Research Article

A Robust Spline Collocation Method for Pricing American Put Options

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In this paper a robust numerical method is proposed for pricing American put options. The Black-Scholes differential operator in the original form is discretized by using a quadratic spline collocation method on a piecewise uniform mesh for the spatial discretization and the implicit Euler scheme for the time discretization. The position of collocation points is chosen so that the spline difference operator satisfies the discrete maximum principle, which guarantees that the scheme is maximum-norm stable. The error estimation is derived by applying the maximum principle to the discrete linear complementarity problem in two mesh sets. It is proved that the scheme is second-order convergent with respect to the spatial variable and first-order convergent with respect to the time variable. Numerical results demonstrate that the scheme is stable and accurate.

1. Introduction

The American option is an important financial instrument that gives the holder the right, but not obligation, to buy (call option) or sell (put option) an asset at any time prior to its maturity date. One way to price American options is to solve a linear complementarity problem involving the Black-Scholes differential operator [1]. Since this complementarity problem is, in general, not analytically solvable, numerical methods are required to obtain the approximate solution.

The spline approximation methods have become interesting and very promising in solving differential equations due to their flexibility in practical applications. The spline solution has its own advantages, for example, once the solution has been computed, the information required for the spline interpolation between mesh points is easy to obtain. A few papers have used the spline approximation methods to solve option pricing problems. Khabir and Patidar [2] applied a B-spline collocation method to solve the heat equation which is obtained from the Black-Scholes equation by an Euler transformation. Kadalbajoo et al. [3, 4] used cubic B-spline collocation methods for the Euler transformed generalized Black-Scholes equation. Mohammadi [5] developed a quintic B-spline collocation method for solving the generalized

Black-Scholes equation governing option pricing. Christara and Leung [6] derived a quadratic spline collocation method on a uniform mesh in the pricing problem under a finite activity jump-diffusion model. Rashidinia and Jamalzadeh [7] proposed a modified B-spline collocation approach for pricing American style Asian options.

In this paper we adopt a quadratic spline collocation method on a piecewise uniform mesh to solve the continuous linear complementarity problem arising from American put option pricing. The Black-Scholes differential operator in the original form is discretized. By applying the technique of Surla et al. [8, 9], the position of collocation points is chosen so that the spline difference operator on a piecewise uniform mesh satisfies the discrete maximum principle, which guarantees that the discretization scheme is maximum-norm stable. The error estimation is derived by applying the maximum principle to the discrete linear complementarity problem in two mesh sets. It is proved that the convergence order of the scheme is second-order with respect to the spatial variable and first-order with respect to the time variable. The scheme examined in the present paper overcomes the difficulty in the derivation of the error estimation for the linear complementarity problem under the difference schemes. Numerical results are presented to support these theoretical results.

The outline of the paper is as follows. In Section 2 the continuous linear complementarity problem for pricing American put options is given. In Section 3 the spline difference scheme is derived. In Section 4 the stability and error analysis for the spline difference scheme are proved. In Section 5 numerical experiments are carried out. Finally conclusion and discussion are indicated in Section 6.

2. The Continuous Problem

It is well known that the value of an American put option $v(x, t)$ satisfies the following continuous linear complementarity problem [1, 10]:

$$Lv(x, t) \geq 0, \quad (1)$$

$$x > 0 \text{ and } t \in [0, T],$$

$$v(x, t) - g(x) \geq 0, \quad (2)$$

$$x > 0 \text{ and } t \in [0, T],$$

$$Lv(x, t) \cdot [v(x, t) - g(x)] = 0, \quad (3)$$

$$x > 0 \text{ and } t \in [0, T],$$

$$v(x, T) = g(x), \quad x \geq 0 \text{ and } t = T, \quad (4)$$

$$v(0, t) = E, \quad (5)$$

$$x = 0 \text{ and } t \in [0, T],$$

$$v(x, t) \longrightarrow 0, \quad (6)$$

$$x \longrightarrow +\infty \text{ and } t \in [0, T],$$

where L represents the Black-Scholes differential operator defined by

$$Lv(x, t) \equiv -\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} - (r - d)x \frac{\partial v}{\partial x} + rv, \quad (7)$$

x is the underlying asset price, t is the time, σ is the volatility of the underlying asset price, r is the risk-free interest rate, d is the continuous dividend rate, E is the strike price, T is the maturity date, and $g(x)$ is the payoff function given by

$$g(x) = \max(E - x, 0). \quad (8)$$

Here we assume that $r > d$.

