

## Research Article

# Pricing of European Currency Options with Uncertain Exchange Rate and Stochastic Interest Rates

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Suppose that the interest rates obey stochastic differential equations, while the exchange rate follows an uncertain differential equation; this paper proposes a new currency model. Under the proposed currency model, the pricing formula of European currency options is then derived. Some numerical examples recorded illustrate the quality of pricing formulas. Meanwhile, this paper analyzes the relationship between the pricing formula and some parameters.

## 1. Introduction

Nowadays, the currency option is one of the best investment tools for companies and individuals to hedge against adverse movements in exchange rates. It can be divided into European currency option, American currency option, Asian currency option, and so forth, where European currency option is a contract giving the owner the right to buy or sell one unit of foreign currency with a specified price at a maturity date [1]. The theoretical models of currency option pricing have been a hot issue in mathematical finance, and the key point of this topic is how to get an appropriate pricing formula.

For European currency option, Garman and Kohlhagen [2] first proposed G-K model, where both domestic and foreign interest rates are assumed to be constant and the exchange rate is governed by a geometric Brownian motion. However, it is unrealistic for the exchange rate to obey the geometric Brownian motion in the subsequent literatures. By modifying the G-K model, more and more methodologies for the currency option pricing have been proposed, such as Bollen and Rasieland [3], Carr and Wu [4], Sun [5], Swishchuk et al. [6], Xiao et al. [7], and Wang, Zhou, and Yang [8].

In the above-mentioned literature, the exchange rate follows a stochastic differential equation under the framework of probability theory. When we use probability theory, the

available probability distribution needs to be close to the true frequency. However, some emergencies coming from wars, political policies, or natural disasters may affect the exchange rate. In this case, it is difficult to obtain available statistical data about exchange rate, and the assumption that the exchange rate follows a stochastic differential equation may be out of work. At this time, belief degrees given by some domain experts are used to estimate values or distributions. To model the belief degree, uncertainty theory was established by Liu [9]. In addition, to describe the evolution of an uncertain phenomenon, uncertain process [10] and a Liu process [11] were subsequently proposed. To further express uncertain dynamic systems, uncertain differential equation was proposed [10] and has been widely applied to control and financial market.

Back to the foreign exchange market, assume that the exchange rate follows an uncertain differential equation; Liu, Chen, and Ralescu [1] proposed Liu-Chen-Ralescu currency model. In addition, Shen and Yao [12] proposed a mean-reverting currency model under uncertain environment. Recently, Ji and Wu [13] provided an uncertain currency model with jumps. Besides that, Sheng and Shi [14] proposed the mean-reverting currency model under Asian currency option. In these currency models listed above, the exchange rate is governed by an uncertain process instead of stochastic process, and the interest rates are taken as constant.

TABLE 1: Some notations and parameters.

$r_t$ : domestic interest rate at $t$	$K$ : strike price
$f_t$ : foreign interest rate at $t$	$T$ : maturity date
$Z_t$ : spot domestic currency price of a unit of foreign exchange at $t$	$C_t$ : a Liu process
$\mu$ : log-drift of the spot currency price $Z_t$	$W_t$ : a Wiener process
$\sigma$ : log-diffusion of the spot currency price $Z_t$	$\mathcal{M}$ : uncertain measure
$C_E$ : European call currency option price	$E[\xi]$ : expected value of $\xi$
$P_E$ : European put currency option price	$\text{var}[\xi]$ : variance of $\xi$
$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$	$N(\mu, \sigma^2)$ : normal distribution

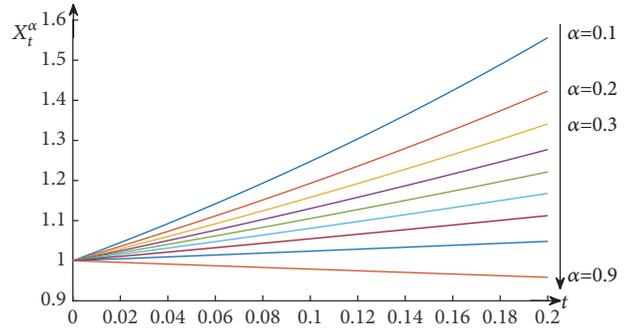
However, considering the fluctuation of interest rate market from time to time, it is unreasonable to regard the interest rates as constant. Up to now, interest rate is mainly studied under the framework of uncertainty theory or probability theory. When the sample size of interest rate is too small (even no-sample) to estimate a probability distribution, interest rate is usually described by an uncertain process under the framework of uncertainty theory. Chen and Gao [15] started from an assumption that the short interest rate follows uncertain process and proposed three equilibrium models. Sun, Yao, and Fu [16] proposed another interest rate model on the basis of exponential Ornstein-Uhlenbeck equation under the uncertain environment. Suppose that both domestic and foreign interest rates follow uncertain differential equations, Wang and Ning [17] proposed an uncertain currency model, where the exchange rate also follows an uncertain differential equation.

