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Research Article

Minimal Wave Speed in a Delayed Lattice Dynamical System with Competitive Nonlinearity

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This paper deals with the minimal wave speed of delayed lattice dynamical systems without monotonicity in the sense of standard partial ordering in $\mathbb{R}^2$. By constructing upper and lower solutions appealing to the exponential ordering, we prove the existence of traveling wave solutions if the wave speed is not smaller than some threshold. The nonexistence of traveling wave solutions is obtained when the wave speed is smaller than the threshold. Therefore, we confirm the threshold is the minimal wave speed, which completes the known results.

1. Introduction

Propagation thresholds including minimal wave speed of traveling wave solutions and asymptotic speed of spreading in population models have attracted much attention; see [1, 2] for their biological backgrounds. For monotone systems, some sharp results were established [3, 4]. If a noncooperative system admits comparison principle in the sense of standard partial ordering in $\mathbb{R}^n$, there are also some results on propagation thresholds, e.g., predator-prey systems [5–7] and competitive system [8, 9]. For some delayed models, the dynamics may be very plentiful [10, 11], and the propagation modes may be complex; see some by [12–17] and recent papers [18–20] and references cited therein.

In a very recent paper, Pan and Shi [21] studied the minimal wave speed of the following lattice dynamical systems with time delay

$$\begin{aligned}\frac{du_n(t)}{dt} &= [\mathscr{D}_1 u]_n(x) \\ &\quad + r_1 u_n(t)\left[1 - u_n(t) - b_1 v_n(t-\tau_1)\right], \\ \frac{dv_n(t)}{dt} &= [\mathscr{D}_2 v]_n(x) \\ &\quad + r_2 v_n(t)\left[1 - b_2 u_n(t-\tau_2) - v_n(t)\right]\end{aligned} \tag{1}$$

with

$$\begin{aligned}[\mathscr{D}_1 u]_n(x) &= d_1\left[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)\right], \\ [\mathscr{D}_2 v]_n(x) &= d_2\left[v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)\right],\end{aligned} \tag{2}$$

in which $n \in \mathbb{Z}$, $t > 0$, d_1, d_2, r_1, r_2 are positive constants and b_1, b_2, τ_1, τ_2 are nonnegative constants. By showing the existence or nonexistence of traveling wave solutions with small wave speed, they completed the conclusions in Lin and Li [22, Example 5.1] if $b_1, b_2 \in [0, 1)$.

Although (1) does not satisfy the quasimonotonicity [23, 24], it admits a comparison principle appealing to competitive systems in the sense of standard partial ordering in $\mathbb{R}^2$. In fact, Lin and Li [22, Example 5.7] also studied the following coupled system:

$$\begin{aligned}\frac{du_n(t)}{dt} &= [\mathscr{D}_1 u]_n(x) \\ &\quad + r_1 u_n(t)\left[1 - u_n(t-\tau_1) - b_1 v_n(t-\tau_2)\right], \\ \frac{dv_n(t)}{dt} &= [\mathscr{D}_2 v]_n(x) \\ &\quad + r_2 v_n(t)\left[1 - b_2 u_n(t-\tau_3) - v_n(t-\tau_4)\right],\end{aligned} \tag{3}$$

where τ_i, $i = 1, 2, 3, 4$, are nonnegative. Evidently, if $\tau_1 + \tau_4 > 0$, then the comparison principle in the sense of standard partial ordering in $\mathbb{R}^2$ does not work. Lin and Li [22] studied the existence of traveling wave solutions of (3) if the wave speed is larger than some threshold clarified later. The purpose of this paper is to confirm the existence or nonexistence of traveling wave solutions if the wave speed is positive when

$$b_1, b_2 \in [0, 1). \tag{4}$$

In this paper, *a traveling wave solution* of (3) is still a special solution taking the following form:

$$\begin{aligned} u_n(t) &= \phi(\xi), \\ v_n(t) &= \psi(\xi), \\ \xi &= n + ct, \end{aligned} \tag{5}$$

where $\phi, \psi \in C^1(\mathbb{R}, \mathbb{R})$ denote the wave profiles and $c > 0$ reflects the wave speed. Therefore, ϕ, ψ, and c must satisfy

$$\begin{aligned} c\phi'(\xi) &= [\mathscr{D}_1\phi](\xi) \\ &\quad + r_1\phi(\xi)[1 - \phi(\xi - c\tau_1) - b_1\psi(\xi - c\tau_2)], \\ c\psi'(\xi) &= [\mathscr{D}_2\psi](\xi) \\ &\quad + r_2\psi(\xi)[1 - b_2\phi(\xi - c\tau_3) - \psi(\xi - c\tau_4)] \end{aligned} \tag{6}$$

with

$$\begin{aligned} [\mathscr{D}_1\phi](\xi) &= d_1[\phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)], \\ [\mathscr{D}_2\psi](\xi) &= d_2[\psi(\xi + 1) + \psi(\xi - 1) - 2\psi(\xi)]. \end{aligned} \tag{7}$$

Because (4), (3) have a positive steady state $K = (k_1, k_2)$ defined by

$$\begin{aligned} k_1 &= \frac{1 - b_1}{1 - b_1 b_2} > 0, \\ k_2 &= \frac{1 - b_2}{1 - b_1 b_2} > 0. \end{aligned} \tag{8}$$

Similar to that in [21, 22], we want to study the existence or nonexistence of (6) satisfying

$$\begin{aligned} \lim_{\xi \to -\infty} (\phi(\xi), \psi(\xi)) &= 0, \\ \lim_{\xi \to +\infty} (\phi(\xi), \psi(\xi)) &= K, \end{aligned} \tag{9}$$

which may formulate the coinvasion-coexistence of two competitors.