The infinite domain $(0, +\infty)$ needs to be truncated into a bounded domain $(0, X)$ for solving the problem by the numerical method, where X is taken as $X = 4E$ based on Willmott et al.'s estimate [1]. The boundary condition at $x = X$ is set to be $v(X, t) = 0$. A detailed discussion of the choice of the linear boundary conditions can be found in [11]. Normally, the error in the computed option price due to the domain truncation is negligible [12]. Therefore, in the

remaining of this paper we will focus on solving the following linear complementarity problem

$$Lv(x, t) \geq 0, \quad (9)$$

$$(x, t) \in (0, X) \times (0, T),$$

$$v(x, t) - g(x) \geq 0, \quad (10)$$

$$(x, t) \in (0, X) \times [0, T],$$

$$Lv(x, t) \cdot [v(x, t) - g(x)] = 0, \quad (11)$$

$$(x, t) \in (0, X) \times [0, T],$$

$$v(x, T) = g(x), \quad x \in [0, X], \quad (12)$$

$$v(0, t) = E, \quad (13)$$

$$v(X, t) = 0, \quad t \in [0, T].$$

3. Discretization

Since the Black-Scholes differential operator at $x = 0$ is degenerative, the Euler transformation is usually used to remove the singularity of the differential operator at $x = 0$ (corresponding to $-\infty$ in the transformed domain); see, e.g., Schwartz [13], Tangman et al. [14], and Zhao et al. [15]. However, this will result in additional computational errors due to the truncation on the left-hand side of the domain. Furthermore, the originally mesh points will concentrate around $x = 0$ inappropriately by using the uniform mesh on the transformed interval. In this paper we will discretize the Black-Scholes equation in the original form, but with a quadratic spline collocation method for the spatial discretization and the implicit Euler scheme for the temporal discretization. It should be remarked that if the commonly quadratic spline collocation method is adopted on a uniform mesh for the original Black-Scholes equation, the stability of the scheme could not be guaranteed. The analogous problems have been discussed in [16, 17].

For the spatial discretization, a piecewise uniform mesh Ω^N is constructed to guarantee the stability of the discretization scheme:

$$x_i = \begin{cases} h & i = 1, \\ h \left[1 + \frac{\sigma^2}{r - d} (i - 1) \right] & i = 2, \dots, N, \end{cases} \quad (14)$$

where

$$h = \frac{X}{1 + (\sigma^2 / (r - d)) (N - 1)}. \quad (15)$$

For the temporal discretization, a uniform mesh Ω^K on $[0, T]$ with K mesh elements is used. Then the domain $\Omega \equiv (0, X) \times (0, T)$ is divided into the piecewise uniform mesh

$\Omega^{N \times K} \equiv \Omega^N \times \Omega^K$. The mesh sizes $h_i = x_i - x_{i-1}$ and $\Delta t = t_j - t_{j-1}$, respectively, satisfy

$$h_i = \begin{cases} h & i = 1, \\ \frac{\sigma^2}{r-d} h & i = 2, \dots, N, \end{cases} \quad (16)$$

and

$$\Delta t = \frac{T}{K}, \quad j = 1, \dots, K. \quad (17)$$

It will be shown that the matrix associated with the spline difference operator on the above piecewise uniform mesh is an M-matrix.

The implicit Euler scheme on Ω^K is used for the time discretization to get the following linear complementarity problem:

$$\begin{aligned} \tilde{L}v^j(x) &\geq \frac{v^{j+1}(x)}{\Delta t}, \\ x &\in (0, X), \end{aligned} \quad (18)$$

$$\begin{aligned} v^j(x) - g(x) &\geq 0, \\ x &\in (0, X), \end{aligned} \quad (19)$$

$$\begin{aligned} \left[\tilde{L}v^j(x) - \frac{v^{j+1}(x)}{\Delta t} \right] \cdot [v^j(x) - g(x)] &= 0, \\ x &\in (0, X), \end{aligned} \quad (20)$$

$$\begin{aligned} v^K(x) &= g(x), \\ x &\in [0, X], \end{aligned} \quad (21)$$

$$\begin{aligned} v^j(0) &= E, \\ v^j(X) &= 0, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \tilde{L}v^j(x) &= -\frac{1}{2}\sigma^2 x^2 \frac{d^2 v^j}{dx^2} - (r-d)x \frac{dv^j}{dx} \\ &\quad + \left(r + \frac{1}{\Delta t}\right)v^j(x), \end{aligned} \quad (23)$$

and $v^j(x)$ denotes the approximation of the exact solution $v(x, t)$ at j -th time level.

The solution $v^j(x)$ of the problem (18)-(22) is approximated by a quadratic spline function $u(x) \in C^1([0, X])$ which on each subinterval $[x_i, x_{i+1}]$ has the following form:

$$\begin{aligned} u(x) &= u_i + (x - x_i)u'_i + \frac{1}{2}(x - x_i)^2 u''_i, \\ x &\in [x_i, x_{i+1}]. \end{aligned} \quad (24)$$

