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Research Article

The Existence Result for a Class of p-Kirchhoff-Type Problem with a Multilinear Growth Nonlinearity

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In this paper, we firstly discuss the existence of the least energy sign-changing solutions for a class of p-Kirchhoff-type problems with a $(2p-1)$-linear growth nonlinearity. The quantitative deformation lemma and Non-Nehari manifold method are used in the paper to prove the main results. Remarkably, we use a new method to verify that $\mathscr{M}_b \neq \emptyset$. The main results of our paper are the existence of the least energy sign-changing solution and its corresponding energy doubling property. Moreover, we also give the convergence property of the least energy sign-changing solution as the parameter $b \searrow 0$.

1. Introduction and the Main Results

In this paper, we are devoted to investigating the existence of the least energy sign-changing solutions for the following p-Kirchhoff-type problem with a (2p-1)-linear growth nonlinearity:

$$-\left(a+b\int_{\Omega}|\nabla u|^{p}\,dx\right)\Delta_{p}u=\lambda|u|^{p-2}u+|u|^{2p-2}u,\quad x\in\Omega,$$
$$u=0,\quad x\in\partial\Omega, \tag{1}$$

where Ω is a bounded domain in $R^{N}(N=1,2,3)$, $a,b>0$, $\lambda<a\lambda_1$, λ_1 is the first eigenvalue of the following problem:

$$-\Delta_{p}\phi=\lambda|\phi|^{p-2}\phi,\quad x\in\Omega,$$
$$\phi=0,\quad x\in\partial\Omega. \tag{2}$$

In fact, the related problems have been studied extensively, especially on the existence of the positive solutions, multiple solutions, ground state solutions, and least energy sign-changing solutions. In [1], Li and Sun studied the existence and multiplicity of solutions for the Kirchhoff equations with asymptotically linear nonlinearities; the mountain pass theorem was used in the paper. Guo, Ma, and Zhang [2] studied a class of autonomous Kirchhoff-type equation. By a simple transformation, they found that the solutions of autonomous Kirchhoff-type equation or system could be obtained by using the known solutions of the corresponding local equation or system, which is very interesting. In [3], Ying Li and Lin Li considered the existence and multiplicity of solutions to a class of p(x)-Laplacian-like equations. They introduced a revised Ambrosetti-Rabinowitz condition and obtained that the problem had a nontrivial solution and infinitely many solutions, respectively. Meanwhile, in [4], Luca Vilasi proved an eigenvalue theorem for a stationary p(x)-Kirchhoff problem by using variational techniques, and the author also provided an estimate for the range of such eigenvalues. For more details, we refer the reader to [5–30].

In [31, 32], the authors studied the following Kirchhoff-type problems in bounded domains:

$$-\left(a+b\int_{\Omega}|\nabla u|^{2}\right)\Delta u=f(u),\quad x\in\Omega,$$
$$u=0,\quad x\in\partial\Omega, \tag{3}$$

under different assumptions on $f(u)$, the authors mainly use the quantitative deformation lemma and the degree theory to get the existence of the least energy sign-changing solution and its corresponding convergence property as the parameter $b \searrow 0$. From the assumptions on $f(u)$, we can easily find that both in [31, 32] $f(u)$ satisfies 3-superlinear growth at infinity and superlinear growth at zero.

Later, some scholars made some expanding work; we can find some details in [33]. In [33], we know that the nonlinearity f satisfies $(2p-1)$-superlinear growth condition at infinity.

Motivated by the above works, a natural question is that if there exists a ground state sign-changing solution for problem (1). However, up to now, no paper has appeared in the literature which discusses the existence and convergence property of the solution for the p-Kirchhoff-type problem with a $(2p-1)$-linear growth nonlinearity. This paper attempts to fill this gap in the literature.

Throughout this paper, we will make full use of the following notations. Let $W = W_0^{1,p}(\Omega)$ be the usual Sobolev space equipped with the following norm:

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}. \tag{4}$$

$\|\cdot\|_s$ denotes the usual Lebesgue space $L^s(\Omega)$ norm. S is the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ in $L^{2p}(\Omega)$; that is,

$$\|u\|_{2p} \le S^{-1/p}\|u\|. \tag{5}$$

From the above definition, we give the energy functional corresponding to problem (1) by

$$\begin{aligned} I_b(u) = {} & \frac{a}{p}\int_\Omega |\nabla u|^p dx + \frac{b}{2p}\left(\int_\Omega |\nabla u|^p dx\right)^2 \\ & - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{1}{2p}\int_\Omega |u|^{2p} dx. \end{aligned} \tag{6}$$

Clearly, I_b is well defined on W and is of C^1 class. For each $u, v \in W$, by a simple calculation, we have

$$\begin{aligned} & \left\langle I_b'(u), v\right\rangle \\ & = \left(a + b\int_\Omega |\nabla u|^p dx\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx \\ & \quad - \int_\Omega \lambda |u|^{p-2} uv dx - \int_\Omega |u|^{2p-2} uv dx. \end{aligned} \tag{7}$$

Obviously, the critical points of I_b are corresponding to the weak solutions of problem (1). If $u \in W$ is a sign-changing solution of problem (1), then

(i) u is a solution of problem (1), that is, u is a critical point of I_b;

(ii) $u^\pm \ne 0$, where $u^+ = \max\{u(x), 0\}$, $u^- = \min\{u(x), 0\}$.