In this paper, we establish the existence of traveling wave solutions by an abstract result in Lin and Li [22, Theorem 4.5] with the help of exponential ordering [25], which will be finished by constructing proper upper and lower solutions if the wave speed is large. Then the nonexistence of traveling wave solutions is proved by the theory of asymptotic spreading if the wave speed is small. It should be noted that our discussion includes all positive wave speed, so we obtain a minimal wave speed and complete the conclusion in [22].

2. Main Result

In this paper, we shall prove the following result.

Theorem 1. *Assume that τ_1, τ_4 are small enough. Then (6) and (9) have a strict positive solution satisfying*

$$0 < \phi(\xi), \psi(\xi) < 1, \quad \xi \in \mathbb{R} \tag{10}$$

if and only if $c \geq c^*$, where $c^* = \max\{c_1^*, c_2^*\}$ and

$$c_i^* = \inf_{\lambda > 0} \frac{d_i[e^{\lambda} + e^{-\lambda} - 2] + r_i}{\lambda} > 0, \quad i = 1, 2. \tag{11}$$

We now prove the above conclusion. And we use the standard partial ordering in $\mathbb{R}^2$. That is, if $u = (u_1, u_2) \in \mathbb{R}^2$ and $v = (v_1, v_2) \in \mathbb{R}^2$, then

$$u \leq v \quad \text{iff } u_i \leq v_i, \ i = 1, 2, \tag{12}$$

and

$$u < v \quad \text{iff } u \leq v \text{ but } u \neq v. \tag{13}$$

If $\lambda > 0, c > 0$, we define

$$\Lambda_i(\lambda, c) = d_i[e^{\lambda} + e^{-\lambda} - 2] - c\lambda + r_i, \quad i = 1, 2. \tag{14}$$

Lemma 2. *Assume that $\Lambda_i(\lambda, c), i = 1, 2$, are defined as the above.*

(1) *$c_i^* > 0$ holds.*

(2) *For $i \in \{1, 2\}$ with any fixed $c > c_i^*$, there exist $0 < \lambda_i^c < \lambda_{i+2}^c$ such that $\Lambda_i(\lambda_i^c, c) = 0$ and $\lambda_i \in (\lambda_i^c, \lambda_{i+2}^c)$ implies $\Lambda_i(\lambda_i, c) < 0$.*

(3) *For $i \in \{1, 2\}$ with any fixed $c \in (0, c_i^*)$, $\Lambda_i(\lambda, c) > 0, \lambda > 0$.*

(4) *For $i \in \{1, 2\}$ with $c = c_i^*$, $\Lambda_i(\lambda, c^*) \geq 0, \lambda > 0$ holds and $\Lambda_i(\lambda, c^*) = 0$ has a unique positive root λ_i.*

When the wave speed is small, we have the following conclusion.

Lemma 3. *If $c < c^*$, then (6) with (9) does not have a positive solution.*

Proof. The proof is similar to that in Pan and Shi [21]. We prove the result if $c^* = c_1^*$. Assume that for some fixed $c < c^*$, (6) has a positive solution $(\phi(\xi), \psi(\xi))$ satisfying (9) such that

$$\begin{aligned} \phi(\xi) &> 0, \\ \psi(\xi) &> 0, \\ \xi &\in \mathbb{R}. \end{aligned} \tag{15}$$

Select $\epsilon > 0$ such that

$$\inf_{\lambda > 0} \frac{d_1[e^{\lambda} + e^{-\lambda} - 2] + r_1(1 - 2\epsilon)}{\lambda} > c. \tag{16}$$

Let ξ' such that

$$\phi(\xi - c\tau_1) + b_1\psi(\xi - c\tau_2) < \epsilon, \quad \xi \le \xi', \tag{17}$$

where ξ' is admissible since

$$\lim_{\xi \longrightarrow -\infty} (\phi(\xi), \psi(\xi)) = (0,0). \tag{18}$$

Define

$$\Phi = \inf_{\xi > \xi'} \phi(\xi - c\tau_1), \tag{19}$$

and then $\Phi > 0$.