The collocation points are chosen in a nonstandard way as that in [8, 9]:

$$\begin{aligned} \xi_i &= \alpha_{1,i}x_{i-1} + (1 - \alpha_{1,i})x_i \quad \text{on } [x_{i-1}, x_i], \\ \eta_i &= \alpha_{2,i}x_i + (1 - \alpha_{2,i})x_{i+1} \quad \text{on } [x_i, x_{i+1}] \end{aligned} \quad (25)$$

for $1 \leq i < N$, where $0 < \alpha_{1,i}, \alpha_{2,i} < 1$. Then the collocation equations are defined by

$$\begin{aligned} &-\frac{1}{2}\sigma^2 \xi_i^2 u''(\xi_i) - (r-d)\xi_i u'(\xi_i) + \left(r + \frac{1}{\Delta t}\right)u(\xi_i) \\ &= \tilde{L}u(\xi_i), \quad \xi_i \in [x_{i-1}, x_i], \end{aligned} \quad (26)$$

$$\begin{aligned} &-\frac{1}{2}\sigma^2 \eta_i^2 u''(\eta_i) - (r-d)\eta_i u'(\eta_i) + \left(r + \frac{1}{\Delta t}\right)u(\eta_i) \\ &= \tilde{L}u(\eta_i), \quad \eta_i \in [x_i, x_{i+1}]. \end{aligned} \quad (27)$$

We can obtain $u''(\xi_i) = u''_{i-1}$, $u''(\eta_i) = u''_i$ from the quadratic spline function (24). Since $\xi_i - x_{i-1} = h_i(1 - \alpha_{1,i})$, $\eta_i - x_i = h_{i+1}(1 - \alpha_{2,i})$, we have

$$\begin{aligned} u(\xi_i) &= u_{i-1} + h_i(1 - \alpha_{1,i})u'_{i-1} \\ &\quad + \frac{1}{2}h_i^2(1 - \alpha_{1,i})^2 u''_{i-1}, \\ u(\eta_i) &= u_i + h_{i+1}(1 - \alpha_{2,i})u'_i + \frac{1}{2}h_{i+1}^2(1 - \alpha_{2,i})^2 u''_i, \\ u'(\xi_i) &= u'_{i-1} + h_i(1 - \alpha_{1,i})u''_{i-1}, \\ u'(\eta_i) &= u'_i + h_{i+1}(1 - \alpha_{2,i})u''_i. \end{aligned} \quad (28)$$

Substituting the above expressions for $u(\eta_i)$, $u'(\eta_i)$, and $u''(\eta_i)$ into (27) on the interval $[x_i, x_{i+1}]$ with

$$\begin{aligned} u_{i+1} &= u_i + h_{i+1}u'_i + \frac{1}{2}h_{i+1}^2 u''_i, \\ u'_{i+1} &= u'_i + h_{i+1}u''_i, \end{aligned} \quad (29)$$

we have

$$\begin{aligned} u'_i &= \frac{(u_{i+1} - u_i)Q_i + (r + 1/\Delta t)h_{i+1}^2 u_i - h_{i+1}^2 \tilde{L}u(\eta_i)}{h_{i+1}P_i}, \\ u'_{i+1} &= \frac{2(u_{i+1} - u_i)}{h_{i+1}} \\ &\quad - \frac{(u_{i+1} - u_i)Q_i + (r + 1/\Delta t)h_{i+1}^2 u_i - h_{i+1}^2 \tilde{L}u(\eta_i)}{h_{i+1}P_i}, \end{aligned} \quad (30)$$

where

$$\begin{aligned} Q_i &= -\sigma^2 \eta_i^2 - 2(r-d)\eta_i h_{i+1}(1 - \alpha_{2,i}) \\ &\quad + \left(r + \frac{1}{\Delta t}\right)h_{i+1}^2(1 - \alpha_{2,i})^2, \end{aligned}$$

$$\begin{aligned}
P_i &= -\sigma^2 \eta_i^2 + (r-d) \eta_i h_{i+1} (-1 + 2\alpha_{2,i}) \\
&\quad - \left(r + \frac{1}{\Delta t}\right) h_{i+1}^2 \alpha_{2,i} (1 - \alpha_{2,i}).
\end{aligned} \tag{32}$$

Using the same technique on the interval $[x_{i-1}, x_i]$ with the collocation equation (26) and replacing i by $i-1$ in (31), we also can obtain

$$\begin{aligned}
u'_i &= \frac{2(u_i - u_{i-1})}{h_i} \\
&\quad - \frac{(u_i - u_{i-1}) W_i + (r + 1/\Delta t) h_i^2 u_{i-1} - h_i^2 \tilde{L}u(\xi_i)}{h_i M_i},
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
W_i &= -\sigma^2 \xi_i^2 - 2(r-d) \xi_i h_i (1 - \alpha_{1,i}) \\
&\quad + \left(r + \frac{1}{\Delta t}\right) h_i^2 (1 - \alpha_{1,i})^2 \\
M_i &= -\sigma^2 \xi_i^2 + (r-d) \xi_i h_i (-1 + 2\alpha_{1,i}) \\
&\quad - \left(r + \frac{1}{\Delta t}\right) h_i^2 \alpha_{1,i} (1 - \alpha_{1,i}).
\end{aligned} \tag{34}$$

