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Research Article

Solvability of Some Fractional Boundary Value Problems with a Convection Term

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This paper is devoted to the research of some Caputo's fractional derivative boundary value problems with a convection term. By the use of some fixed-point theorems and the properties of Green function, the existence results of at least one or triple positive solutions are presented. Finally, two examples are given to illustrate the main results.

1. Introduction

Fractional differential equations (FDEs) present new models for many applications in physics, biomathematics, environmental issues, control theory, image processing, chemistry, mechanics, and so on [1–17]. Recently, researchers focus on studying various aspects of fractional differential equations, such as stability analysis, existence, multiplicity, and uniqueness of solutions [1–40]. Among all these topics, the existence and multiplicity results of positive solutions represent a topic of high interest in fractional calculus.

Some authors studied the existence and uniqueness of solutions for fractional differential equations with Caputo or Riemann-Liouville derivatives based on the Banach contraction principle and investigate the stability results for various fractional problems [4, 5, 16, 17]. Others studied the existence and multiplicity results of positive solutions or the iterative scheme. By the use of the Krasnoesel-skii's fixed-point theorem, Zhang [34] obtained some existence results of positive solutions of following problem without the convection term:

$$\begin{aligned} &{}^{C}D_{0+}^{\alpha}y(x) = f(x, y(x)), \quad 0 < x < 1, \\ &y(0) + y'(0) = 0, \\ &y(1) + y'(1) = 0. \end{aligned} \tag{1}$$

Wang and Liu [29] deduced the Green function and some interesting properties for the Dirichlet BVPs

$$\begin{gathered} -{}^{RL}D_{0+}^{\alpha}y(x) + by(x) = f(t, y(t)), \quad x \in (0,1), \\ y(0) = y(1) = 0, \end{gathered} \tag{2}$$

where ${}^{RL}D_{0+}^{\alpha}$ is the Riemann-Liouville (R-L) fractional derivative, $\alpha \in (1,2)$, and $b > 0$. And they established an iterative scheme to approximate the unique positive solution under the singular conditions.

Meng and Stynes [23] considered the following linear two-point fractional differential equation BVPs with general Robin type boundary condition:

$$\begin{aligned} &-{}^{C}D_{0+}^{\alpha}y(x) + by'(x) = f(x), \quad x \in (0,1), \\ &y(0) - \beta_0 y'(0) = \gamma_0, \\ &y(1) + \beta_1 y'(1) = \gamma_1, \end{aligned} \tag{3}$$

where ${}^{C}D_{0+}^{\alpha}$ denotes the Caputo derivative, $\alpha \in (1,2]$, b, β_0, β_1 are constant, and $f \in C[0,1]$. Meng used two parameter Mittag-Leffler functions to establish explicitly Green's function for the problems. They obtained the nonnegativity of Green's function.

This paper is devoted to the research of the solvability of the following nonlinear fractional BVPs:

$$-{}^{C}D_{0+}^{\alpha}y(x) + by'(x) = f(x, y(x)), \quad x \in (0,1), \tag{4}$$

$$\begin{aligned} &y(0)-\beta_0 y'(0)=0,\\ &y(1)+\beta_1 y'(1)=0, \end{aligned} \tag{5}$$

where $1<\alpha\le 2$ and $b,\beta_0,\beta_1\in\mathbb{R}$ are constants. ${}^{C}D_{0+}^{\alpha}$ is the Caputo's fractional derivative. Compared to the existing literature, the interesting point here is that the convection term is involved in the study of the solvability of fractional differential boundary value problems. By applying some fixed-point theorems, some existence and multiplicity results of positive solutions are given. Some examples are presented in last section to illustrate the main theorems.

In the sequel, the following conditions will be used:

(H_1) $f\in C([0,1]\times[0,\infty),[0,\infty))$;

(H_2) the constants $\beta_1\ge 0,\beta_0\ge -F_2(1)+F_1(1)(F_\alpha(1)/F_{\alpha-1}(1))$. The function $F_c(x)$ is defined in Section 2.

2. Background Material and Definitions

In order to solve the BVPs (4), (5), the following definitions and lemmas are needed.