By what we have done, $\phi(\xi)$ satisfies

$$c\phi'(\xi) \ge [\mathscr{D}_1\phi](\xi) + r_1\phi(\xi)[1 - \epsilon - M\phi(\xi)] \tag{20}$$

with

$$M = \sup_{\xi > \xi'} \frac{\phi(\xi - c\tau_1) + b_1\psi(\xi - c\tau_2)}{\Phi} \le \frac{1 + b_1}{\Phi}. \tag{21}$$

By the definition of $\phi(\xi) = u_n(t)$, we see that

$$\begin{aligned} \frac{du_n(t)}{dt} &\ge [\mathscr{D}_1 u]_n(x) + r_1 u_n(t)[1 - \epsilon - Mu_n(t)], \\ u_n(0) &= \phi(n) > 0. \end{aligned} \tag{22}$$

According to the theory of asymptotic spreading, we have

$$\liminf_{t \longrightarrow \infty} \inf_{|n| \le c't} u_n(t) \ge \frac{1 - \epsilon}{M} \tag{23}$$

with

$$c' = \inf_{\lambda > 0} \frac{d_1\left[e^{\lambda} + e^{-\lambda} - 2\right] + r_1(1 - 3\epsilon/2)}{\lambda}. \tag{24}$$

For any $-n \in \mathbb{R}$, select $t > 0$ with $-2n = (c + c')t$, and then $n \longrightarrow -\infty$ implies

$$n + ct \longrightarrow -\infty \tag{25}$$

such that $\lim_{n \longrightarrow \infty} u_n(t) = \lim_{\xi \longrightarrow -\infty} \phi(\xi) = 0$, and a contradiction occurs between the above and (23). The proof is complete. □

Lin and Li [22, Theorem 4.5] proved the following abstract conclusion on the existence of (6) with (9).

Lemma 4. *Assume that $c > 0$ is fixed. Further suppose that continuous functions*

$$\left(\underline{\phi}(\xi), \underline{\psi}(\xi)\right), \left(\overline{\phi}(\xi), \overline{\psi}(\xi)\right) \in C\left(\mathbb{R}, \mathbb{R}^2\right) \tag{26}$$

such that

(1) $(0,0) \le (\underline{\phi}(\xi), \underline{\psi}(\xi)) \le (\overline{\phi}(\xi), \overline{\psi}(\xi)) \le (1,1), \xi \in \mathbb{R}$,

(2) $\lim_{\xi \longrightarrow -\infty}(\overline{\phi}(\xi), \overline{\psi}(\xi)) = (0,0)$,

(3) $\lim_{\xi \longrightarrow \infty}(\underline{\phi}(\xi), \underline{\psi}(\xi)) = \lim_{\xi \longrightarrow \infty}(\overline{\phi}(\xi), \overline{\psi}(\xi)) = (k_1, k_2)$,

(4) *there exists $\beta > 0$ such that*

$$\begin{aligned} &e^{\beta\xi}\left[\overline{\phi}(\xi) - \underline{\phi}(\xi)\right], \\ &e^{\beta\xi}\left[\overline{\psi}(\xi) - \underline{\psi}(\xi)\right] \end{aligned} \tag{27}$$

are nondecreasing for $\xi \in \mathbb{R}$.

Moreover, except for several points, they are differentiable such that

$$\begin{aligned} c\overline{\phi}'(\xi) \ge{} & \left[\mathscr{D}_1\overline{\phi}\right](\xi) \\ & + r_1\overline{\phi}(\xi)\left[1 - \overline{\phi}(\xi - c\tau_1) - b_1\underline{\psi}(\xi - c\tau_2)\right], \end{aligned} \tag{28}$$

$$\begin{aligned} c\overline{\psi}'(\xi) \ge{} & \left[\mathscr{D}_2\overline{\psi}\right](\xi) \\ & + r_2\overline{\psi}(\xi)\left[1 - b_2\underline{\phi}(\xi - c\tau_3) - \overline{\psi}(\xi - c\tau_4)\right], \end{aligned} \tag{29}$$

$$\begin{aligned} c\underline{\phi}'(\xi) \le{} & \left[\mathscr{D}_1\underline{\phi}\right](\xi) \\ & + r_1\underline{\phi}(\xi)\left[1 - \underline{\phi}(\xi - c\tau_1) - b_1\overline{\psi}(\xi - c\tau_2)\right], \end{aligned} \tag{30}$$

$$\begin{aligned} c\underline{\psi}'(\xi) \le{} & \left[\mathscr{D}_2\underline{\psi}\right](\xi) \\ & + r_2\underline{\psi}(\xi)\left[1 - b_2\overline{\phi}(\xi - c\tau_3) - \underline{\psi}(\xi - c\tau_4)\right]. \end{aligned} \tag{31}$$

If τ_1, τ_4 are small enough, then (6) with (9) has a positive solution $(\phi(\xi), \psi(\xi))$ such that

$$\left(\underline{\phi}(\xi), \underline{\psi}(\xi)\right) \le (\phi(\xi), \psi(\xi)) \le \left(\overline{\phi}(\xi), \overline{\psi}(\xi)\right), \quad \xi \in \mathbb{R}. \tag{32}$$

Remark 5. The continuous functions satisfying (28)-(31) are a pair of upper and lower solutions of (6). In Lin and Li [22], there are also some sufficient conditions on the bounds of τ_1, τ_4 and β. At least if τ_1, τ_4 are small, there exists a bounded $\beta > 0$, and we do not focus on precise conditions on τ_1, τ_4, β in this paper. Moreover, the existence of (6) and (9) has been confirmed if $c > c^*$; we refer to Lin and we refer to Lin and Li [22, Example 5.2]. Therefore, it suffices to consider $c = c^*$.