Combining (30) with (33), we have the following spline difference scheme

$$a_i^- u_{i-1} + a_i^c u_i + a_i^+ u_{i+1} = q_i^- \tilde{L}u(\xi_i) + q_i^+ \tilde{L}u(\eta_i), \tag{35}$$

where

$$\begin{aligned}
a_i^- &= -\frac{Z_i}{h_i M_i}, \\
a_i^+ &= -\frac{Q_i}{h_{i+1} P_i}, \\
a_i^c &= \frac{2}{h_i} - \left(r + \frac{1}{\Delta t}\right) \frac{h_{i+1}}{P_i} + \frac{Q_i}{h_{i+1} P_i} - \frac{W_i}{h_i M_i}, \\
q_i^- &= -\frac{h_i}{M_i}, \\
q_i^+ &= -\frac{h_{i+1}}{P_i}, \\
Z_i &= -\sigma^2 \xi_i^2 + 2(r-d) \xi_i h_i \alpha_{1,i} + \left(r + \frac{1}{\Delta t}\right) h_i^2 \alpha_{1,i}^2.
\end{aligned} \tag{36}$$

Two degrees of freedom are provided by parameters $\alpha_{1,i}$ and $\alpha_{2,i}$, which can be used to guarantee that the matrix associated with the discrete operator is an M-matrix. Here we choose $\alpha_{1,i} = \Delta t$ and $\alpha_{2,i} = 1 - \Delta t$ for $1 \leq i < N$. In the next section we will prove that $a_i^- < 0$, $a_i^+ < 0$, $q_i^- > 0$, $q_i^+ > 0$ and $a_i^- + a_i^c + a_i^+ - q_i^-/\Delta t - q_i^+/\Delta t > 0$.

Combining (18)-(22) with (35) we can obtain the following fully discretization scheme

$$L^{N,K} V_i^j \geq 0, \quad (i, j) \in \tilde{\Omega}_h, \tag{37}$$

$$V_i^j - g_i \geq 0, \quad (i, j) \in \tilde{\Omega}_h, \tag{38}$$

$$L^{N,K} V_i^j \cdot [V_i^j - g_i] = 0, \quad (i, j) \in \tilde{\Omega}_h, \tag{39}$$

$$V_i^K = g_i, \quad i = 0, 1, \dots, N, \tag{40}$$

$$V_0^j = E,$$

$$V_N^j = 0, \tag{41}$$

$$j = K-1, \dots, 1, 0,$$

where

$$\begin{aligned}
L^{N,K} V_i^j &\equiv a_i^- V_{i-1}^j + a_i^c V_i^j + a_i^+ V_{i+1}^j \\
&\quad - \frac{q_i^-}{\Delta t} [\alpha_{1,i} V_{i-1}^{j+1} + (1 - \alpha_{1,i}) V_i^{j+1}] \\
&\quad - \frac{q_i^+}{\Delta t} [\alpha_{2,i} V_i^{j+1} + (1 - \alpha_{2,i}) V_{i+1}^{j+1}].
\end{aligned} \tag{42}$$

Here we have used the linear interpolation to get the approximated solutions V^{j+1} at mesh points ξ_i and η_i since we only know the numerical solution on mesh points at $j+1$ -th time level.

There exists a unique solution V for the above linear complementarity problem (37)-(41). A detailed discussion of existence and uniqueness of the solution for problem (37)-(41) can be found in [18]. Then from the above solution V we can obtain the optimal stopping price which is the maximum asset price such that $V_i^j = g_i$ for each t_j .

Remark 1. In general the points $x_{i+1/2}$ are used as the collocation points [19, 20], i.e., $\alpha_{1,i} = \alpha_{2,i} = 1/2$. But for the problem (18)-(22) the discrete operator with $\alpha_{1,i} = \alpha_{2,i} = 1/2$ does not satisfy the discrete maximum principle. Hence, in order to apply the discrete maximum principle and barrier function technique to analyze the stability and error estimate we use collocation points ξ_i on $[x_{i-1}, x_i]$ and η_i on $[x_i, x_{i+1}]$ for $1 \leq i < N$.

4. Analysis of the Method

The discrete maximum principle, truncation error analysis techniques, and barrier function techniques are used for analysis of the scheme.

Lemma 2 (discrete maximum principle). *When $\alpha_{1,i} = \Delta t$ and $\alpha_{2,i} = 1 - \Delta t$ for $1 \leq i < N$, the operator $L^{N,K}$ on $\Omega^{N \times K}$ satisfies a discrete maximum principle; i.e., if w_i^j is a mesh function satisfying $w_0^j \geq 0$, $w_N^j \geq 0$ ($0 \leq j < K$), $w_i^K \geq 0$ ($0 \leq i \leq N$) and $L^{N,K} w_i^j \geq 0$ ($1 \leq i < N$, $0 \leq j < K$), then $w_i^j \geq 0$ for all i, j .*