For $u = u^+ + u^-$, from (6) and (7), we have

$$I_b(u) = I_b(u^+) + I_b(u^-) + \frac{b}{p}\|\nabla u^+\|_p^p \|\nabla u^-\|_p^p; \tag{8}$$

$$\left\langle I_b'(u), u^+\right\rangle = \left\langle I_b'(u^+), u^+\right\rangle + b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p; \tag{9}$$

$$\left\langle I_b'(u), u^-\right\rangle = \left\langle I_b'(u^-), u^-\right\rangle + b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p. \tag{10}$$

When $b = 0$, problem (1) reduces to the following problem:

$$\begin{aligned} & -a\Delta_p u = \lambda |u|^{p-2} u + |u|^{2p-2} u, \quad x \in \Omega, \\ & u = 0, \quad x \in \partial\Omega. \end{aligned} \tag{11}$$

The corresponding energy functional $I_0 : W \longrightarrow R$ is defined by

$$\begin{aligned} I_0(u) = {} & \frac{a}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx \\ & - \frac{1}{2p}\int_\Omega |u|^{2p} dx. \end{aligned} \tag{12}$$

Also, we can compute that

$$\begin{aligned} \left\langle I_0'(u), v\right\rangle = {} & \int_\Omega a|\nabla u|^{p-2}\nabla u\cdot\nabla v dx \\ & - \int_\Omega \lambda |u|^{p-2} uv dx - \int_\Omega |u|^{2p-2} uv dx. \end{aligned} \tag{13}$$

For $b > 0$, problem (1) is called a nonlocal problem since the appearance of the nonlocal term $(\int_\Omega |\nabla u|^p dx)\Delta_p u$. The differences posed by the nonlocal term make the method in solving problem (11) cannot be applied to solve problem (1), which makes the study of our paper very interesting and meaningful.

In our paper, we restrict u in the following sets to find the ground state sign-changing solutions of (1) and (11),

$$\begin{aligned} \mathcal{M}_b = {} & \left\{u \in W : u^\pm \ne 0, \left\langle I_b'(u), u^+\right\rangle = \left\langle I_b'(u), u^-\right\rangle \right. \\ & \left. = 0\right\}, \\ \mathcal{M}_0 = {} & \left\{u \in W : u^\pm \ne 0, \left\langle I_0'(u), u^+\right\rangle = \left\langle I_0'(u), u^-\right\rangle \right. \\ & \left. = 0\right\}, \end{aligned} \tag{14}$$

and we define $m_b = \inf_{u\in\mathcal{M}_b} I_b(u)$ and $m_0 = \inf_{u\in\mathcal{M}_0} I_0(u)$.

To get the ground state solutions, we define the following sets:

$$\begin{aligned} \mathcal{N}_b &= \left\{u \in W : u \ne 0, \left\langle I_b'(u), u\right\rangle = 0\right\}, \\ \mathcal{N}_0 &= \left\{u \in W : u \ne 0, \left\langle I_0'(u), u\right\rangle = 0\right\} \end{aligned} \tag{15}$$

and consider the following minimization problem:

$$\begin{aligned} c_b &= \inf_{u\in\mathcal{N}_b} I_b(u), \\ c_0 &= \inf_{u\in\mathcal{N}_0} I_0(u). \end{aligned} \tag{16}$$

Since $\mathcal{M}_b \subset \mathcal{N}_b$, we can immediately get $m_b \ge c_b$. The main results of the paper are described as follows.

Theorem 1. *For $b \in (0, 1/2S^2)$ and $\lambda < a\lambda_1$, problem (1) has at least one ground state sign-changing solution, which precisely has two nodal domains. Moreover, $m_b > 2c_b$.*

Theorem 2. *For each $\lambda < a\lambda_1$, for any sequence $\{b_n\}$ small enough with $b_n \searrow 0$ as $n \longrightarrow \infty$, there exists a subsequence still denoted by $\{b_n\}$, such that u_{b_n} convergent to u_0 strongly in $W_0^{1,p}(\Omega)$, where u_0 is a ground state sign-changing solution of problem (11), which changes sign only once.*

Our paper is organized as follows. In Section 2, some preliminary lemmas are given to prove the main results. In Sections 3 and 4, we are devoted to proving the main results of the paper.

2. Some Critical Preliminaries

The following several lemmas are crucial to prove our main results.

Lemma 3. *If $b > 0$, $\lambda < a\lambda_1$, $u \in W$ satisfies $u^{\pm} \neq 0$ and*

$$b\|\nabla u^+\|_p^{2p} + b\|\nabla u^+\|_p^p \|\nabla u^-\|_p^p < \int_\Omega |u^+|^{2p}\,dx,$$
$$b\|\nabla u^-\|_p^{2p} + b\|\nabla u^+\|_p^p \|\nabla u^-\|_p^p < \int_\Omega |u^-|^{2p}\,dx, \tag{17}$$

then there exists a unique pair (s_u, t_u) of positive numbers such that

(i) $s_u u^+ + t_u u^- \in \mathscr{M}_b$;

(ii) $I_b(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} I_b(su^+ + tu^-)$.