Definition 1. Define the two-parameter Mittag-Leffler function by

$$E_{\tau,\upsilon}(x):=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\tau k+\upsilon)},\quad \text{for } \tau>0,\ \upsilon\ge 0,\ x\in\mathbb{R}. \tag{6}$$

Definition 2. Assume $g\in C^{m-1}[0,1]$ and $g^{(m-1)}(x)\in AC[0,1]$. The Caputo fractional derivative of order $\alpha\in(m-1,m]$ is defined as

$${}^{C}D_{0+}^{\alpha}g(x):=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}g^{(m)}(t)\,dt,\quad \text{for } 0<x\le 1. \tag{7}$$

In particularly, for $g\in C^2[0,1]$

$$\lim_{\alpha\longrightarrow 2}{}^{C}D_{0+}^{\alpha}g(x)=g''(x),\quad \text{for each } x\in(0,1]. \tag{8}$$

From the above equation, for $x\ge 0$ there is $E_{\tau,\upsilon}(x)>0$ and

$$E_{\alpha-1,1}(x):=xE_{\alpha-1,\alpha}(x)+1,\quad \text{for all } x\in\mathbb{R}. \tag{9}$$

Remark 3. For simplicity, let

$$F_c(x):=x^{c-1}E_{\alpha-1,c}\left(bx^{\alpha-1}\right),\quad \text{for } c\ge 0,\ x\ge 0, \tag{10}$$

the particular,

$$F_1(x)=bF_\alpha(x)+1. \tag{11}$$

Lemma 4 (see [23]). *Suppose $1<\alpha\le 2$ and the conditions (H_1) and (H_2) hold. For $h\in C[0,1]$, the problem*

$${}^{C}D_{0+}^{\alpha}y(x)+h(x)=0,\quad 1<x<1, \tag{12}$$

$$\begin{aligned} &y(0)-\beta_0 y'(0)=0,\\ &y(1)+\beta_1 y'(1)=0, \end{aligned} \tag{13}$$

has a unique solution

$$y(x)=\int_0^1 G(x,t)h(t)\,dt, \tag{14}$$

where $G(x,t)$ is Green's function

$$\begin{aligned} G(x,t)&=\begin{cases}\sigma(x)\left[F_\alpha(1-t)+\beta_1F_{\alpha-1}(1-t)\right]-F_\alpha(x-t), & \text{for } 0\le t\le x;\\ \sigma(x)\left[F_\alpha(1-t)+\beta_1F_{\alpha-1}(1-t)\right], & \text{for } x<t\le 1,\end{cases}\\ \sigma(x)&=\frac{\beta_0+F_2(x)}{\beta_0+\beta_1F_1(1)+F_2(1)}, \end{aligned} \tag{15}$$

and $F_c(x)$ is defined by Remark 3.

By the use of some interesting properties of the Mittag-Leffler function, Meng and Stynes proved the following result.

Lemma 5 (see [23]). *Suppose $\beta_1\ge 0$. Then Green's function $G(x,t)\ge 0$ for $x,t\in[0,1]$ if and only if*

$$\beta_0\ge -F_2(1)+F_1(1)\frac{F_\alpha(1)}{F_{\alpha-1}(1)}. \tag{16}$$

We pointed out here that this lemma comes from Theorem 5.1 and Remark 5.2 of Ref. [23] with some equivalent changes.

The following theorems are fundamental for proving the main results.

Lemma 6 (see [41]). *Let E be a Banach space, $K\subseteq E$ a cone, and Ω_1,Ω_2 two bounded sets of E with $0\in\overline{\Omega}_1\subset\Omega_2$. Suppose that $A:K\cap(\overline{\Omega}_2\backslash\Omega_2)\longrightarrow K$ is completely continuous such that either $(R1)$ $\|Ax\|\le\|x\|,x\in K\cap\partial\Omega_1$ and $\|Ax\|\ge\|x\|,x\in K\cap\partial\Omega_2$, or*

$(R2)$ $\|Ax\|\ge\|x\|,x\in K\cap\partial\Omega_1$ and $\|Ax\|\le\|x\|,x\in K\cap\partial\Omega_2$ holds.

Then the operator A has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Lemma 7 (see [42]). *Let P be a cone in a real Banach space E, $P_c=\{u\in P\mid\|u\|\le c\}$, θ is a nonnegative continuous* *concave functional on P such that $\theta(u) \le \|u\|$, for $u \in \overline{P_c}$, and $P(\theta, b, d) = \{u \in P \mid b \le \theta(u), \|u\| \le d\}$. Suppose $A : \overline{P_c} \longrightarrow \overline{P_c}$ is completely continuous and there exist four constants $0 < a < b < d \le c$ satisfying*

(C1) $\{u \in P(\theta, b, d) \mid \theta(u) > b\} \ne \emptyset$ and $\theta(Au) > b$ for $u \in P(\theta, b, d)$;

(C2) $\|Au\| < a$ for $\|u\| \le a$;

(C3) $\theta(Au) > b$ for $u \in P(\theta, b, d)$ with $\|Au\| > b$.