Lemma 6. *If $c = c^*$ and τ_1, τ_4 are small enough, then (6) with (9) has a positive solution.*

Proof. We now prove the conclusion by constructing upper and lower solutions. Firstly, we show the definition of potential upper and lower solutions and show the logistic sequence and the admissible of several parameters in upper and lower solutions; then we shall verify the necessary inequalities.

When $c = c_1^* > c_2^*$, by selecting parameters, we define positive continuous functions

$$\overline{\phi}(\xi) = \begin{cases} -L\xi e^{\lambda_1 \xi}, & \xi < \xi_1, \\ \min\left\{1, k_1 + k_1 e^{\lambda \xi}\right\}, & \xi \geq \xi_1, \end{cases}$$
$$\overline{\psi}(\xi) = \begin{cases} e^{\lambda_2^c \xi}, & \xi < \xi_2, \\ \min\left\{1, k_2 + k_2 e^{\lambda \xi}\right\}, & \xi \geq \xi_2, \end{cases} \tag{33}$$

and

$$\underline{\phi}(\xi) = \begin{cases} \left(-L\xi - q\sqrt{-\xi}\right) e^{\lambda_1 \xi}, & \xi < \xi_3 < 0, \\ \epsilon, & \xi \in [\xi_3, 0], \\ k_1 - (k_1 - \epsilon) e^{\lambda \xi}, & \xi > 0, \end{cases}$$
$$\underline{\psi}(\xi) = \begin{cases} e^{\lambda_2^c \xi} - p e^{\eta \lambda_2^c}, & \xi < \xi_4 < 0, \\ \epsilon, & \xi \in [\xi_4, 0], \\ k_2 - (k_2 - \epsilon) e^{\lambda \xi}, & \xi > 0, \end{cases} \tag{34}$$

where all the parameters are clarified later. We now show the key logistic sequence on the selection and verification of parameters and functions as follows.

(P1) Fix a constant $L > 1$ such that

$$\max_{\xi < 0} \left\{-L\xi e^{\lambda_1 \xi}\right\} > 3,$$
$$\xi_1'' - \xi_1' \geq 1, \tag{35}$$

where ξ_1'', ξ_1' with $\xi_1'' - \xi_1' > 1 + c$ are two roots of $-L\xi e^{\lambda_1^* \xi} = 3$.

By the selection of L, we see that $\xi < \xi_1'$ such that $\phi(\xi) = -L\xi e^{\lambda_1^* \xi}$ and

$$c\overline{\phi}'(\xi) = -c^* L\left[\lambda_1^* \xi + 1\right] e^{\lambda_1^* \xi} \tag{36}$$

and

$$\begin{aligned} & d_1\left[\overline{\phi}(\xi+1) + \overline{\phi}(\xi-1) - 2\overline{\phi}(\xi)\right] + r_1 \overline{\phi}(\xi)\left[1 \right. \\ & \left. - \overline{\phi}(\xi - c\tau_1) - b_1 \underline{\psi}(\xi - c\tau_2)\right] \leq d_1\left[\overline{\phi}(\xi+1) \right. \\ & \left. + \overline{\phi}(\xi-1) - 2\overline{\phi}(\xi)\right] + r_1 \overline{\phi}(\xi) \\ & \leq -Le^{\lambda_1 \xi}\left\{d_1\left[(\xi+1) e^{\lambda_1} + (\xi-1) e^{-\lambda_1} - 2\xi\right] \right. \\ & \left. + r_1 \xi\right\}. \end{aligned} \tag{37}$$

Thus (28) is true if

$$\begin{aligned} & -c\left[\lambda_1 \xi + 1\right] \\ & \geq -\left\{d_1\left[(\xi+1) e^{\lambda_1} + (\xi-1) e^{-\lambda_1} - 2\xi\right] + r_1 \xi\right\} \\ & = -\xi\left\{d_1\left[e^{\lambda_1} + e^{-\lambda_1} - 2\right] + r_1\right\} - d_1\left[e^{\lambda_1} - e^{-\lambda_1}\right], \end{aligned} \tag{38}$$

which is evident by Λ_1. This completes the verification of (28) when $\xi + c < \xi_1 < \xi_1'$.