While there is a large amount of historical data about interest rate, the short interest rate is usually described by a stochastic process. Under the framework of probability theory, Morton [18] proposed the first interest rate model and Ho and Lee [19] then extended it. In addition, many other economists have built other models, such as Hull and White [20] and Vasicek [21]. Taking into account two factors, randomness and uncertainty, we propose a new currency model in this paper. In detail, both domestic and foreign interest rates follow stochastic differential equations, while the exchange rate follows an uncertain differential equation.

The paper is organized as follows. In Section 2, we mainly introduce uncertain differential equation, Liu-Chen-Ralescu model, Vasicek model, and moment generating function. In Section 3, we propose a new currency model with uncertain exchange rate and stochastic. The pricing formula of European currency option under the proposed model is derived in Section 4. Some numerical examples are carried out in Section 5. Finally, Section 6 makes a brief conclusion. For convenience, some notations and parameters employed in the later sections are shown in Table 1.

## 2. Preliminaries

In this section, we first introduce uncertain differential equation. Then, an uncertain currency model and Vasicek model

FIGURE 1: A spectrum of  $\alpha$ -paths of  $dX_t = X_t dt + X_t dC_t$ .

are recalled. Finally, we introduce the moment generating function.

### 2.1. Uncertain Differential Equation

*Definition 1* (see Liu [10]). Suppose that  $C_t$  is a Liu process, and  $f$  and  $g$  are two functions. Given an initial value  $X_0$ ,

$$dX_t = f(t, X_t) dt + g(t, X_t) dC_t \quad (1)$$

is called an *uncertain differential equation* with an initial value  $X_0$ .

*Definition 2* (see Yao and Chen [22]). Let  $\alpha$  be a number with  $0 < \alpha < 1$ . Uncertain differential equation (1) is said to have an  $\alpha$ -path  $X_t^\alpha$  if it solves the corresponding ordinary differential equation

$$dX_t^\alpha = f(t, X_t^\alpha) dt + |g(t, X_t^\alpha)| \Phi^{-1}(\alpha) dt \quad (2)$$

where  $\Phi^{-1}(\alpha) = (\sqrt{3}/\pi) \ln(\alpha/(1-\alpha))$ .

*Example 3.* Uncertain differential equation  $dX_t = X_t dt + X_t dC_t$  with  $X_0 = 1$  has an  $\alpha$ -path

$$X_t^\alpha = \exp(t + \Phi^{-1}(\alpha)t) \quad (3)$$

and its spectrum is shown in Figure 1.

**Theorem 4** (see Liu [23]). *Let  $\eta$  and  $\tau$  be an uncertain variable and a random variable, respectively. Then*

$$E[\eta\tau] = E[\eta] E[\tau]. \quad (4)$$

**Theorem 5** (see Yao and Chen [22]). Let  $X_t$  and  $X_t^\alpha$  be the solution and  $\alpha$ -path of uncertain differential equation (1), respectively. Then, for any monotone function  $J$ , we have

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha) d\alpha. \quad (5)$$

**2.2. Uncertain Currency Model with Fixed Interest Rates.** Assume that the exchange rate  $Z_t$  follows an uncertain differential equation and the domestic and foreign interest rates are constant; Liu, Chen, and Ralescu [1] proposed Liu-Chen-Ralescu model:

$$\begin{aligned} dX_t &= rX_t dt \\ dY_t &= fY_t dt \\ dZ_t &= \mu Z_t dt + \sigma Z_t dC_t \end{aligned} \quad (6)$$

where  $X_t$  represents the riskless domestic currency with the fixed domestic interest rate  $r$  and  $Y_t$  represents the riskless foreign currency with the fixed foreign interest rate  $f$ . The meaning of remaining parameters in model (6) can be seen in Table 1.

**2.3. Vasicek Model.** Vasicek model [21] is a classical stochastic interest rate model and is defined by a stochastic differential equation of the form

$$dr_t = a(b - r_t) dt + \sigma dW_t \quad (7)$$

describing the interest rate process  $r_t$ , where  $\sigma$  determines the volatility of the interest rate,  $a$  represents the rate of adjustment, and  $b$  is the long run average value.

By solving stochastic differential equation (7), we have

$$\begin{aligned} r_t &= r_0 \exp(-at) + b(1 - \exp(-at)) \\ &+ \sigma \exp(-at) \int_0^t \exp(as) dW_s, \end{aligned} \quad (8)$$

and

$$\begin{aligned} r_t &\sim N\left(r_0 \exp(-at) \right. \\ &\left. + b(1 - \exp(-at)), \frac{\sigma^2}{2a} (1 - \exp(-2at))\right). \end{aligned} \quad (9)$$

**2.4. Moment Generating Function**

**Definition 6** (see Shaked and Shanthikumar [24]). The moment generating function of a random variable  $\xi$  is  $M_\xi[t] = E[\exp(t\xi)]$ ,  $t \in \mathfrak{R}$  wherever this expectation exists.

**Remark 7.** If  $\xi \sim N(\mu, \sigma^2)$ , then  $E[\exp(t\xi)] = \exp(\mu t + (1/2)\sigma^2 t^2)$ ,  $t \in \mathfrak{R}$ .