Then the operator A has three fixed-points u_1, u_2, u_3 such that

$$\|u_1\| < a,\ b < \theta(u_2),\ a < \|u_3\| \quad \text{with } \theta(u_3) < b. \tag{17}$$

Remark 8. Specially, if $d = c$, then condition $(C1)$ of Lemma 7 implies condition $(C3)$.

3. Existence and Multiplicity

In this section, by applying Lemma 6 and Lemma 7, some solvability results for BVPs (4), (5) will be obtained.

Let $E = C[0,1]$ be a Banach space; $\|u\| = \max_{0\le t\le 1}|u(t)|$. The cone $P \subset E$ is defined as

$$P = \{u \in E \mid u(t) \ge 0,\ t \in [0,1]\}. \tag{18}$$

Let the concave functional θ be defined by

$$\theta(u) = \min_{1/4\le t\le 3/4} |u(t)|. \tag{19}$$

Define the operator $T : P \longrightarrow E$ by

$$(Ty)(x) := \int_0^1 G(x,t) f(t, y(t))\, dt, \quad \text{for } 0 \le x \le 1. \tag{20}$$

It is clear that $y(x)$ is a positive solution of BVPs (4), (5) equivalent to that $y \in P$ is a fixed-point of T.

Lemma 9. *The operator $T : P \longrightarrow P$ is completely continuous.*

Proof. Taking into account that the functions $G(x,t)$ and $f(x,y)$ are all continuous and nonnegative, the operator $T : P \longrightarrow E$ is continuous and nonnegative, i.e., $T : P \longrightarrow P$. Suppose $\Omega \subset P$ is a bounded set, and for all $y \in \Omega$ there holds $\|y\| \le M_1$ for some $M_1 > 0$. Let $\overline{M} = \max_{0\le x\le 1, 0\le y\le M_1}|f(x,y)| + 1$; then, for $y \in \Omega$, there is

$$\begin{aligned} |(Ty)(x)| &\le \left|\int_0^1 G(x,t) f(t, y(t))\, dt\right| \\ &\le \overline{M} \int_0^1 \max_{0\le x\le 1} G(x,t)\, dt. \end{aligned} \tag{21}$$

Thus, the set $T(\Omega)$ is bounded.

By the fact that the Green function $G(x,t)$ is continuous on $[0,1] \times [0,1]$, one has the fact that it is uniformly continuous. Therefor, for given $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in \Omega, x_1, x_2 \in [0,1], x_1 < x_2$, and $x_2 - x_1 < \delta$; there holds $|G(x_2,t) - G(x_1,t)| < \epsilon/\overline{M}$. Thus

$$\begin{aligned} &|(Ty)(x_2) - (Ty)(x_1)| \\ &\quad \le \left|\int_0^1 [G(x_2,t) - G(x_1,t)] f(t, y(t))\, dt\right| \\ &\quad \le \int_0^1 \frac{\epsilon}{\overline{M}} \cdot \overline{M}\, dt = \epsilon. \end{aligned} \tag{22}$$

Hence, the set $T(\Omega)$ is equicontinuous. Thus, using the Arzela-Ascoli theorem, we claim that $T : P \longrightarrow P$ is a completely continuous operator. □

Let

$$\begin{aligned} M &= \left(\int_0^1 \max_{0\le x\le 1} G(x,t)\, dt\right)^{-1}; \\ N &= \left(\int_{1/4}^{3/4} \min_{1/4\le x\le 3/4} G(x,t)\, dt\right)^{-1}. \end{aligned} \tag{23}$$

Theorem 10. *Assume conditions (H_1) and (H_2) hold. If there exist two different positive constants $0 < r_2 < r_1$ such that*

(A1) $f(x,y) \le Mr_1$, for $(x,y) \in [0,1] \times [0, r_1]$;

(A2) $f(x,y) \ge Nr_2$, for $(x,y) \in [1/4, 3/4] \times [0, r_2]$,

then the BVPs (4), (5) have one positive solution y such that $r_2 \le \|y\| \le r_1$.