(P2) Let $\lambda' \in (\lambda_1/2, \lambda_1)$ be a constant such that

$$\lambda' + \lambda_2^c > \lambda_1. \tag{39}$$

Denote $\underline{\lambda} = \min\{2\lambda' - \lambda_1, \lambda_2^c\}$. Fix

$$q > 2L + \sup_{\xi \leq -2} \frac{16 r_1 e^{\underline{\lambda} \xi}\left(-(\xi+1)\right)^{3/2}}{d_1\left[e^{\lambda_1} + e^{-\lambda_1}\right]} \tag{40}$$

such that $\xi < -q^2/L^2 := \overline{\xi}_3 < -2$ implies

$$\overline{\phi}(\xi - c\tau_1) < e^{\lambda' \xi},$$
$$\overline{\phi}(\xi) < e^{\lambda' \xi}, \tag{41}$$

which is admissible by simple limit analysis as $\xi \longrightarrow -\infty$.

We now verify (30) if $\xi < \overline{\xi}_3$. By the definition, we see that

$$\begin{aligned} & d_1\left[\underline{\phi}(\xi+1) + \underline{\phi}(\xi-1) - 2\underline{\phi}(\xi)\right] \\ & \geq d_1 e^{\lambda_1 \xi}\left[\left(-L(\xi+1) - q\sqrt{-(\xi+1)}\right) e^{\lambda_1} \right. \\ & + \left(-L(\xi-1) - q\sqrt{-(\xi-1)}\right) e^{-\lambda_1} \\ & \left. - 2\left(-L\xi - q\sqrt{-\xi}\right)\right] = -L d_1 e^{\lambda_1 \xi}\left[(\xi+1) e^{\lambda_1} \right. \\ & \left. + (\xi-1) e^{-\lambda_1} - 2\xi\right] - q d_1 e^{\lambda_1 \xi}\left[\sqrt{-(\xi+1)} e^{\lambda_1} \right. \\ & \left. + \sqrt{-(\xi-1)} e^{-\lambda_1} - 2\sqrt{-\xi}\right]. \end{aligned} \tag{42}$$

By the definition of λ', we see

$$\begin{aligned} & \underline{\phi}(\xi - c\tau_1) + b_1 \overline{\psi}(\xi - c\tau_2) \\ & \leq \overline{\phi}(\xi - c\tau_1) + b_1 \overline{\psi}(\xi - c\tau_2) \leq e^{\lambda' \xi} + e^{\lambda_2^c \xi} \end{aligned} \tag{43}$$

and so

$$\begin{aligned} & r_1 \underline{\phi}(\xi)\left[1 - \underline{\phi}(\xi) - b_1 \overline{\psi}(\xi - c\tau_1)\right] \\ & \geq e^{\lambda_1 \xi}\left[r_1\left(-L\xi - q\sqrt{-\xi}\right) - 2 r_1 e^{\underline{\lambda} \xi}\right]. \end{aligned} \tag{44}$$

Then it suffices to verify

$$\begin{aligned} & c\underline{\phi}'(\xi) = c e^{\lambda_1 \xi}\left[\lambda_1\left(-L\xi - q\sqrt{-\xi}\right) - L + \frac{q}{2\sqrt{-\xi}}\right] \\ & \leq -L d_1 e^{\lambda_1 \xi}\left[(\xi+1) e^{\lambda_1} + (\xi-1) e^{-\lambda_1} - 2\xi\right] \\ & - q d_1 e^{\lambda_1 \xi}\left[\sqrt{-(\xi+1)} e^{\lambda_1} + \sqrt{-(\xi-1)} e^{-\lambda_1} \right. \\ & \left. - 2\sqrt{-\xi}\right] + r_1\left(-L\xi - q\sqrt{-\xi}\right) e^{\lambda_1 \xi} - 2 r_1 e^{\lambda_1 \xi} e^{\underline{\lambda} \xi} \end{aligned} \tag{45}$$

or

$$\begin{aligned} & c\left[\lambda_1\left(-L\xi-q\sqrt{-\xi}\right)-L+\frac{q}{2\sqrt{-\xi}}\right] \\ & \quad\leq -Ld_1\left[(\xi+1)e^{\lambda_1}+(\xi-1)e^{-\lambda_1}-2\xi\right] \\ & \quad\quad -qd_1\left[\sqrt{-(\xi+1)}e^{\lambda_1}+\sqrt{-(\xi-1)}e^{-\lambda_1}-2\sqrt{-\xi}\right] \\ & \quad\quad +r_1\left(-L\xi-q\sqrt{-\xi}\right)-2r_1e^{\underline{\lambda}\xi}. \end{aligned} \tag{46}$$

By the properties of $\Lambda_1(\lambda, c)$, the above is true if

$$\begin{aligned} c\left[\frac{q}{2\sqrt{-\xi}}\right] & \leq -qd_1\left[\left(\sqrt{-(\xi+1)}-\sqrt{-\xi}\right)e^{\lambda_1}\right. \\ & \quad \left.+\left(\sqrt{-(\xi-1)}-\sqrt{-\xi}\right)e^{-\lambda_1}\right]-2r_1e^{\underline{\lambda}\xi} \end{aligned} \tag{47}$$

or

$$\begin{aligned} q & \left[\frac{c}{2\sqrt{-\xi}}+d_1\left[\left(\sqrt{-(\xi+1)}-\sqrt{-\xi}\right)e^{\lambda_1}\right.\right. \\ & \quad \left.\left.+\left(\sqrt{-(\xi-1)}-\sqrt{-\xi}\right)e^{-\lambda_1}\right]\right] \leq -2r_1e^{\underline{\lambda}\xi}. \end{aligned} \tag{48}$$