### 3. Model Establishment

In this part, we generalize the Liu-Chen-Ralescu model through the use of stochastic interest rates. We employ

Vasicek model for the domestic and foreign interest rates. Besides, assume that the exchange rate follows an uncertain differential equation; a new currency model is then proposed as follows:

$$\begin{aligned} dr_t &= a_1(b_1 - r_t) dt + \sigma_1 dW_{1t} \\ df_t &= a_2(b_2 - f_t) dt + \sigma_2 dW_{2t} \\ dZ_t &= \mu Z_t dt + \sigma Z_t dC_t \end{aligned} \quad (10)$$

where  $\sigma_1$  is the diffusion of  $r_t$ ;  $\sigma_2$  is the diffusion of  $f_t$ ;  $a_1, a_2, b_1$ , and  $b_2$  are the constant parameters;  $W_{1t}$  and  $W_{2t}$  are independent Wiener processes and  $W_{it}$  and  $C_t$  are independent for  $i = 1, 2$ .

By using formula (9), we have

$$\begin{aligned} r_t &\sim N\left(r_0 \exp(-a_1 t) \right. \\ &\left. + b_1(1 - \exp(-a_1 t)), \frac{\sigma_1^2}{2a_1} (1 - \exp(-2a_1 t))\right) \end{aligned} \quad (11)$$

and

$$\begin{aligned} f_t &\sim N\left(f_0 \exp(-a_2 t) \right. \\ &\left. + b_2(1 - \exp(-a_2 t)), \frac{\sigma_2^2}{2a_2} (1 - \exp(-2a_2 t))\right). \end{aligned} \quad (12)$$

By Definition 2, we obtain that the  $\alpha$ -path of the exchange rate  $Z_t$  is

$$Z_t^\alpha = Z_0 \cdot \exp(\mu t + \sigma t \Phi^{-1}(\alpha)). \quad (13)$$

**Remark 8.** When uncertain differential equation has no analytic solution, the solution can be calculated by some numerical methods. Interested readers can refer to [22, 25, 26].

### 4. European Currency Option Pricing

In this section, we study the European currency option pricing problem and provide the pricing formula of European currency option under model (10). For your convenience, the current time is set to 0.

**4.1. Pricing Formula of European Call Currency Option.** In [1], we can see that European call currency option is a contract endowing the holder the right to buy one unit of foreign currency at a maturity date  $T$  for  $K$  units of domestic currency, where  $K$  is commonly called a strike price.

Let  $C_E$  represent this contract price in domestic currency. The investor needs to pay  $C_E$  to buy this contract at time 0. The payoff of the investor is  $(Z_T - K)^+$  in domestic currency at the maturity date  $T$ . So the expected profit of the investor at time 0 is

$$-C_E + E\left[(Z_T - K)^+ \exp\left(-\int_0^T r_s ds\right)\right]. \quad (14)$$

Due to selling the contract, the bank can receive  $C_E$  at time 0. At the maturity date  $T$ , the bank also pays  $(1 - K/Z_T)^+$  in foreign currency. The expected profit of the bank at time 0 is

$$C_E - Z_0 E \left[ \left( 1 - \frac{K}{Z_T} \right)^+ \exp \left( - \int_0^T f_s ds \right) \right]. \quad (15)$$

To ensure the fairness of the contract, we have

$$\begin{aligned} -C_E + E \left[ (Z_T - K)^+ \exp \left( - \int_0^T r_s ds \right) \right] \\ = C_E - Z_0 E \left[ \left( 1 - \frac{K}{Z_T} \right)^+ \exp \left( - \int_0^T f_s ds \right) \right]. \end{aligned} \quad (16)$$

In this way, the investor and the bank have an identical expected profit.

*Definition 9.* Under model (10), given a strike price  $K$  and a maturity date  $T$ , the European currency call option price  $C_E$  is

$$\begin{aligned} C_E = \frac{1}{2} E \left[ (Z_T - K)^+ \exp \left( - \int_0^T r_s ds \right) \right] \\ + \frac{Z_0}{2} E \left[ \left( 1 - \frac{K}{Z_T} \right)^+ \exp \left( - \int_0^T f_s ds \right) \right]. \end{aligned} \quad (17)$$

**Theorem 10.** Under model (10), given a strike price  $K$  and a maturity date  $T$ , the European currency call option price is

$$\begin{aligned} C_E = \frac{1}{2} \int_0^1 (Z_T^\alpha - K)^+ d\alpha \exp \left( -b_1 T \right. \\ \left. - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1} + \frac{\sigma_1^2}{2a_1^2} \left( T \right. \right. \\ \left. \left. - \frac{2(1 - \exp(-a_1 T))}{a_1} + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \right) + \frac{Z_0}{2} \\ \cdot \int_0^1 \left( 1 - \frac{K}{Z_T^\alpha} \right)^+ d\alpha \exp \left( -b_2 T \right. \\ \left. - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2} + \frac{\sigma_2^2}{2a_2^2} \left( T \right. \right. \\ \left. \left. - \frac{2(1 - \exp(-a_2 T))}{a_2} + \frac{1 - \exp(-2a_2 T)}{2a_2} \right) \right) \end{aligned} \quad (18)$$

where

$$(Z_T^\alpha - K)^+ = (Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) - K)^+ \quad (19)$$

and

$$\left( 1 - \frac{K}{Z_T^\alpha} \right)^+ = \left( 1 - \frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} \right)^+. \quad (20)$$