Proof. Define two open sets

$$\begin{aligned} \Omega_1 &:= \{y \in P \mid \|y\| < r_1\}, \\ \Omega_2 &:= \{y \in P \mid \|y\| < r_2\}. \end{aligned} \tag{24}$$

For $y \in \partial\Omega_1$, there is $0 \le y(x) \le r_1$ for $x \in [0,1]$. The condition $(A1)$ yields that

$$\begin{aligned} \|Ty\| &= \max_{0\le x\le 1}\left|\int_0^1 G(x,t) f(t, y(t))\, dt\right| \\ &\le Mr_1 \int_0^1 \max_{0\le x\le 1} G(x,t)\, dt = r_1 = \|y\|. \end{aligned} \tag{25}$$

So

$$\|Ty\| \le \|y\|, \quad \text{for } y \in \partial\Omega_1. \tag{26}$$

For $y \in \partial\Omega_2$, there is $0 \le y(x) \le r_2$ for $x \in [0,1]$. According to $(A2)$, for $x \in [1/4, 3/4]$,

$$\begin{aligned} (Ty)(x) &= \int_0^1 G(x,t) f(t, y(t))\, dt \\ &\ge \int_0^1 \min_{1/4\le x\le 3/4} G(x,t) f(t, y(t))\, dt \\ &\ge \int_{1/4}^{3/4} \min_{1/4\le x\le 3/4} G(x,t) f(t, y(t))\, dt \\ &\ge Nr_2 \int_{1/4}^{3/4} \min_{1/4\le x\le 3/4} G(x,t)\, dt = r_2 = \|y\|. \end{aligned} \tag{27}$$

So

$$\|Ty\| \geq \|y\|, \quad \text{for } y \in \partial\Omega_2. \tag{28}$$

And the proof also holds when $0 < r_2 < r_1$. Therefore, by Lemma 6, the proof is complete. □

Theorem 11. *Assume conditions* (H_1) *and* (H_2) *hold. If there exist constants* $0 < a < b < c$ *such that*

(B1) $f(x,y) < Ma$, *for* $(x,y) \in [0,1] \times [0,a]$;

(B2) $f(x,y) \geq Nb$, *for* $(x,y) \in [1/4,3/4] \times [b,c]$;

(B3) $f(x,y) \leq Mc$, *for* $(x,y) \in [0,1] \times [0,c]$,

then the BVPs (4), (5) have three positive solutions y_1, y_2, y_3 *with*

$$\begin{aligned} &\max_{0\leq x\leq 1} |y_1(x)| < a, \\ &b < \min_{1/4\leq x\leq 3/4} |y_2(x)| < \max_{0\leq x\leq 1} |y_2(x)| \\ &\leq c, \\ &a < \max_{0\leq x\leq 1} |y_3(x)| \leq c, \\ &\min_{1/4\leq x\leq 3/4} |y_3(x)| < b. \end{aligned} \tag{29}$$

Proof. We just need to prove that all the conditions of Lemma 7 hold.

For $y \in \overline{P_c} = \{y \in P \mid \|y\| \leq c\}$, there is $\|y\| \leq c$. Assumption $(B3)$ shows $f(x,y(x)) \leq Mc$ for $0 \leq x \leq 1$. Taking into account the definition of M, there is

$$\begin{aligned} \|Ty\| &= \max_{0\leq x\leq 1} \left| \int_0^1 G(x,t) f(t,y(t))\, dt \right| \\ &\leq \int_0^1 \max_{0\leq x\leq 1} G(x,t) f(t,y(t))\, dt \\ &\leq Mc \int_0^1 \max_{0\leq x\leq 1} G(x,t)\, dt \leq c. \end{aligned} \tag{30}$$

Hence $T : \overline{P_c} \longrightarrow \overline{P_c}$. Similarly, if $y \in \overline{P_a}$, then assumption $(B1)$ shows that $f(x,y(x)) < Ma$ for $0 \leq x \leq 1$. Therefore, $(C2)$ of Lemma 7 holds.

For $0 \leq x \leq 1$, choose

$$y(x) = \frac{c+b}{2}, \quad 0 \leq x \leq 1. \tag{31}$$

It is clear that $y(x) \in P(\theta,b,c)$, $\theta(y) = \theta((b+c)/2) > b$. Thus the set $\{y \in P(\theta,b,c) \mid \theta(y) > b\}$ is nonempty. If $y \in P(\theta,b,c)$, then $b \leq y(x) \leq c$ for $1/4 \leq x \leq 3/4$. And assumption $(B2)$ shows $f(x,y(x)) \geq Nb$ for $1/4 \leq x \leq 3/4$. So

$$\begin{aligned} \theta(Ty) &= \min_{1/4\leq x\leq 3/4} |(Ty)(x)| \\ &\geq \int_0^1 \min_{1/4\leq x\leq 3/4} G(x,t) f(t,y(t))\, dt \\ &> Nb \int_{1/4}^{3/4} \min_{1/4\leq x\leq 3/4} G(x,t)\, dt = b. \end{aligned} \tag{32}$$

That is to say that $\theta(Ty) > b$ for $y \in P(\theta,b,c)$. I.e., the condition $(C1)$ of Lemma 7 holds. By Remark 8, one has the fact that the condition $(C3)$ holds, too.