Proof. (i) If $su^+ + tu^- \in \mathscr{M}_b$, then from (7), (9), and (10), we have

$$\begin{aligned}\left\langle I_b'(su^+ + tu^-), su^+\right\rangle &= as^p\|\nabla u^+\|_p^p + bs^{2p}\|\nabla u^+\|_p^{2p} \\ &\quad + bs^p t^p \|\nabla u^+\|_p^p \|\nabla u^-\|_p^p - s^p\int_\Omega \lambda |u^+|^p\,dx \\ &\quad - s^{2p}\int_\Omega |u^+|^{2p}\,dx = 0\end{aligned} \tag{18}$$

and

$$\begin{aligned}\left\langle I_b'(su^+ + tu^-), tu^-\right\rangle &= at^p\|\nabla u^-\|_p^p + bt^{2p}\|\nabla u^-\|_p^{2p} \\ &\quad + bs^p t^p \|\nabla u^+\|_p^p \|\nabla u^-\|_p^p - t^p\int_\Omega \lambda |u^-|^p\,dx \\ &\quad - t^{2p}\int_\Omega |u^-|^{2p}\,dx = 0.\end{aligned} \tag{19}$$

Let $S = s^p$ and $T = t^p$, the above equations correspond to the following system:

$$\begin{aligned}S\int_\Omega |u^+|^{2p}\,dx - bS\|\nabla u^+\|_p^{2p} - bT\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p &= a\|\nabla u^+\|_p^p - \int_\Omega \lambda|u^+|^p\,dx, \\ T\int_\Omega |u^-|^{2p}\,dx - bT\|\nabla u^-\|_p^{2p} - bS\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p &= a\|\nabla u^-\|_p^p - \int_\Omega \lambda|u^-|^p\,dx.\end{aligned} \tag{20}$$

Obviously, if we can prove that system (20) has the unique solution (S, T), then $(s = S^{1/p}, t = T^{1/p})$ is the unique solution for (18) and (19).

Let

$$D = \begin{vmatrix}\int_\Omega |u^+|^{2p}\,dx - b\|\nabla u^+\|_p^{2p} & -b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p \\ -b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p & \int_\Omega |u^-|^{2p}\,dx - b\|\nabla u^-\|_p^{2p}\end{vmatrix} > 0. \tag{21}$$

For $\lambda < a\lambda_1$, we have $a\|\nabla u^{\pm}\|_p^p > \lambda\int_\Omega |u^{\pm}|^p$. Since $\int_\Omega |u^-|^{2p} - b\|\nabla u^-\|_p^{2p} > 0$, then

$$D_S = \begin{vmatrix}a\|\nabla u^+\|_p^p - \int_\Omega \lambda|u^+|^p\,dx & -b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p \\ a\|\nabla u^-\|_p^p - \int_\Omega \lambda|u^-|^p\,dx & \int_\Omega |u^-|^{2p}\,dx - b\|\nabla u^-\|_p^{2p}\end{vmatrix} > 0. \tag{22}$$

Similarly, we have

$$D_T = \begin{vmatrix}\int_\Omega |u^+|^{2p}\,dx - b\|\nabla u^+\|_p^{2p} & a\|\nabla u^+\|_p^p - \int_\Omega \lambda|u^+|^p\,dx \\ -b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p & a\|\nabla u^-\|_p^p - \int_\Omega \lambda|u^-|^p\,dx\end{vmatrix} > 0. \tag{23}$$

From (21)-(23), we have $S = D_S/D > 0, T = D_T/D > 0$, and (S, T) is the unique solution for system (20). Accordingly, $(s = S^{1/p}, t = T^{1/p})$ is the unique positive solution for (18) and (19). Thus, (i) is proved.

(ii) Next, we give the proof of (ii).

From (6), we have

$$\begin{aligned}I_b(su^+ + tu^-) &= \frac{as^p}{p}\|\nabla u^+\|_p^p + \frac{bs^{2p}}{2p}\|\nabla u^+\|_p^{2p} \\ &\quad - \frac{s^p}{p}\int_\Omega \lambda|u^+|^p\,dx \\ &\quad - \frac{s^{2p}}{2p}\int_\Omega |u^+|^{2p}\,dx + \frac{at^p}{p}\|\nabla u^-\|_p^p\end{aligned}$$

$$+\frac{bt^{2p}}{2p}\left\|\nabla u^{-}\right\|_{p}^{2p}-\frac{t^{p}}{p}\int_{\Omega}\lambda\left|u^{-}\right|^{p}dx$$
$$-\frac{t^{2p}}{2p}\int_{\Omega}\left|u^{-}\right|^{2p}dx$$
$$+\frac{bs^{p}t^{p}}{p}\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}. \tag{24}$$

By a simple computation, we have

$$\begin{aligned}\frac{\partial^2 I_b}{\partial s^2}&=(p-1)s^{p-2}\left\{a\left\|\nabla u^{+}\right\|_{p}^{p}-\lambda\int_{\Omega}\left|u^{+}\right|^{p}dx\right.\\&\left.+bt^{p}\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}\right\}+(2p-1)\\&\cdot s^{2p-2}\left(b\left\|\nabla u^{+}\right\|_{p}^{2p}-\int_{\Omega}\left|u^{+}\right|^{2p}dx\right),\end{aligned} \tag{25}$$

and

$$\begin{aligned}\frac{\partial^2 I_b}{\partial t^2}&=(p-1)t^{p-2}\left\{a\left\|\nabla u^{-}\right\|_{p}^{p}-\lambda\int_{\Omega}\left|u^{-}\right|^{p}dx\right.\\&\left.+bs^{p}\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}\right\}+(2p-1)\\&\cdot t^{2p-2}\left(b\left\|\nabla u^{-}\right\|_{p}^{2p}-\int_{\Omega}\left|u^{-}\right|^{2p}dx\right).\end{aligned} \tag{26}$$