Since

$$\begin{aligned} & \frac{c}{2\sqrt{-\xi}}+d_1\left[\left(\sqrt{-(\xi+1)}-\sqrt{-\xi}\right)e^{\lambda_1}\right. \\ & \quad \left.+\left(\sqrt{-(\xi-1)}-\sqrt{-\xi}\right)e^{-\lambda_1}\right]=\frac{c}{2\sqrt{-\xi}} \\ & \quad +d_1\left[\frac{-1}{\sqrt{-(\xi+1)}+\sqrt{-\xi}}e^{\lambda_1}\right. \\ & \quad \left.+\frac{1}{\sqrt{-(\xi-1)}+\sqrt{-\xi}}e^{-\lambda_1}\right]=\frac{c}{2\sqrt{-\xi}} \\ & \quad +d_1\left[\frac{-1}{2\sqrt{-\xi}}e^{\lambda_1}+\frac{1}{2\sqrt{-\xi}}e^{-\lambda_1}\right] \\ & \quad +d_1\left[\frac{-1}{\sqrt{-(\xi+1)}+\sqrt{-\xi}}+\frac{1}{2\sqrt{-\xi}}\right]e^{\lambda_1} \\ & \quad +d_1\left[\frac{1}{\sqrt{-(\xi-1)}+\sqrt{-\xi}}-\frac{1}{2\sqrt{-\xi}}\right]e^{-\lambda_1} \\ & \quad =d_1\left[\frac{-1}{\sqrt{-(\xi+1)}+\sqrt{-\xi}}+\frac{1}{2\sqrt{-\xi}}\right]e^{\lambda_1} \\ & \quad +d_1\left[\frac{1}{\sqrt{-(\xi-1)}+\sqrt{-\xi}}-\frac{1}{2\sqrt{-\xi}}\right]e^{-\lambda_1} \\ & \quad =d_1\left[\frac{\sqrt{-(\xi+1)}-\sqrt{-\xi}}{2\sqrt{-\xi}\left[\sqrt{-(\xi+1)}+\sqrt{-\xi}\right]}\right]e^{\lambda_1} \\ & \quad +d_1\left[\frac{\sqrt{-\xi}-\sqrt{-(\xi-1)}}{2\sqrt{-\xi}\left[\sqrt{-(\xi-1)}+\sqrt{-\xi}\right]}\right]e^{-\lambda_1} \\ & \quad =\frac{-d_1e^{-\lambda_1}}{2\sqrt{-\xi}\left[\sqrt{-(\xi-1)}+\sqrt{-\xi}\right]^2} \\ & \quad -\frac{d_1e^{\lambda_1}}{2\sqrt{-\xi}\left[\sqrt{-(\xi+1)}+\sqrt{-\xi}\right]^2}\leq\frac{-d_1\left[e^{-\lambda_1}+e^{\lambda_1}\right]}{8\left(-(\xi+1)\right)^{3/2}}, \end{aligned} \tag{49}$$

(30) is true if

$$\frac{qd_1\left[e^{\lambda_1}+e^{-\lambda_1}\right]}{8\left(-(\xi+1)\right)^{3/2}}\geq 2r_1e^{\underline{\lambda}\xi}. \tag{50}$$

Note that $\xi_2<-2$, and then

$$q\geq\sup_{\xi\leq-2}\frac{16r_1e^{\underline{\lambda}\xi}\left(-(\xi+1)\right)^{3/2}}{d_1\left[e^{\lambda_1}+e^{-\lambda_1}\right]} \tag{51}$$

implies what we wanted.

(P3) Fix a constant $\eta>1$ such that

$$\lambda_2^c<\eta\lambda_2^c<\min\left\{\lambda_4^c,2\lambda_2^c,\frac{\lambda_1}{2}+\lambda_2^c\right\}. \tag{52}$$

Let

$$p>1+\frac{r_2+r_2b_2}{-\Lambda_2\left(c,\eta\lambda_2^c\right)} \tag{53}$$

such that $e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}>0$ implies

$$-L\xi e^{\lambda_1\xi}<e^{\lambda_1\xi/2}, \tag{54}$$

which is admissible by the limit as $\xi\longrightarrow-\infty$.