Then by Lemma 7, the BVPs (4), (5) have three positive solutions y_1, y_2, y_3 such that

$$\begin{aligned} &\max_{0\leq x\leq 1} |y_1(x)| < a, \\ &b < \min_{1/4\leq x\leq 3/4} |y_2(x)| < \max_{0\leq x\leq 1} |y_2(x)| \\ &\leq c, \\ &a < \max_{0\leq x\leq 1} |y_3(x)| \leq c, \\ &\min_{1/4\leq x\leq 3/4} |y_3(x)| < b. \end{aligned} \tag{33}$$

The theorem is proven. □

4. Two Examples

Now, we present two examples to check the main results. Readers can easily find that the following problem cannot be solved with existing literature. Choose $\alpha = 3/2, b = 1, \beta_1 = 1$, and

$$\beta_0 = 1 \geq -F_2(1) + \frac{F_1(1) F_\alpha(1)}{F_{\alpha-1}(1)} = 0.56. \tag{34}$$

Example 1. Consider the following problem:

$$-{}^C D_{0+}^{3/2} y(x) + y'(x) = \sqrt{y(x)} - \frac{x^2}{2} + 1, \quad x \in (0,1), \tag{35}$$

$$\begin{aligned} &y(0) - y'(0) = 0, \\ &y(1) + y'(1) = 0. \end{aligned} \tag{36}$$

Clearly, conditions (H_1) and (H_2) are satisfied. A simple computation shows $M \approx 1.6, N \approx 2.8$. Let $r_1 = 1, r_2 = 1/4$, and there are

$$\begin{aligned} &f(x,y) = \sqrt{y} - \frac{x^2}{2} + 1 \leq 1.5 < Mr_1 = 1.6, \\ &\qquad \text{for } (x,y) \in [0,1] \times [0,1], \\ &f(x,y) = \sqrt{y} - \frac{x^2}{2} + 1 \geq 1.1 > Nr_2 = 0.7, \\ &\qquad \text{for } (x,y) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[0, \frac{1}{4}\right]. \end{aligned} \tag{37}$$

By Theorem 10, the BVPs (35), (36) have one positive solution y and $1/4 \leq \|y\| \leq 1$.

Example 2. Consider the following fractional problem with the convection term:

$$-{}^{C}D_{0+}^{3/2}y(x) + y'(x) = f(x, y(x)), \quad x \in (0, 1), \tag{38}$$

$$\begin{aligned} y(0) - y'(0) &= 0, \\ y(1) + y'(1) &= 0, \end{aligned} \tag{39}$$

where

$$f(x, y) = \begin{cases} y^2 + \dfrac{1}{4}x^2, & \text{for } y \le 1; \\ \dfrac{1}{2}y + \dfrac{4}{x^2 + 1}, & \text{for } y \ge 1. \end{cases} \tag{40}$$

Clearly, conditions (H_1) and (H_2) hold. A simple computation shows that $M \approx 1.6, N \approx 2.8$. Choose $a = 1/2, b = 1, c = 4$, and there hold

$$\begin{aligned} f(x, y) &= y^2 + \frac{1}{4}x^2 \le 0.5 < Ma = 0.8, \\ &\qquad \text{for } (x, y) \in [0, 1] \times \left[0, \frac{1}{2}\right], \\ f(x, y) &= \frac{1}{2}y + \frac{4}{x^2 + 1} \ge 3.06 > Nb = 2.8, \\ &\qquad \text{for } (x, y) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [1, 4], \\ f(x, y) &\le 1.25 + 4 = 5.25 < Nc = 6.4, \\ &\qquad \text{for } (x, y) \in [0, 1] \times [0, 4]. \end{aligned} \tag{41}$$

By Theorem 11, the BVPs (38), (39) have three positive solutions y_1, y_2, y_3 such that

$$\begin{aligned} &\max_{0 \le x \le 1} |y_1(x)| < \frac{1}{2}, \\ &1 < \min_{1/4 \le x \le 3/4} |y_2(x)| < \max_{0 \le x \le 1} |y_2(x)| \\ &\le 4, \\ &\frac{1}{2} < \max_{0 \le x \le 1} |y_3(x)| \le 4, \\ &\min_{1/4 \le x \le 3/4} |y_3(x)| < 1. \end{aligned} \tag{42}$$

Data Availability

No data were used to support this study.

Additional Points

Results and Discussion. Under the nonlinear term satisfying some growth condition, we studied some fractional differential equation boundary value problems in Caputo sense with the convection term. By applying fixed-point theory, some solvability criteria of positive solution are obtained. Two examples are given to illustrate the existence and multiplicity results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper.

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