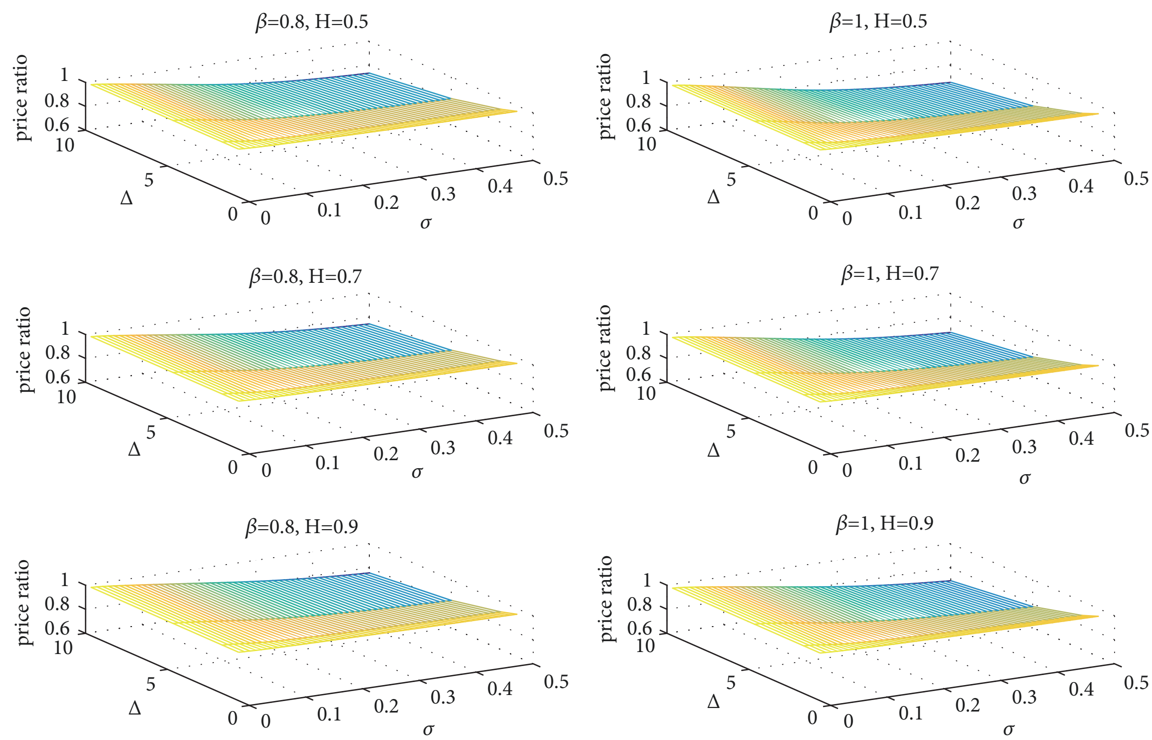

Figure 4: The exotic option price ratios between the proactive hedging European option and the continuous linear hedging strategy under different parameter settings of Δ, σ, β, and H values.

price of the underlying asset, the discrete position strategy generally enjoys a significant price advantage. Although making the continuous strategy discrete will somewhat sacrifice the price advantage, simulation studies show that a small-step discrete strategy can mostly maintain the price advantage through proactive hedging actions under most of the market conditions considered here. Overall, the discrete strategy greatly improves the feasibility of this exotic option. However, future work should examine the effects of relaxation of the assumptions involved in the derivation and discussion, such as the no trading cost assumption, on the feasibility of this option strategy.

Data Availability

The simulation code and data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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