From $s_u u^{+}+t_u u^{-}\in\mathscr{M}_b$, we have

$$\begin{aligned}\left.\frac{\partial^2 I_b}{\partial s^2}\right|_{(s_u,t_u)}&=ps_u^{2p-2}\left(b\left\|\nabla u^{+}\right\|_{p}^{2p}-\int_{\Omega}\left|u^{+}\right|^{2p}dx\right)<0,\\\left.\frac{\partial^2 I_b}{\partial t^2}\right|_{(s_u,t_u)}&=pt_u^{2p-2}\left(b\left\|\nabla u^{-}\right\|_{p}^{2p}-\int_{\Omega}\left|u^{-}\right|^{2p}dx\right)<0,\end{aligned} \tag{27}$$

and

$$\left.\frac{\partial^2 I_b}{\partial t\partial s}\right|_{(s_u,t_u)}=pbs_u^{p-1}t_u^{p-1}\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}. \tag{28}$$

We consider the Hessian matrix of $I_b(su^{+}+tu^{-})$; then from (17), we have

$$\begin{aligned}\left|H(s_u,t_u)\right|&=\begin{vmatrix}\dfrac{\partial^2 I_b}{\partial s^2}&\dfrac{\partial^2 I_b}{\partial s\partial t}\\\dfrac{\partial^2 I_b}{\partial s\partial t}&\dfrac{\partial^2 I_b}{\partial t^2}\end{vmatrix}\\&=p^2s_u^{2p-2}t_u^{2p-2}\left(\int_{\Omega}\left|u^{+}\right|^{2p}dx-b\left\|\nabla u^{+}\right\|_{p}^{2p}\right)\\&\cdot\left(\int_{\Omega}\left|u^{-}\right|^{2p}dx-b\left\|\nabla u^{-}\right\|_{p}^{2p}\right)\\&-p^2b^2s_u^{2p-2}t_u^{2p-2}\left\|\nabla u^{+}\right\|_{p}^{2p}\left\|\nabla u^{-}\right\|_{p}^{2p}>0.\end{aligned} \tag{29}$$

The above deduction implies that (s_u,t_u) is a maximal point of $I_b(su^{+}+tu^{-})$ for $s,t\geq 0$. Since we cannot get the maximal point of I_b on the boundary of R^{+}, (s_u,t_u) is the unique maximal point; that is, $I_b(s_u u^{+}+t_u u^{-})=\max_{s,t\geq 0}I_b(su^{+}+tu^{-})$. □

Lemma 4. *Assume that $\lambda<a\lambda_1$ and $u\in\mathscr{M}_b$, then (17) holds.*

Proof. For $u=u^{+}+u^{-}\in\mathscr{M}_b$, we have

$$\begin{aligned}\left\langle I_b'(u),u^{+}\right\rangle&=a\left\|\nabla u^{+}\right\|_{p}^{p}+b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{+}\right\|_{p}^{p}\\&-\int_{\Omega}\lambda\left|u^{+}\right|^{p}dx-\int_{\Omega}\left|u^{+}\right|^{2p}dx=0,\\\left\langle I_b'(u),u^{-}\right\rangle&=a\left\|\nabla u^{-}\right\|_{p}^{p}+b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}\\&-\int_{\Omega}\lambda\left|u^{-}\right|^{p}dx-\int_{\Omega}\left|u^{-}\right|^{2p}dx=0.\end{aligned} \tag{30}$$

Since $\lambda<a\lambda_1$, we have $a\|\nabla u^{\pm}\|_p^p>\int_{\Omega}\lambda|u^{\pm}|^p dx$. Thus, we have

$$\begin{aligned}b\left\|\nabla u^{+}\right\|_{p}^{2p}+b\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}&<\int_{\Omega}\left|u^{+}\right|^{2p}dx\\b\left\|\nabla u^{-}\right\|_{p}^{2p}+b\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}&<\int_{\Omega}\left|u^{-}\right|^{2p}dx.\end{aligned} \tag{31}$$

□

Lemma 5. *Assume that $b>0,\lambda<a\lambda_1$, $u\in W$ with $u^{\pm}\neq 0$ and $\langle I_b'(u),u^{\pm}\rangle\leq 0$, there exists a unique pair $(s_u,t_u)\in(0,1]\times(0,1]$ such that $s_u u^{+}+t_u u^{-}\in\mathscr{M}_b$.*

Proof. If $u\in W$ with $u^{\pm}\neq 0$ and $\langle I_b'(u),u^{\pm}\rangle\leq 0$, we have

$$\begin{aligned}&a\left\|\nabla u^{+}\right\|_{p}^{p}+b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{+}\right\|_{p}^{p}\\&\quad\leq\int_{\Omega}\lambda\left|u^{+}\right|^{p}dx+\int_{\Omega}\left|u^{+}\right|^{2p}dx\\&a\left\|\nabla u^{-}\right\|_{p}^{p}+b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}\\&\quad\leq\int_{\Omega}\lambda\left|u^{-}\right|^{p}dx+\int_{\Omega}\left|u^{-}\right|^{2p}dx.\end{aligned} \tag{32}$$