Proof. When $\alpha_{1,i} = \Delta t$ and $\alpha_{2,i} = 1 - \Delta t$, it is easy to check that

$$\begin{aligned} Z_i &< 0, \\ M_i &< 0, \\ \text{and } Q_i &< 0 \end{aligned} \quad (43)$$

for sufficiently small Δt and h . We also have

$$\begin{aligned} P_i &< -\sigma^2 \eta_i^2 + (r-d) \eta_i h_{i+1} \\ &= \eta_i \left[-\sigma^2 \eta_i + (r-d) h_{i+1} \right] \\ &< \eta_i \left[-\sigma^2 x_i + (r-d) h_{i+1} \right] \\ &< \eta_i \left[-\sigma^2 h + (r-d) \frac{\sigma^2}{r-d} h \right] = 0. \end{aligned} \quad (44)$$

Hence we can conclude that

$$\begin{aligned} a_i^- &< 0, \\ a_i^+ &< 0, \\ q_i^- &> 0, \\ q_i^+ &> 0, \end{aligned} \quad (45)$$

and

$$\begin{aligned} a_i^c &= -a_i^- - a_i^+ + \left(r + \frac{1}{\Delta t}\right) q_i^- + \left(r + \frac{1}{\Delta t}\right) q_i^+ > 0, \\ a_i^- + a_i^c + a_i^+ - \frac{1}{\Delta t} (q_i^- + q_i^+) &= r (q_i^- + q_i^+) > 0 \end{aligned} \quad (46)$$

for $1 \leq i < N$ and $0 \leq j < K$. Hence the matrix associated with $L^{N,K}$ is an M-matrix, which reveals that the operator $L^{N,K}$ satisfies the discrete maximum principle [21]. $\square$

By using Taylor expansions we can get the following bound of the truncation error

$$\begin{aligned} \tau_i^j(v) &= L^{N,K} v_i^j - q_i^- L v(\xi_i, t_j) - q_i^+ L v(\eta_i, t_j) \\ &= p_0 v(x_i, t_j) + p_1 \frac{\partial v}{\partial x}(x_i, t_j) + p_2 \frac{\partial^2 v}{\partial x^2}(x_i, t_j) \\ &\quad + p_3 \frac{\partial v}{\partial t}(x_i, t_j) + \sum_{n=1}^{12} \pi_{i,n}^j(v), \end{aligned} \quad (47)$$

where $p_0 = p_1 = p_2 = p_3 = 0$ and

$$\begin{aligned} \pi_{i,1}^j(v) &= a_i^+ R_2(x_i, x_{i+1}, v_j(x)), \\ \pi_{i,2}^j(v) &= -a_i^- R_2(x_i, x_{i-1}, v_j(x)), \end{aligned}$$

$$\begin{aligned} \pi_{i,3}^j(v) &= -\frac{1}{2} \sigma^2 x_i^2 q_i^- R_0\left(x_i, \xi_i, \frac{d^2 v_j}{dx^2}\right), \\ \pi_{i,4}^j(v) &= -\frac{1}{2} \sigma^2 x_i^2 q_i^+ R_0\left(x_i, \eta_i, \frac{d^2 v_j}{dx^2}\right), \\ \pi_{i,5}^j(v) &= -(r-d) x_i q_i^- R_1\left(x_i, \xi_i, \frac{dv_j}{dx}\right), \\ \pi_{i,6}^j(v) &= -(r-d) x_i q_i^+ R_1\left(x_i, \eta_i, \frac{dv_j}{dx}\right), \\ \pi_{i,7}^j(v) &= \left(r + \frac{1}{\Delta t}\right) q_i^- R_2(x_i, \xi_i, v_j(x)), \\ \pi_{i,8}^j(v) &= \left(r + \frac{1}{\Delta t}\right) q_i^+ R_2(x_i, \eta_i, v_j(x)), \\ \pi_{i,9}^j(v) &= -\frac{q_i^-}{\Delta t} [\alpha_{1,i} R_1(\xi_i, x_{i-1}, v(x, t_{j+1})) \\ &\quad + (1 - \alpha_{1,i}) R_1(\xi_i, x_i, v(x, t_{j+1}))], \\ \pi_{i,10}^j(v) &= -\frac{q_i^-}{\Delta t} [\alpha_{2,i} R_1(\eta_i, x_i, v(x, t_{j+1})) \\ &\quad + (1 - \alpha_{2,i}) R_1(\eta_i, x_{i+1}, v(x, t_{j+1}))], \\ \pi_{i,11}^j(v) &= -\frac{q_i^-}{\Delta t} R_1(x_i, \xi_i, v(x, t_j)) - \frac{q_i^+}{\Delta t} \\ &\quad \cdot R_1(x_i, \eta_i, v(x, t_j)) \\ \pi_{i,12}^j(v) &= -q_i^- \Delta t R_1(t_j, t_{j+1}, v(\xi_i, t)) \\ &\quad - q_i^+ \Delta t R_1(t_j, t_{j+1}, v(\eta_i, t)), \\ R_n(x, z, u) &= \frac{(z-x)^{n+1}}{(n+1)!} u^{(n+1)}(s), \quad s \in (x, z). \end{aligned} \quad (48)$$

Thus, when $\alpha_{1,i} = \Delta t$ and $\alpha_{2,i} = 1 - \Delta t$ for $1 \leq i < N$, from (47) we have

$$|\tau_i^j| \leq C (q_i^- + q_i^+) (h^2 + \Delta t) \quad 1 \leq i < N, \quad 0 \leq j < K, \quad (49)$$

where C is a positive constant independent of the mesh.