*Proof.* By Theorem 4, we have

$$\begin{aligned} E \left[ (Z_T - K)^+ \exp \left( - \int_0^T r_s ds \right) \right] \\ = E \left[ (Z_T - K)^+ \right] E \left[ \exp \left( - \int_0^T r_s ds \right) \right]. \end{aligned} \quad (21)$$

By Theorem 5 and formula (13), we have

$$E \left[ (Z_T - K)^+ \right] = \int_0^1 (Z_T^\alpha - K)^+ d\alpha, \quad (22)$$

where

$$(Z_T^\alpha - K)^+ = (Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) - K)^+. \quad (23)$$

According to formula (8), we get

$$\begin{aligned} \int_0^T r_t dt &= \int_0^T r_0 \exp(-a_1 t) dt \\ &+ b_1 \int_0^T (1 - \exp(-a_1 t)) dt \\ &+ \int_0^T \left( \sigma_1 \exp(-a_1 t) \int_0^t \exp(a_1 s) dW_s \right) dt \\ &= b_1 T + (1 - \exp(-a_1 T)) \frac{r_0 - b_1}{a_1} \\ &+ \sigma_1 \int_0^T \int_0^t \exp(a(s-t)) dW_s dt. \end{aligned} \quad (24)$$

Since the expectation of the stochastic integral is zero, the expected value of  $\int_0^T r_t dt$  is provided by

$$E \left[ \int_0^T r_t dt \right] = b_1 T + \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1}. \quad (25)$$

Since the variance is determined from the diffusion term, namely, the stochastic integral, the two deterministic drift terms in  $\int_0^T r_t dt$  make no contribution to the variance. This gives

$$\begin{aligned} \text{var} \left[ \int_0^T r_t dt \right] \\ = \text{var} \left[ \sigma_1 \int_0^T \int_0^t \exp(a_1(s-t)) dW_s dt \right]. \end{aligned} \quad (26)$$

Since  $\text{var}[\xi] = E[\xi^2] - (E[\xi])^2$ , we have

$$\begin{aligned} \text{var} \left[ \int_0^T r_t dt \right] \\ = E \left[ \left( \sigma_1 \int_0^T \int_0^t \exp(a_1(s-t)) dW_s dt \right)^2 \right] \\ - \left( E \left[ \sigma_1 \int_0^T \int_0^t \exp(a_1(s-t)) dW_s dt \right] \right)^2. \end{aligned} \quad (27)$$

Observe that  $E[\sigma_1 \int_0^T \int_0^t \exp(a_1(s-t))dW_s dt] = 0$ , and this implies that

$$\begin{aligned} \text{var} \left[ \int_0^T r_t dt \right] \\ = E \left[ \left( \sigma_1 \int_0^T \int_0^t \exp(a_1(s-t)) dW_s dt \right)^2 \right]. \end{aligned} \quad (28)$$

By use of Fubini theorem as well as that for the stochastic integral, we have

$$\begin{aligned} E \left[ \left( \sigma_1 \int_0^T \int_0^t \exp(a_1(s-t)) dW_s dt \right)^2 \right] \\ = E \left[ \left( \sigma_1 \int_0^T \int_s^T \exp(a_1(s-t)) dt dW_s \right)^2 \right] \\ = E \left[ \sigma_1^2 \int_0^T \left( \int_s^T \exp(a_1(s-t)) dt \right)^2 ds \right]. \end{aligned} \quad (29)$$

Since  $(dW_s)^2 = ds$ , the square term eliminates the source of randomness, and

$$\begin{aligned} E \left[ \sigma_1^2 \int_0^T \left( \int_s^T \exp(a_1(s-t)) dt \right)^2 ds \right] \\ = \sigma_1^2 \int_0^T \left( \int_s^T \exp(a_1(s-t)) dt \right)^2 ds. \end{aligned} \quad (30)$$

Continuing with simple integration leads to

$$\begin{aligned} \text{var} \left[ \int_0^T r_t dt \right] dt = \sigma_1^2 \int_0^T \left( \frac{1 - \exp(a_1(s-T))}{a_1} \right)^2 ds \\ = \frac{\sigma_1^2}{a_1^2} \left( T - \frac{2(1 - \exp(-a_1 T))}{a_1} \right. \\ \left. + \frac{1 - \exp(-2a_1 T)}{2a_1} \right), \end{aligned} \quad (31)$$

and

$$\begin{aligned} - \int_0^T r_t dt \sim N \left( -b_1 T - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1}, \right. \\ \left. \frac{\sigma_1^2}{a_1^2} \left( T - \frac{2(1 - \exp(-a_1 T))}{a_1} \right. \right. \\ \left. \left. + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \right). \end{aligned} \quad (32)$$