Since $a\|\nabla u^{\pm}\|_p^p>\lambda\int_{\Omega}|u^{\pm}|^p dx$, then

$$\begin{aligned}b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{+}\right\|_{p}^{p}&<\int_{\Omega}\left|u^{+}\right|^{2p}dx\\b\left\|\nabla u\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}&<\int_{\Omega}\left|u^{-}\right|^{2p}dx.\end{aligned} \tag{33}$$

From Lemma 3, there is a unique pair (s_u,t_u) of positive numbers such that $s_u u^{+}+t_u u^{-}\in\mathscr{M}_b$, which implies that (s_u^p,t_u^p) is the solution of system (20). Then, we have

$$\begin{aligned}D_{s_u^p}&=\left(a\left\|\nabla u^{+}\right\|_{p}^{p}-\int_{\Omega}\lambda\left|u^{+}\right|^{p}dx\right)\\&\cdot\left(\int_{\Omega}\left|u^{-}\right|^{2p}dx-b\left\|\nabla u^{-}\right\|_{p}^{2p}\right)+b\left\|\nabla u^{+}\right\|_{p}^{p}\left\|\nabla u^{-}\right\|_{p}^{p}\end{aligned}$$

$$\begin{aligned}
&\cdot \left(a\left\|\nabla u^-\right\|_p^p - \int_\Omega \lambda \left|u^-\right|^p dx\right) \\
&\leq \left(\int_\Omega \left|u^+\right|^{2p} dx - b\left\|\nabla u\right\|_p^p \left\|\nabla u^+\right\|_p^p\right) \\
&\cdot \left(\int_\Omega \left|u^-\right|^{2p} dx - b\left\|\nabla u^-\right\|_p^{2p}\right) + b\left\|\nabla u^+\right\|_p^p \left\|\nabla u^-\right\|_p^p \\
&\cdot \left(\int_\Omega \left|u^-\right|^{2p} dx - b\left\|\nabla u\right\|_p^p \left\|\nabla u^-\right\|_p^p\right) \\
&= \left(\int_\Omega \left|u^+\right|^{2p} dx - b\left\|\nabla u^+\right\|_p^{2p}\right) \\
&\cdot \left(\int_\Omega \left|u^-\right|^{2p} dx - b\left\|\nabla u^-\right\|_p^{2p}\right) - b^2\left\|\nabla u^+\right\|_p^{2p} \\
&\cdot \left\|\nabla u^-\right\|_p^{2p} = D.
\end{aligned} \tag{34}$$

Therefore, we have $s_u^p = D_{s_u^p}/D \leq 1$. Similarly, we have $t_u^p = D_{t_u^p}/D \leq 1$. Thus, there exists a unique pair $(s_u, t_u) \in (0,1] \times (0,1]$ such that $s_u u^+ + t_u u^- \in \mathscr{M}_b$. □

Lemma 6. *If $\lambda < a\lambda_1$ for any $u \in W$ with $b\|\nabla u\|_p^{2p} < \int_\Omega |u|^{2p} dx$, there exists a unique $\overline{s_u} > 0$ such that $\overline{s_u} u \in \mathscr{N}_b$. Moreover, $I_b(\overline{s_u} u) > I_b(su)$ for all $s \geq 0$ and $s \neq \overline{s_u}$.*

Proof. If $\lambda < a\lambda_1$ and $u \in W$ satisfies $b\|\nabla u\|_p^{2p} < \int_\Omega |u|^{2p} dx$, $su \in \mathscr{N}_b$ implies that

$$\begin{aligned}
\left\langle I_b'(su), su\right\rangle &= as^p\|\nabla u\|_p^p + bs^{2p}\|\nabla u\|_p^{2p} \\
&\quad - s^p \int_\Omega \lambda |u|^p dx - s^{2p}\int_\Omega |u|^{2p} dx \\
&= 0.
\end{aligned} \tag{35}$$

Thus, there exists a unique $\overline{s_u} = ((a\|\nabla u\|_p^p - \int_\Omega \lambda |u|^p dx)/(\int_\Omega |u|^{2p} dx - b\|\nabla u\|_p^{2p}))^{1/p} > 0$ satisfying (35). From (6), we have

$$\begin{aligned}
I_b(su) &= \frac{as^p}{p}\|\nabla u\|_p^p + \frac{bs^{2p}}{2p}\|\nabla u\|_p^{2p} - \frac{s^p}{p}\int_\Omega \lambda |u|^p dx \\
&\quad - \frac{s^{2p}}{2p}\int_\Omega |u|^{2p} dx.
\end{aligned} \tag{36}$$

By a simple deduction, we have

$$\begin{aligned}
\left.\frac{\partial^2 I_b}{\partial s^2}\right|_{\overline{s_u}} &= \left((p-1)s^{p-2}\left\{a\|\nabla u\|_p^p - \int_\Omega \lambda |u|^p dx\right\}\right. \\
&\quad \left.\left. + (2p-1)s^{2p-2}\left(b\|\nabla u\|_p^{2p} - \int_\Omega |u|^{2p} dx\right)\right)\right|_{\overline{s_u}} \\
&= -p\overline{s_u}^{p-2}\left(a\|\nabla u\|_p^p - \int_\Omega \lambda |u|^p dx\right) < 0.
\end{aligned} \tag{37}$$