The following theorem is our main result for the quadratic spline collocation scheme.

Theorem 3. Let $v(x, t)$ be the solution of problem (9)-(13) and V_i^j the solution of problem (37)-(41). Then if we set $\alpha_{1,i} = \Delta t$ and $\alpha_{2,i} = 1 - \Delta t$ for $1 \leq i < N$, there exists a positive constant C independent of the mesh such that

$$|v(x_i, t_j) - V_i^j| \leq C (h^2 + \Delta t), \quad 0 \leq i \leq N, \quad 0 \leq j \leq K. \quad (50)$$

Proof. The error estimate will be derived by applying the maximum principle to the discrete linear complementarity problem in two mesh sets as that in [22]. Set

$$\begin{aligned}\overline{\Omega}_h &= \{(i, j) \mid 0 \leq i \leq N, 0 \leq j \leq K\}, \\ \widetilde{\Omega}_h &= \{(i, j) \mid 1 \leq i \leq N-1, 0 \leq j \leq K-1\}, \\ \Omega^{(1)} &= \{(i, j) \in \widetilde{\Omega}_h \mid v(x_i, t_j) = g_i\}, \\ \Omega^{(2)} &= \overline{\Omega}_h \setminus \Omega^{(1)}, \\ \partial\Omega_h &= \overline{\Omega}_h \setminus \widetilde{\Omega}_h.\end{aligned}\quad (51)$$

From (9)-(13) we have

$$\begin{aligned}Lv(x_i, t_j) &\geq 0, \quad (i, j) \in \Omega^{(1)}, \\ Lv(x_i, t_j) &= 0, \quad (i, j) \in \Omega^{(2)}.\end{aligned}\quad (52)$$

Let

$$\begin{aligned}\Omega_h^{(1)} &= \{(i, j) \in \widetilde{\Omega}_h \mid V_i^j = g_i\}, \\ \Omega_h^{(2)} &= \widetilde{\Omega}_h \setminus \Omega_h^{(1)}.\end{aligned}\quad (53)$$

It is easy to check that

$$L^{N,K}V_i^j = 0, \quad (i, j) \in \Omega_h^{(2)}. \quad (54)$$

Define the barrier function

$$\Phi_i^j = C(h^2 + \Delta t) > 0, \quad (i, j) \in \widetilde{\Omega}_h, \quad (55)$$

where C is a sufficiently large constant independent of h and Δt .

For $(i, j) \in \Omega_h^{(2)}$ we have

$$\begin{aligned}L^{N,K}(v(x_i, t_j) - V_i^j + \Phi_i^j) &= L^{N,K}v(x_i, t_j) \\ &+ L^{N,K}\Phi_i^j = [L^{N,K}v(x_i, t_j) - q_i^-Lv(\xi_i, t_j) \\ &- q_i^+Lv(\eta_i, t_j) + L^{N,K}\Phi_i^j] + q_i^-Lv(\xi_i, t_j) \\ &+ q_i^+Lv(\eta_i, t_j) \geq 0\end{aligned}\quad (56)$$

by using $Lv(\xi_i, t_j) \geq 0, Lv(\eta_i, t_j) \geq 0$ and (49), (54), and (55).

On the “boundary” of the $\Omega_h^{(2)}$ we have

$$\begin{aligned}v(x_i, t_j) - V_i^j + \Phi_i^j &= v(x_i, t_j) - g_i + \Phi_i^j \geq 0 \\ &\text{for } (i, j) \in \Omega_h^{(1)},\end{aligned}\quad (57)$$

and

$$v(x_i, t_j) - V_i^j + \Phi_i^j = \Phi_i^j \geq 0 \quad \text{for } (i, j) \in \partial\Omega_h. \quad (58)$$

Then we have

$$v(x_i, t_j) - V_i^j + \Phi_i^j \geq 0, \quad (i, j) \in \Omega_h^{(2)} \quad (59)$$

by using the maximum principle (Lemma 2) to $\Omega_h^{(2)}$. Hence, we have

$$v(x_i, t_j) - V_i^j + \Phi_i^j \geq 0, \quad (i, j) \in \overline{\Omega}_h. \quad (60)$$