According to Remark 7, we have

$$\begin{aligned} E \left[ \exp \left( - \int_0^T r_t dt \right) \right] = \exp \left( -b_1 T \right. \\ \left. - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1} + \frac{\sigma_1^2}{2a_1^2} \left( T \right. \right. \\ \left. \left. - \frac{2(1 - \exp(-a_1 T))}{a_1} + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \right). \end{aligned} \quad (33)$$

Thus

$$\begin{aligned} E \left[ (Z_T - K)^+ \exp \left( - \int_0^T r_s ds \right) \right] = \int_0^1 (Z_T^\alpha - K)^+ d\alpha \\ \cdot \exp \left( -b_1 T - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1} \right. \\ \left. + \frac{\sigma_1^2}{2a_1^2} \left( T - \frac{2(1 - \exp(-a_1 T))}{a_1} \right. \right. \\ \left. \left. + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \right). \end{aligned} \quad (34)$$

By the same way, we get

$$\begin{aligned} E \left[ \exp \left( - \int_0^T f_s ds \right) \right] = \exp \left( -b_2 T \right. \\ \left. - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2} + \frac{\sigma_2^2}{2a_2^2} \left( T \right. \right. \\ \left. \left. - \frac{2(1 - \exp(-a_2 T))}{a_2} + \frac{1 - \exp(-2a_2 T)}{2a_2} \right) \right). \end{aligned} \quad (35)$$

By Theorem 5 and formula (13), we have

$$E \left[ \left( 1 - \frac{K}{Z_T} \right)^+ \right] = \int_0^1 \left( 1 - \frac{K}{Z_T^\alpha} \right)^+ d\alpha, \quad (36)$$

where

$$\left( 1 - \frac{K}{Z_T^\alpha} \right)^+ = \left( 1 - \frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} \right)^+. \quad (37)$$

Hence,

$$\begin{aligned} E \left[ \left( 1 - \frac{K}{Z_T} \right)^+ \exp \left( - \int_0^T f_s ds \right) \right] = \int_0^1 \left( 1 - \frac{K}{Z_T^\alpha} \right)^+ \\ d\alpha \exp \left( -b_2 T \right. \\ \left. - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2} + \frac{\sigma_2^2}{2a_2^2} \left( T \right. \right. \\ \left. \left. - \frac{2(1 - \exp(-a_2 T))}{a_2} + \frac{1 - \exp(-2a_2 T)}{2a_2} \right) \right). \end{aligned} \quad (38)$$

According to Definition 9 and formulas (34) and (38), the pricing formula of European call currency option is immediately obtained.  $\square$

Herein we discuss the properties of this pricing formula, and the result is shown by the following theorem.

**Theorem 11.** Let  $C_E$  be the European call currency option price under model (10). Then

- (1)  $C_E$  is a decreasing function of  $b_1, b_2, r_0, f_0$ , and  $K$ .
- (2)  $C_E$  is an increasing function of  $\mu$  and  $Z_0$ .

*Proof.* Theorem 10 tells us that  $C_E$  can be expressed as

$$\begin{aligned}
C_E = & \frac{1}{2} \int_0^1 \left( Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) - K \right)^+ d\alpha \\
& \cdot \exp \left( -b_1 T - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1} \right. \\
& + \frac{\sigma_1^2}{2a_1^2} \left( T - \frac{2(1 - \exp(-a_1 T))}{a_1} \right. \\
& \left. \left. + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \right) + \frac{Z_0}{2} \int_0^1 \left( 1 \right. \\
& \left. - \frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} \right)^+ d\alpha \\
& \cdot \exp \left( -b_2 T - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2} \right. \\
& + \frac{\sigma_2^2}{2a_2^2} \left( T - \frac{2(1 - \exp(-a_2 T))}{a_2} \right. \\
& \left. \left. + \frac{1 - \exp(-2a_2 T)}{2a_2} \right) \right). \tag{39}
\end{aligned}$$

(1) Since  $a_1 T + \exp(-a_1 T) - 1 > 0$  ( $a_1, T > 0$ ),

$$\begin{aligned}
& \exp \left( -b_1 T - \frac{b_1}{a_1} (\exp(-a_1 T) - 1) \right) \\
& = \exp \left( -\frac{b_1}{a_1} (a_1 T + \exp(-a_1 T) - 1) \right) \tag{40}
\end{aligned}$$

is decreasing with  $b_1$  and  $C_E$  is decreasing with  $b_1$ .

Since  $a_2 T + \exp(-a_2 T) - 1 > 0$  ( $a_2, T > 0$ ),

$$\begin{aligned}
& \exp \left( -b_2 T - \frac{b_2}{a_2} (\exp(-a_2 T) - 1) \right) \\
& = \exp \left( -\frac{b_2}{a_2} (a_2 T + \exp(-a_2 T) - 1) \right) \tag{41}
\end{aligned}$$

is decreasing with  $b_2$  and  $C_E$  is decreasing with  $b_2$ .

Since  $\exp(-r_0)$  is decreasing with  $r_0$ ,  $C_E$  is decreasing with  $r_0$ .

Since  $\exp(-f_0)$  is decreasing with  $f_0$ ,  $C_E$  is decreasing with  $f_0$ .