Thus, $I_b(su)$ attains its maximal point at $s = \overline{s_u}$. In other words, we have $I_b(\overline{s_u} u) > I_b(su)$ for all $s \geq 0$ and $s \neq \overline{s_u}$. □

Lemma 7. *Assume $\lambda < a\lambda_1$; we have that*

(i) if $0 < b < 1/S^2$, $c_b > 0$ is attained by some $v_b \in \mathscr{N}_b$ and v_b is a constant sign critical point of I_b, where S is given by (5);

(ii) if $0 < b < 1/2S^2$, $m_b > 0$ is attained by some $u_b \in \mathscr{M}_b$ and u_b is a sign-changing critical point of I_b.

Proof. (i) Firstly, we will show that for all $0 < b < 1/S^2$, there exists $u \in W$ such that $b\|\nabla u\|_p^{2p} < \int_\Omega |u|^{2p} dx$, which implies $\mathscr{N}_b \neq \emptyset$. From (5), we know that there exists $e_1 \in W$ such that $\|e_1\|_{2p} = S^{-1/p}\|e_1\|$. For $0 < b < 1/S^2$, we have

$$b\left\|\nabla e_1\right\|_p^{2p} < \frac{1}{S^2}\left\|\nabla e_1\right\|_p^{2p} = \int_\Omega \left|e_1\right|^{2p} dx. \tag{38}$$

Thus, we have $\mathscr{N}_b \neq \emptyset$.

For each $u \in \mathscr{N}_b$, it follows from $\lambda < a\lambda_1$ and (5) that

$$\begin{aligned}
a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} &= \lambda\int_\Omega |u|^p dx + \int_\Omega |u|^{2p} dx \\
&\leq \frac{\lambda}{\lambda_1}\|\nabla u\|_p^p + \frac{1}{S^2}\|\nabla u\|_p^{2p}.
\end{aligned} \tag{39}$$

Then, $\|\nabla u\|_p^p \geq (a - \lambda/\lambda_1)/(1/S^2 - b)$. Thus, we have

$$\begin{aligned}
I_b(u) &= I_b(u) - \frac{1}{2p}\left\langle I_b'(u), u\right\rangle \\
&= \frac{1}{2p}\left(a\|\nabla u\|_p^p - \int_\Omega \lambda |u|^p dx\right) \\
&\geq \frac{1}{2p}\left(a - \frac{\lambda}{\lambda_1}\right)\|\nabla u\|_p^p \geq \frac{\left(a - \lambda/\lambda_1\right)^2 S^2}{2p\left(1 - bS^2\right)},
\end{aligned} \tag{40}$$

that is, $c_b = \inf_{u\in\mathscr{N}_b} I_b(u) \geq (a - \lambda/\lambda_1)^2 S^2/2p(1 - bS^2) > 0$ and I_b is coercive and bounded below on $\mathscr{N}_b$ for $0 < b < 1/S^2$ and $\lambda < a\lambda_1$.

Let $\{v_n\} \subset \mathscr{N}_b$ be a minimizing sequence for I_b. From $I_b(v_n) = I_b(|v_n|)$ and $|v_n| \in \mathscr{N}_b$, we assume that $v_n(x) \geq 0$ in Ω for all $n \in \mathbb{N}$. Since I_b is coercive and bounded below on $\mathscr{N}_b$, the sequence $\{v_n\}$ is bounded in W, so that, up to subsequences, $v_n \rightharpoonup v_b$ in W and $v_b \geq 0$. Next, we will prove that $v_n \longrightarrow v_b$ strongly in W. We suppose by contradiction that $\|v_b\| < \liminf_{n\longrightarrow\infty}\|v_n\|$. Therefore, we have

$$a\left\|\nabla v_b\right\|_p^p + b\left\|\nabla v_b\right\|_p^{2p} < \lambda\int_\Omega \left|v_b\right|^p dx + \int_\Omega \left|v_b\right|^{2p} dx. \tag{41}$$

If $v_b = 0$, the above inequality makes a contradiction. Thus, we have $v_b \neq 0$ in Ω. From the fact that $a\|\nabla v_b\|_p^p > \lambda\int_\Omega |v_b|^p dx$, we have $b\|\nabla v_b\|_p^{2p} < \int_\Omega |v_b|^{2p} dx$. By Lemma 6, there exists a unique $s_v > 0$ such that $s_v v_b \in \mathcal{N}_b$ and $I_b(s_v v_n) \leq I_b(v_n)$ for all $v_n \in \mathcal{N}_b$. Thus, we have