For $(i, j) \in \Omega^{(2)}$, $Lv(x_i, t_j) = 0$, but $L^{N,K}V_i^j \geq 0$; thus for the sufficiently large constant C we have

$$\begin{aligned}L^{N,K}(v(x_i, t_j) - V_i^j - \Phi_i^j) &= [L^{N,K}v(x_i, t_j) \\ &- q_i^-Lv(\xi_i, t_j) - q_i^+Lv(\eta_i, t_j) - L^{N,K}\Phi_i^j] \\ &+ q_i^-Lv(\xi_i, t_j) + q_i^+Lv(\eta_i, t_j) - L^{N,K}V_i^j \\ &\leq \{L^{N,K}v(x_i, t_j) - q_i^-Lv(\xi_i, t_j) - q_i^+Lv(\eta_i, t_j) \\ &- L^{N,K}\Phi_i^j \\ &+ [q_i^- \alpha_{1,i}(1 - \alpha_{1,i}) + q_i^+ \alpha_{2,i}(1 - \alpha_{2,i})] O(h^2)\} \\ &+ q_i^- [\alpha_{1,i}Lv(x_{i-1}, t_j) + (1 - \alpha_{1,i})Lv(x_i, t_j)] \\ &+ q_i^+ [\alpha_{2,i}Lv(x_i, t_j) + (1 - \alpha_{2,i})Lv(x_{i+1}, t_j)] \\ &- L^{N,K}V_i^j \leq \{L^{N,K}v(x_i, t_j) - q_i^-Lv(\xi_i, t_j) \\ &- q_i^+Lv(\eta_i, t_j) - L^{N,K}\Phi_i^j \\ &+ [q_i^- \alpha_{1,i}(1 - \alpha_{1,i}) + q_i^+ \alpha_{2,i}(1 - \alpha_{2,i})] O(h^2) \\ &+ (q_i^- + q_i^+) O(\Delta t)\} - L^{N,K}V_i^j \leq 0,\end{aligned}\quad (61)$$

where we have used $\alpha_{1,i} = 1 - \alpha_{2,i} = \Delta t$ and (49). On the “boundary” of the $\Omega^{(2)}$ we have

$$\begin{aligned}v(x_i, t_j) - V_i^j - \Phi_i^j &= g_i - V_i^j - \Phi_i^j \leq 0 \\ &\text{for } (i, j) \in \Omega^{(1)},\end{aligned}\quad (62)$$

and

$$v(x_i, t_j) - V_i^j - \Phi_i^j = -\Phi_i^j \leq 0 \quad \text{for } (i, j) \in \partial\Omega_h. \quad (63)$$

Then we have

$$v(x_i, t_j) - V_i^j - \Phi_i^j \leq 0, \quad (i, j) \in \Omega^{(2)} \quad (64)$$

by using the maximum principle (Lemma 2) to $\Omega^{(2)}$. Hence

$$v(x_i, t_j) - V_i^j - \Phi_i^j \leq 0, \quad (i, j) \in \overline{\Omega}_h. \quad (65)$$

Combining (60) with (65) we have

$$\max_{(i,j) \in \overline{\Omega}_h} |v(x_i, t_j) - V_i^j| \leq \max_{(i,j) \in \overline{\Omega}_h} \Phi_i^j \leq C(h^2 + \Delta t), \quad (66)$$

which completes the proof. $\square$

FIGURE 1: Computed option value V for Example 1. FIGURE 2: The constraint $V - g$ for Example 1.

5. Numerical Experiments

In this section we perform the numerical experiments to illustrate the applicability and accuracy of the method obtained in the preceding section. Errors and convergence rates for the spline difference scheme are presented for two examples.

Example 1. Compute the value of an American put option with $\sigma = 0.1, r = 0.08, d = 0.03, T = 1, E = 25$, and $X = 100$.

Example 2. Compute the value of an American put option with $\sigma = 0.5, r = 0.1, d = 0.02, T = 1, E = 25$, and $X = 100$.

The projection scheme [23] is used to solve the linear inequality system (37)-(41). Assume that V^{j+1} are known; then we first solve the following discretization problem for $\{V^{j+1/2}\}$:

$$\begin{aligned} L^{N,K} V_i^{j+1/2} &= 0, \quad i = 1, \dots, N-1, \\ V_0^{j+1/2} &= E, \\ V_N^{j+1/2} &= 0. \end{aligned} \quad (67)$$

Next we set

$$V_i^j = \max \{V_i^{j+1/2}, g_i\}, \quad i = 0, 1, \dots, N, \quad (68)$$

which ensure that (38) and (39) are satisfied.

The computed option values V and the constraints $V - g$ with $N = 128$ and $K = 64$ are plotted in Figures 1-4 for Examples 1 and 2, respectively.

Since the exact solutions of our examples are not available, the quadratic spline approximated solution of $N = 2048, K = 4096$ is used as the exact solution denoted by $\tilde{V}(x, t)$. The error estimates for different N at $t = 0$ are presented. The accuracy in the discrete maximum-norm,

$$e^{N,K} = \max_{i,j} |V_{i,0}^{N,K} - \tilde{V}(x_i, 0)|, \quad (69)$$

FIGURE 3: Computed option value V for Example 2.

TABLE 1: Error estimates and convergence rates for Example 1.

K	N	error	rate
1024	128	2.3532e-2	2.009
	256	5.8452e-3	1.914
	512	1.5506e-3	2.342
	1024	3.0592e-4	-

and the convergence rate

$$R^{N,K} = \log_2 \left(\frac{e^{N,K}}{e^{2N,K}} \right) \quad (70)$$

are computed. The error estimates and convergence rates for Examples 1 and 2 are listed in Tables 1 and 2, respectively.