Since

$$Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) - K \tag{42}$$

and

$$Z_0 - \frac{K}{\exp(\mu T + \sigma T \Phi^{-1}(\alpha))} \tag{43}$$

are decreasing with  $K$ ,  $C_E$  is decreasing with  $K$ .

(2) Since

$$Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) - K \tag{44}$$

and

$$Z_0 - \frac{K}{\exp(\mu T + \sigma T \Phi^{-1}(\alpha))} \quad (K, Z_0, T > 0) \tag{45}$$

are increasing with  $\mu$  and  $Z_0$ ,  $C_E$  is increasing with  $\mu$  and  $Z_0$ .  $\square$

**4.2. Pricing Formula of European Put Currency Option.** Similarity to European call currency option, the definition, formula and property of European put currency option pricing with a strike price  $K$ , and a maturity date  $T$  are handy to get and are shown as below.

**Definition 12.** Under model (10), given a strike price  $K$  and a maturity date  $T$ , the European put currency option price  $P_E$  is

$$\begin{aligned}
P_E = & \frac{1}{2} E \left[ (K - Z_T)^+ \exp \left( - \int_0^T r_s ds \right) \right] \\
& + \frac{Z_0}{2} E \left[ \left( \frac{K}{Z_T} - 1 \right)^+ \exp \left( - \int_0^T f_s ds \right) \right]. \tag{46}
\end{aligned}$$

**Theorem 13.** Under model (10), given a strike price  $K$  and a maturity date  $T$ , the European put currency option price is

$$\begin{aligned}
P_E = & \frac{1}{2} \int_0^1 (K - Z_T^\alpha)^+ d\alpha \exp \left( -b_1 T \right. \\
& - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1} + \frac{\sigma_1^2}{2a_1^2} \left( T \right. \\
& \left. - \frac{2(1 - \exp(-a_1 T))}{a_1} + \frac{1 - \exp(-2a_1 T)}{2a_1} \right) \Bigg) \\
& + \frac{Z_0}{2} \int_0^1 \left( \frac{K}{Z_T^\alpha} - 1 \right)^+ d\alpha \exp \left( -b_2 T \right. \\
& - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2} + \frac{\sigma_2^2}{2a_2^2} \left( T \right. \\
& \left. - \frac{2(1 - \exp(-a_2 T))}{a_2} + \frac{1 - \exp(-2a_2 T)}{2a_2} \right) \Bigg) \tag{47}
\end{aligned}$$

where

$$(K - Z_T^\alpha)^+ = (K - Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)))^+ \tag{48}$$

and

$$\left(\frac{K}{Z_T^\alpha} - 1\right)^+ = \left(\frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} - 1\right)^+. \quad (49)$$

*Proof.* By Theorem 5 and formula (13), we have

$$E[(K - Z_T)^+] = \int_0^1 (K - Z_T^\alpha)^+ d\alpha \quad (50)$$

and

$$E\left[\left(\frac{K}{Z_T} - 1\right)^+\right] = \int_0^1 \left(\frac{K}{Z_T^\alpha} - 1\right)^+ d\alpha, \quad (51)$$

where

$$(K - Z_T^\alpha)^+ = (K - Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)))^+ \quad (52)$$

and

$$\left(\frac{K}{Z_T^\alpha} - 1\right)^+ = \left(\frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} - 1\right)^+. \quad (53)$$

By Theorem 4, we have

$$\begin{aligned} E\left[(K - Z_T)^+ \exp\left(-\int_0^T r_s ds\right)\right] \\ = E[(K - Z_T)^+] E\left[\exp\left(-\int_0^T r_s ds\right)\right] \end{aligned} \quad (54)$$

and

$$\begin{aligned} E\left[\left(\frac{K}{Z_T} - 1\right)^+ \exp\left(-\int_0^T f_s ds\right)\right] \\ = E\left[\left(\frac{K}{Z_T} - 1\right)^+\right] E\left[\exp\left(-\int_0^T f_s ds\right)\right]. \end{aligned} \quad (55)$$

From Definition 12 and formulas (33), (35), (54), and (55), the pricing formula of the European put currency option is immediately obtained.  $\square$

**Theorem 14.** Let  $P_E$  be the European put currency option price under model (10). Then

- (1)  $P_E$  is a decreasing function of  $b_1$ ,  $b_2$ ,  $\mu$ ,  $r_0$ ,  $f_0$ , and  $Z_0$ ,
- (2)  $P_E$  is an increasing function of  $K$ .