$$\begin{aligned} c_b &\leq I_b(s_v v_b) = \frac{a}{p}\|\nabla(s_v v_b)\|_p^p + \frac{b}{2p}\|\nabla(s_v v_b)\|_p^{2p} - \frac{\lambda}{p} \\ &\cdot \int_\Omega |s_v v_b|^p\,dx - \frac{1}{2p}\int_\Omega |s_v v_b|^{2p}\,dx \\ &< \liminf_{n\to\infty}\left(\frac{a}{p}\|\nabla(s_v v_n)\|_p^p + \frac{b}{2p}\|\nabla(s_v v_n)\|_p^{2p}\right. \\ &\left. - \frac{\lambda}{p}\int_\Omega |s_v v_n|^p\,dx - \frac{1}{2p}\int_\Omega |s_v v_n|^{2p}\,dx\right) \\ &= \liminf_{n\to\infty} I_b(s_v v_n) \leq \liminf_{n\to\infty} I_b(v_n) = c_b, \end{aligned} \tag{42}$$

which leads to a contradiction. Therefore, we have $\|v_b\| = \liminf_{n\to\infty}\|v_n\|$, $v_n \longrightarrow v_b$ strongly in W and $I_b(v_b) = c_b$. Then, by a standard argument, which is similar to the discussion in [34], we can deduce that v_b is a constant sign critical point of I_b.

(ii) From a similar deduction as (i), we know that for $0 < b < 1/2S^2$, there exists $u_1 \in W$ such that

$$b\|\nabla u_1\|_p^{2p} < \frac{1}{2S^2}\|\nabla u_1\|_p^{2p} = \frac{1}{2}\int_\Omega |u_1|^{2p}\,dx. \tag{43}$$

Obviously, if $u \in W$ such that u_1 satisfies (43), then $|u_1| \in W$ also satisfies (43). Therefore, we assume that $u_1(x) \geq 0$ a.e. in W. We let $\operatorname{supp} u_1 \subset B_\rho(x_0)$ and define $u_2(x) = -u_1(-x)$ for all $x \in B_\rho(-x_0)$, where $B_\rho(x_0) = \{x \in \Omega : |x - x_0| < \rho\}$ and $\rho > 0$. Then, from (43), we have

$$\frac{\int_\Omega |u_1|^{2p}\,dx}{\|\nabla u_1\|_p^{2p}} = \frac{\int_\Omega |u_2|^{2p}\,dx}{\|\nabla u_2\|_p^{2p}} > 2b. \tag{44}$$

Let $u = u_1 + u_2$; we can obtain that $u \in W$ and $u^+ = u_1, u^- = u_2$ and

$$\begin{aligned} &b\|\nabla u^+\|_p^{2p} + b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p \\ &\quad < \frac{1}{2}\int_\Omega |u_1|^{2p}\,dx + \frac{1}{2}\|u_1\|_{2p}^p\|u_2\|_{2p}^p \\ &\quad = \int_\Omega |u_1|^{2p}\,dx, \end{aligned} \tag{45}$$

that is,

$$b\|\nabla u^+\|_p^{2p} + b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p < \int_\Omega |u^+|^{2p}\,dx. \tag{46}$$

Similarly, we also have

$$b\|\nabla u^-\|_p^{2p} + b\|\nabla u^+\|_p^p\|\nabla u^-\|_p^p < \int_\Omega |u^-|^{2p}\,dx. \tag{47}$$

By Lemma 3, we know that $\mathcal{M}_b \neq \emptyset$ for $0 < b < 1/2S^2$.

Assume that $\{u_n\} \subset \mathcal{M}_b$ is a minimizing sequence for I_b, such that $I_b(u_n) \longrightarrow m_b$. Since I_b is coercive on $\mathcal{N}_b$, the sequence $\{u_n\}$ is bounded in W; going if necessary to a subsequence, still denoted by $\{u_n\}$, we can assume that there exists a $u_b \in W$ such that for n sufficiently large,

$$\begin{aligned} &u_n^\pm \rightharpoonup u_b^\pm \quad \text{weakly in } W, \\ &u_n(x) \longrightarrow u_b(x) \quad \text{almost everywhere on } \Omega, \\ &u_n^\pm \longrightarrow u_b^\pm \quad \text{strongly in } L^s(\Omega) \text{ for } p \leq s < p^*. \end{aligned} \tag{48}$$

From $\{u_n\} \subset \mathcal{M}_b$, we have $\langle I_b'(u_n), u_n^\pm\rangle = 0$; that is,

$$\begin{aligned} &a\|\nabla u_n^\pm\|_p^p + b\|\nabla u_n\|_p^p\|\nabla u_n^\pm\|_p^p \\ &\quad = \lambda\int_\Omega |u_n^\pm|^p\,dx + \int_\Omega |u_n^\pm|^{2p}\,dx. \end{aligned} \tag{49}$$

Therefore,

$$a\|\nabla u_n^\pm\|_p^p \leq \lambda\int_\Omega |u_n^\pm|^p\,dx + \int_\Omega |u_n^\pm|^{2p}\,dx. \tag{50}$$