We now prove (31) if $\xi<0$ such that $e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}>0$. In fact, we have

$$\begin{aligned} & r_2\underline{\psi}(\xi)\left[1-b_2\overline{\phi}(\xi-c\tau_3)-\underline{\psi}(\xi-c\tau_4)\right] \\ & \quad =r_2\underline{\psi}(\xi)-r_2\underline{\psi}(\xi)\underline{\psi}(\xi)-r_2b_2\underline{\psi}(\xi)\overline{\phi}(\xi-c\tau_2) \\ & \quad \geq r_2\left(e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}\right)-r_2e^{2\lambda_2\xi}-r_2b_2e^{\lambda_1\xi/2+\lambda_2^c\xi}. \end{aligned} \tag{55}$$

Then it suffices to verify that

$$\begin{aligned} & c\underline{\psi}'(\xi)=c\left(\lambda_2^ce^{\lambda_2^c\xi}-p\eta\lambda_2e^{\eta\lambda_2^c\xi}\right)\leq d_2\left[\underline{\psi}(\xi+1)\right. \\ & \quad \left.+\underline{\psi}(\xi-1)-2\underline{\psi}(\xi)\right]+r_2\left(e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}\right)-r_2e^{2\lambda_2\xi} \\ & \quad -r_2b_2e^{\lambda_1\xi/2+\lambda_2^c\xi}\leq d_2\left\{\left[e^{\lambda_2^c(\xi+1)}-pe^{\eta\lambda_2^c(\xi+1)}\right]\right. \\ & \quad \left.+\left[e^{\lambda_2^c(\xi-1)}-pe^{\eta\lambda_2^c(\xi-1)}\right]-2\left(e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}\right)\right\} \\ & \quad +r_2\left(e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c\xi}\right)-r_2e^{2\lambda_2\xi}-r_2b_2e^{\lambda_1\xi/2+\lambda_2^c\xi}, \end{aligned} \tag{56}$$

which is equivalent to

$$-p\Lambda_2\left(c,\eta\lambda_2^c\right)e^{\eta\lambda_2^c\xi}\geq r_2e^{2\lambda_2\xi}+r_2b_2e^{\lambda_1\xi/2+\lambda_2^c\xi}. \tag{57}$$

Clearly, the above is true if

$$p>\frac{r_2+r_2b_2}{-\Lambda_2\left(c,\eta\lambda_2^c\right)}+1. \tag{58}$$

(P4) Let

$$\varepsilon=\sup_{\eta>0}\left\{\eta:k_1-b_1\left(k_2+\eta\right)>0,k_2-b_2\left(k_1+\eta\right)>0\right\}. \tag{59}$$

For any given $\epsilon>0$, define

$$\begin{aligned}A&=\left\{\xi<0:\left(-L\xi-q\sqrt{-\xi}\right)e^{\lambda_1\xi}>\epsilon\right\},\\B&=\left\{\xi<0:e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c}>\epsilon\right\}.\end{aligned} \tag{60}$$

Then there exists $\epsilon\in(0,\varepsilon)$ such that

$$\begin{aligned}|A|&>2,\\|B|&>2.\end{aligned} \tag{61}$$

(P5) Fix $\lambda>0$ small enough, which will be clarified later.

With the above parameters, we now illustrate the other parameters in upper and lower solutions.

(X1) Let ξ_1 be the smallest root of

$$-L\xi e^{\lambda_1\xi}=\min\left\{1,k_1+k_1e^{\lambda\xi}\right\}. \tag{62}$$

(X2) Let ξ_2 be the smallest root of

$$e^{\lambda_2^c\xi}=\min\left\{1,k_2+k_2e^{\lambda\xi}\right\}. \tag{63}$$

(X3) Let ξ_3 be the larger root of

$$\left(-L\xi-q\sqrt{-\xi}\right)e^{\lambda_1\xi}=\epsilon. \tag{64}$$

(X4) Let ξ_4 be the larger root of

$$e^{\lambda_2^c\xi}-pe^{\eta\lambda_2^c}=\epsilon. \tag{65}$$

We now complete the verification of (28)-(31). On (28), since $\xi_1<\xi_1'$, the verification of $\xi<\xi_1$ has been given by (P1). If $\overline{\phi}=1$ is differentiable, the conclusion is also true if $\tau_1>0$ is small. If $\overline{\phi}(\xi-c\tau_1)=k_1+k_1e^{-\lambda(\xi-c\tau_1)}$, then $\underline{\psi}(\xi-c\tau_2)\geq k_2-(k_2-\epsilon)e^{-\lambda(\xi-c\tau_2)}$ such that

$$\begin{aligned}c\overline{\phi}'(\xi)&=-c\lambda k_1e^{-\lambda\xi},\\\left[\mathscr{D}_1\overline{\phi}\right](\xi)&\geq d_1e^{-\lambda\xi}\left[e^{-\lambda}+e^{-\lambda}-2\right]\end{aligned} \tag{66}$$

and

$$\begin{aligned}H\left(\overline{\phi},\underline{\psi}\right)(\xi)&=r_1\overline{\phi}(\xi)\left[1-\overline{\phi}(\xi-c\tau_1)\right.\\&\quad\left.-b_1\underline{\psi}(\xi-c\tau_2)\right]\leq r_1\left(k_1+k_1e^{-\lambda\xi}\right)\left[1\right.\\&\quad-\left(k_1+k_1e^{-\lambda(\xi-c\tau_1)}\right)\\&\quad\left.-b_1\left(k_2-(k_2-\epsilon)e^{-\lambda(\xi-c\tau_2)}\right)\right]=r_1e^{-\lambda\xi}\left(k_1\right.\\&\quad\left.+k_1e^{-\lambda\xi}\right)\left[-k_1e^{\lambda c\tau_1}+b_1(k_2-\epsilon)e^{\lambda c\tau_2}\right].\end{aligned} \tag{67}$$