*Proof.* By Theorem 13,  $P_E$  can be written as

$$\begin{aligned} P_E = & \frac{1}{2} \int_0^1 (K - Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)))^+ d\alpha \\ & \cdot \exp\left(-b_1 T - \frac{(1 - \exp(-a_1 T))(r_0 - b_1)}{a_1}\right) \\ & + \frac{\sigma_1^2}{2a_1^2} \left(T - \frac{2(1 - \exp(-a_1 T))}{a_1}\right. \\ & \left. + \frac{1 - \exp(-2a_1 T)}{2a_1}\right) + \frac{Z_0}{2} \\ & \cdot \int_0^1 \left(\frac{K}{Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha))} - 1\right)^+ d\alpha \\ & \cdot \exp\left(-b_2 T - \frac{(1 - \exp(-a_2 T))(f_0 - b_2)}{a_2}\right) \\ & + \frac{\sigma_2^2}{2a_2^2} \left(T - \frac{2(1 - \exp(-a_2 T))}{a_2}\right. \\ & \left. + \frac{1 - \exp(-2a_2 T)}{2a_2}\right). \end{aligned} \quad (56)$$

(1) Since  $a_1 T + \exp(-a_1 T) - 1 > 0$  ( $a_1, T > 0$ ),

$$\begin{aligned} & \exp\left(-b_1 T - \frac{b_1}{a_1}(\exp(-a_1 T) - 1)\right) \\ & = \exp\left(-\frac{b_1}{a_1}(a_1 T + \exp(-a_1 T) - 1)\right) \end{aligned} \quad (57)$$

is decreasing with  $b_1$  and  $P_E$  is decreasing with  $b_1$ .

Since  $a_2 T + \exp(-a_2 T) - 1 > 0$  ( $a_2, T > 0$ ),

$$\begin{aligned} & \exp\left(-b_2 T - \frac{b_2}{a_2}(\exp(-a_2 T) - 1)\right) \\ & = \exp\left(-\frac{b_2}{a_2}(a_2 T + \exp(-a_2 T) - 1)\right) \end{aligned} \quad (58)$$

is decreasing with  $b_2$  and  $P_E$  is decreasing with  $b_2$ .

Since

$$K - Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) \quad (59)$$

and

$$\frac{K}{\exp(\mu T + \sigma T \Phi^{-1}(\alpha))} - Z_0 \quad (K, Z_0, T > 0) \quad (60)$$

are decreasing with  $\mu$  and  $Z_0$ ,  $P_E$  is decreasing with  $\mu$  and  $Z_0$ .

Since  $\exp(-r_0)$  is decreasing with  $r_0$ ,  $P_E$  is decreasing with  $r_0$ .

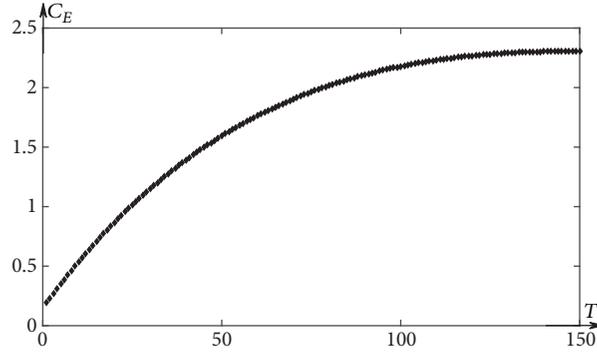
Since  $\exp(-f_0)$  is decreasing with  $f_0$ ,  $P_E$  is decreasing with  $f_0$ .

(2) Since

$$K - Z_0 \exp(\mu T + \sigma T \Phi^{-1}(\alpha)) \quad (61)$$

TABLE 2: Parameters setting in model (10).

$a_1$	0.1	$\mu$	0.0065
$a_2$	0.15	$\sigma$	0.0001
$b_1(\%)$	1.1	$r_0(\%)$	1.2
$b_2(\%)$	0.35	$f_0(\%)$	0.32
$\sigma_1$	0.0002	$Z_0$	6.65
$\sigma_2$	0.0003	$K$	6.5

FIGURE 2: European call currency option price  $C_E$  under model (10).

and

$$\frac{K}{\exp(\mu T + \sigma T \Phi^{-1}(\alpha))} - Z_0 \quad (62)$$

are increasing with  $K$ ,  $P_E$  is decreasing with  $K$ .  $\square$

## 5. Numerical Examples

In this section, some numerical examples are included to illustrate the pricing formulas under model (10). In addition, we mine the influence of other parameters ( $T, a_1, a_2, \sigma_1, \sigma_2, \sigma$ ) on the pricing formulas. For the sake of simplicity, this part just considers the case of European call currency option.

Table 2 presents the required parameters. Under model (10), we calculate the European call currency option prices and depict them in Figure 2, where  $T$  is the maturity date.

According to common sense, the European call currency option price is increasing with the maturity date. From Figure 2, we can see that the price computed by model (10) suits this principle. If the maturity date  $T$  is taken as 100, the European call currency option price is  $C_E = 2.1782$ .

In what follows, we discuss the influence of the parameters ( $a_1, a_2, \sigma_1, \sigma_2, \sigma$ ) on  $C_E$  and show the results by a series of experiments. Consider the first case, where the parameters  $a_1, a_2, T$  are fixed and the other parameters ( $\sigma_1, \sigma_2, \sigma$ ) are changing. The second case is considering  $C_E$  under different parameters  $a_1$  and  $a_2$ , where the other parameters are not changing. Figures 3 and 4 show  $C_E$  versus its parameters  $\sigma_1, \sigma_2, \sigma, a_1$ , and  $a_2$ . The default parameters can be referred to Table 2 and the maturity date  $T = 100$ .