From Figures 1 and 3 we can conclude that the numerical solutions are nonoscillatory, and from Figures 2 and 4 we also can conclude that the numerical solutions satisfy the constraint conditions (38) and (39). Tables 1 and 2 show that

FIGURE 4: The constraint $V - g$ for Example 2.

TABLE 2: Error estimates and convergence rates for Example 2.

K	N	error	rate
1024	128	8.6028e-3	2.031
	256	2.1045e-3	2.097
	512	4.9176e-4	2.451
	1024	8.9918e-5	-

$R^{N,K}$ is close to 2 for sufficiently large N . These results support the convergence estimate of Theorem 3.

Finally, our scheme is compared with the binomial method, analytic approximation method, and compact finite difference methods. The numerical example and numerical results in Zhao [15] are used. The parameters for the American put option are $\sigma = 0.3$, $T = 1$, $r = 0.04$, $d = 0.02$, and $E = 100$. The results for the binomial method are obtained with the time mesh size $\Delta t = 0.01$. The results for the analytical approximation method are obtained with the time mesh size $\Delta t = 0.02$. The results for the compact finite difference methods 1, 2, and 3 are obtained with the spatial mesh size $h = 0.02$ and the time mesh size $\Delta t = 0.0005$. For our spline collocation scheme we use the time mesh size $\Delta t = 0.0005$ and $N = 20000$ mesh points for the spatial discretization which has almost same number of mesh points as that of the compact methods. The true option values are obtained with the trinomial method using the time mesh size $\Delta t = 0.00005$. Table 3 shows that our scheme is more accurate than other methods.

As discussed above, the optimal stopping boundary can be obtained as the maximum asset price such that $V_i^j = g_i$ for each t_j . We plot the optimal stopping boundary for the two examples with $N = K = 2048$ in Figures 5 and 6, respectively.

6. Conclusion and Discussion

In this paper the linear complementarity problem arising from American put option pricing is treated by a quadratic spline collocation method on a piecewise uniform mesh. The main advantage of the spline collocation method examined

FIGURE 5: The optimal stopping boundary for Example 1.

FIGURE 6: The optimal stopping boundary for Example 2.

in the present paper lies in the possibility of simple and direct constructing of a continuous approximate solution and normalized flux between the mesh points. The suitable choice of collocation points guarantees that the spline difference operator on the piecewise uniform mesh satisfies the discrete maximum principle. Hence the scheme is maximum-norm stable. The error estimation is derived by applying the maximum principle to the discrete linear complementarity problem in two mesh sets. It is proved that the scheme is second-order convergent with respect to the spatial variable and first-order convergent with respect to the time variable. Numerical results demonstrate that the scheme is stable and accurate.

For the time discretization, although the Crank-Nicolson scheme can be used to improve the truncation error scheme, the discretization scheme does not satisfy the maximum principle without additional constraints about the time sizes, which leads to the difficulty in the derivation of the error estimation for the linear complementarity problem under the difference schemes. In future we plan to overcome this

TABLE 3: Comparison of the American put option values at $t = 0$ for some different values of x with parameters: $\sigma = 0.3, T = 1, r = 0.04, d = 0.02, E = 100, X = 400$.

Stock price x	Binomial method	Analytic approx. method	Compact method 1	Compact method 2	Compact method 3	Our stable scheme	True values
75.9572	25.33949	25.4509	25.10042	25.32570	25.32739	25.32892	25.32986
83.9457	error	0.00963	0.12104	0.00416	0.00247	0.00094	19.49691
		19.49101	19.34597	19.49193	19.49383	19.49565	
92.7743	error	0.00590	0.16479	0.00498	0.00308	0.00126	14.26265
		14.27957	14.16375	14.25707	14.25914	14.26121	
102.5315	error	0.01692	0.18505	0.00558	0.00351	0.00144	9.84354
		9.87092	9.78167	9.83789	9.84000	9.84216	
113.3148	error	0.02738	0.18426	0.00565	0.00354	0.00138	6.36558
		6.35580	6.3401	6.36044	6.36241	6.36458	
125.2323	error	0.00978	0.16843	0.00514	0.00317	0.00100	3.83337
		3.84473	3.97728	3.82898	3.83064	3.83268	
138.4031	error	0.01136	0.14391	0.00439	0.00273	0.00069	2.13784
		2.14801	2.25467	2.13452	2.13578	2.13757	
	error	0.01017	0.11683	0.00332	0.00206	0.00027	

difficulty in the error analysis for the Crank-Nicolson scheme in discretizing the linear complementarity problem.

Since the probability distribution of realized asset returns may exhibit features such as heavy tails, volatility clustering, and volatility smile, two different classes of models have been studied in the finance literature: the jump-diffusion models [24, 25] and the stochastic volatility models [26, 27]. In future we extend this technique to construct spline collocation schemes and study the stability and error analysis of these schemes for the jump-diffusion models and the stochastic volatility models.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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