Let $\lambda>0$ be small such that $-k_1e^{\lambda c\tau_1}+b_1(k_2-\epsilon)e^{\lambda c\tau_2}<0$, and then (28) is true if

$$\begin{aligned}-c\lambda k_1&\geq d_1\left[e^{-\lambda}+e^{-\lambda}-2\right]\\&\quad+r_1k_1\left[-k_1e^{\lambda c\tau_1}+b_1(k_2-\epsilon)e^{\lambda c\tau_2}\right].\end{aligned} \tag{68}$$

Clearly, (68) holds if $\lambda=0$, and then it is true if $\lambda>0$ is small. In particular, we may fix $\gamma_1>0$ such that

$$\begin{aligned}-c\lambda k_1&\geq d_1\left[e^{-\lambda}+e^{-\lambda}-2\right]\\&\quad+\frac{r_1k_1\left[-k_1e^{\lambda c\tau_1}+b_1(k_2-\epsilon)e^{\lambda c\tau_2}\right]}{2}\end{aligned} \tag{69}$$

for all $\lambda\in(0,\gamma_1)$.

If $\overline{\phi}(\xi)=k_1+k_1e^{-\lambda\xi}$ but $\overline{\phi}(\xi-c\tau_1)\neq k_1+k_1e^{-\lambda(\xi-c\tau_1)}$, we see that ξ varies in a bounded interval, of which the length is $c\tau_1$. Then it suffices to verify

$$-c\lambda k_1\geq d_1\left[e^{-\lambda}+e^{-\lambda}-2\right]+e^{\lambda\xi}H\left(\overline{\phi},\underline{\psi}\right)(\xi), \tag{70}$$

which holds by the continuity if τ_1 is small. We complete the proof of (28).

Similarly, we can confirm that (29)-(31) are true if τ_1,τ_4 are small and $\lambda>0$ is small. The proof is complete if $c=c_1^*>c_2^*$.

If $c=c_2^*>c_1^*$, the proof can be finished similar to that for $c=c_1^*>c_2^*$. When $c=c_1^*=c_2^*$, define

$$\begin{aligned}\overline{\phi}(\xi)&=\begin{cases}-L\xi e^{\lambda_1\xi}, & \xi<\xi_1,\\\min\left\{1,k_1+k_1e^{\lambda\xi}\right\}, & \xi\geq\xi_1,\end{cases}\\\overline{\psi}(\xi)&=\begin{cases}-L\xi e^{\lambda_2\xi}, & \xi<\xi_2,\\\min\left\{1,k_2+k_2e^{\lambda\xi}\right\}, & \xi\geq\xi_2,\end{cases}\end{aligned} \tag{71}$$

and

$$\begin{aligned}\underline{\phi}(\xi)&=\begin{cases}\left(-L\xi-q\sqrt{-\xi}\right)e^{\lambda_1\xi}, & \xi<\xi_3<0,\\\epsilon, & \xi\in[\xi_3,0],\\k_1-(k_1-\epsilon)e^{\lambda\xi}, & \xi>0,\end{cases}\\\underline{\psi}(\xi)&=\begin{cases}\left(-L\xi-q\sqrt{-\xi}\right)e^{\lambda_2\xi}, & \xi<\xi_4<0,\\\epsilon, & \xi\in[\xi_4,0],\\k_2-(k_2-\epsilon)e^{\lambda\xi}, & \xi>0,\end{cases}\end{aligned} \tag{72}$$

in which all the parameters are similar to that in the case of $c_1^* > c_2^*$. By combining the above analysis with Pan and Shi [21], the upper and lower solutions can be verified. We complete the proof. □

3. Conclusion and Discussion

To obtain the existence and asymptotic behavior of traveling wave solutions of nonmonotone systems is not easy; at least the oscillation of traveling wave solutions has been observed in some delayed equations without quasimonotonicity [26]. In Li et al. [27], the authors constructed upper and lower solutions to confirm the existence and asymptotic behavior of traveling wave solutions of delayed competitive systems. Furthermore, Lin and Li [22] also gave some similar conclusions for lattice dynamical systems without quasimonotonicity.

However, to construct upper and lower solutions in [22, 27], several parameters are needed. What is more, the logistic sequence of selection of these parameters is very important. Although Lin and Li [22] showed the existence of desirable upper and lower solutions, the logistic sequence was not given. For large wave speed, Lin et al. [28] presented the sequence of a competitive reaction-diffusion system with time delay. In this paper, we show the sequence of parameters for minimal wave speed of model (3), and the sequence can be utilized to illustrate that in [21].

Moreover, for the size of time delay, it should be further investigated to show the existence of traveling wave solutions satisfying desirable asymptotic behavior. On this topic, very likely the recipe in Kwong and Ou [15] is a good motivation. In the future, we shall try to fix the bounds of time delays such that (6) and (9) have a strict positive solution.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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