Figure 3 shows that the European call currency option price under model (10) is increasing with  $\sigma_1, \sigma_2$ , and  $\sigma$ . From Figure 4, we can obtain the following conclusion: the European call currency option price is increasing with  $a_1$  while it is decreasing with respect to  $a_2$ .

## 6. Conclusion

This paper proposed a new currency model, in which the exchange rate follows an uncertain differential equation, while the domestic and foreign interest rates follow stochastic differential equations. Subsequently, we provided the pricing formulas of European currency option under this currency model. Meanwhile, we find that the European call currency option price under the proposed model is increasing with the initial exchange rate, the log-drift, the diffusion, the log-diffusion, the parameter  $a_1$ , and the maturity date, while it is decreasing with the initial domestic interest rate, the initial foreign interest rate, the strike price, and the parameters  $a_2, b_1, b_2$ . At last, some numerical examples were included to illustrate that the pricing formulas under the proposed currency model are reasonable.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest related to this work.

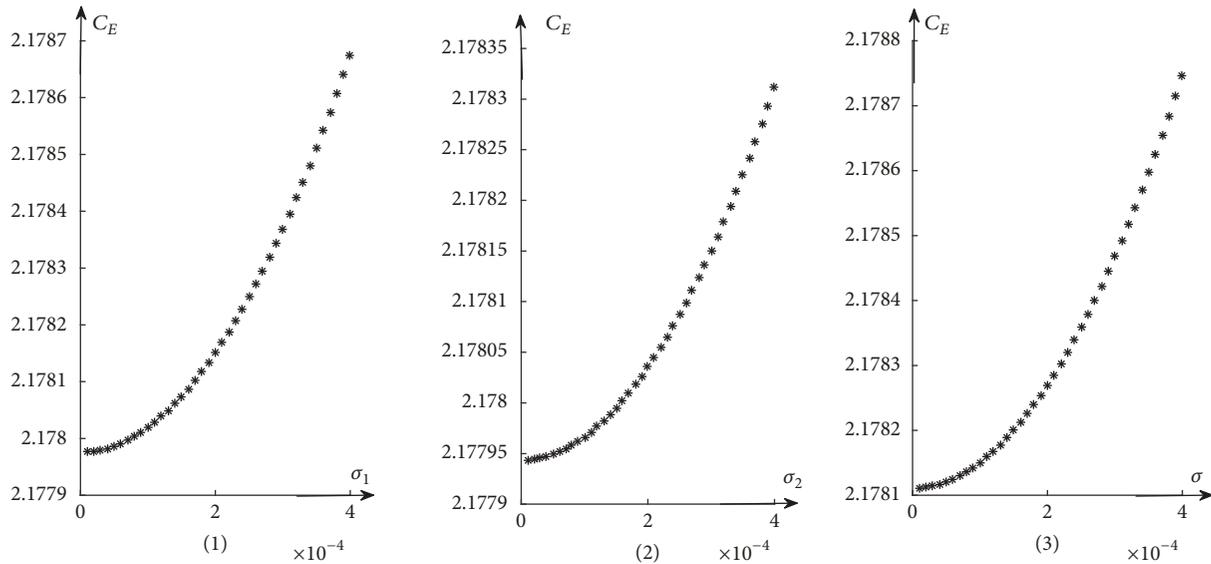


FIGURE 3: European call currency option price  $C_E$  versus parameters  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma$ , where (1)  $C_E$  versus  $\sigma_1$ ; (2)  $C_E$  versus  $\sigma_2$ ; (3)  $C_E$  versus  $\sigma$ .

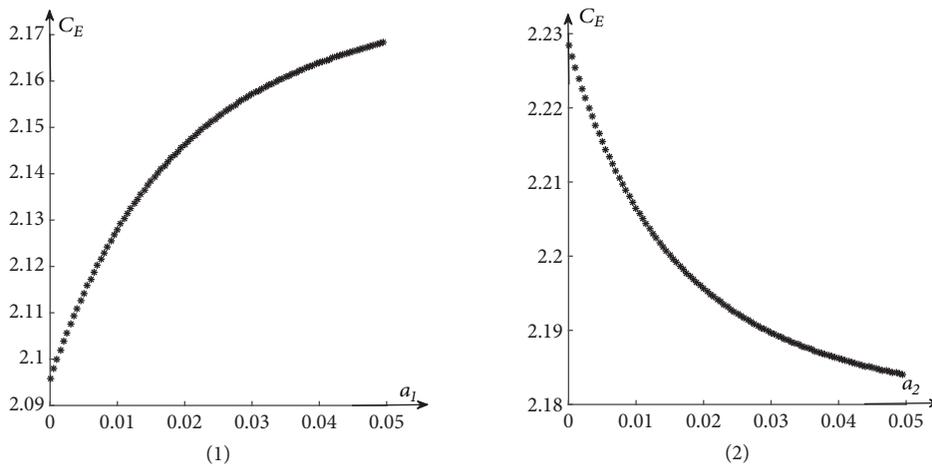


FIGURE 4: European call currency option price  $C_E$  versus parameters  $a_1$  and  $a_2$ , where (1)  $C_E$  versus  $a_1$ ; (2)  $C_E$  versus  $a_2$ .

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