In the same way, we have $(a - \lambda/\lambda_1)\|\nabla u_n^\pm\|_p^p \leq (1/S^2)\|\nabla u_n^\pm\|_p^{2p}$ and $\|\nabla u_n^\pm\|_p^p \geq S^2(a - \lambda/\lambda_1) > 0$. Passing to the limit, we have

$$\begin{aligned} 0 &< S^2\left(a - \frac{\lambda}{\lambda_1}\right)^2 \leq \liminf_{n\to\infty}\left(a - \frac{\lambda}{\lambda_1}\right)\|\nabla u_n^\pm\|_p^p \\ &\leq \int_\Omega |u_b^\pm|^{2p}\,dx, \end{aligned} \tag{51}$$

which implies that $u_b^\pm \neq 0$ and

$$\begin{aligned} &a\|\nabla u_b^+\|_p^p + b\|\nabla u_b\|_p^p\|\nabla u_b^+\|_p^p \\ &\quad \leq \lambda\int_\Omega |u_b^+|^p\,dx + \int_\Omega |u_b^+|^{2p}\,dx, \\ &a\|\nabla u_b^-\|_p^p + b\|\nabla u_b\|_p^p\|\nabla u_b^-\|_p^p \\ &\quad \leq \lambda\int_\Omega |u_b^-|^p\,dx + \int_\Omega |u_b^-|^{2p}\,dx. \end{aligned} \tag{52}$$

From $a\|\nabla u_b^\pm\|_p^p > \lambda\int_\Omega |u_b^\pm|^p dx$, Lemmas 3 and 5, there exists a unique pair $(s_u, t_u) \in (0,1] \times (0,1]$ such that

$$s_u u_b^+ + t_u u_b^- \in \mathcal{M}_b. \tag{53}$$

From the definition of m_b, we have

$$\begin{aligned} m_b &\leq I_b(s_u u_b^+ + t_u u_b^-) = I_b(s_u u_b^+ + t_u u_b^-) \\ &- \frac{1}{2p}\left\langle I_b'(s_u u_b^+ + t_u u_b^-), s_u u_b^+ + t_u u_b^-\right\rangle \\ &= \frac{1}{2p}\left(a\|\nabla(s_u u_b^+ + t_u u_b^-)\|_p^p\right. \\ &\left. - \lambda\int_\Omega |s_u u_b^+ + t_u u_b^-|^p\,dx\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2p}\left\{s_u^p\left(a\left\|\nabla u_b^+\right\|_p^p - \lambda\int_\Omega \left|u_b^+\right|^p dx\right)\right.\\
&\quad \left. + t_u^p\left(a\left\|\nabla u_b^-\right\|_p^p - \lambda\int_\Omega \left|u_b^-\right|^p dx\right)\right\}\\
&\le \frac{1}{2p}\left\{\left(a\left\|\nabla u_b^+\right\|_p^p - \lambda\int_\Omega \left|u_b^+\right|^p dx\right)\right.\\
&\quad \left. + \left(a\left\|\nabla u_b^-\right\|_p^p - \lambda\int_\Omega \left|u_b^-\right|^p dx\right)\right\} = \frac{1}{2p}\left(a\left\|\nabla u_b\right\|_p^p\right.\\
&\quad \left. - \lambda\int_\Omega \left|u_b\right|^p dx\right)\\
&\le \liminf_{n\longrightarrow\infty}\left\{I_b\left(u_n\right) - \frac{1}{2p}\left\langle I_b'\left(u_n\right), u_n\right\rangle\right\} = m_b.
\end{aligned} \tag{54}
$$

Thus, $s_u = t_u = 1$, $u_b \in \mathscr{M}_b$, and $I_b(u_b) = m_b$, u_b is the required minimizer.

Next, we will prove that u_b is indeed a sign-changing solution; that is, $I_b'(u_b) = 0$. We mainly use the quantitative deformation lemma [35] to prove the results.

If $I_b'(u_b) \neq 0$, there exists $\delta > 0$ and $\alpha > 0$, such that

$$
\begin{gathered}
u \in W,\\
\left\|I_b'(u)\right\| \ge \alpha,\\
\forall \left\|u - u_b\right\| \le 3\delta.
\end{gathered} \tag{55}
$$

Let $D = (1/2, 3/2)\times(1/2, 3/2)$, $\psi(s,t) = su_b^+ + tu_b^-$, and $(s,t) \in D$. It follows from Lemma 3 that

$$
\overline{m_b} = \max_{\partial D} I_b \circ \psi < m_b. \tag{56}
$$

Let $\varepsilon = \min\{(m_b - \overline{m_b})/3, \alpha\delta/8\}$ and $S_\delta = \{u \in W : \|u - u_b\| \le \delta\}$; there exists a deformation $\eta \in C([0,1]\times W, W)$ such that

(i) $\eta(1,u) = u$ if $u \notin I_b^{-1}([m_b - 2\varepsilon, m_b + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I_b^{m_b+\varepsilon} \cap S_\delta) \subset I_b^{m_b-\varepsilon}$;

(iii) $I_b(\eta(1,u)) \le I_b(u), \forall u \in W$.

From (56), Lemma 3 and (ii), we can easily get

$$
\max_{(s,t)\in\overline{D}} I_b\left(\eta\left(1, \psi\left(s,t\right)\right)\right) < m_b. \tag{57}
$$

We prove that $\eta(1, \psi(D)) \cap \mathscr{M}_b \neq \emptyset$, which contradicts the definition of m_b. We define $g(s,t) = \eta(1, \psi(s,t))$ and

$$
\begin{aligned}
\Phi_0(s,t) &= \left(I_b'\left(\psi\left(s,t\right)\right)u_b^+, I_b'\left(\psi\left(s,t\right)\right)u_b^-\right)\\
&= \left(I_b'\left(su_b^+ + tu_b^-\right)u_b^+, I_b'\left(su_b^+ + tu_b^-\right)u_b^-\right),\\
\Phi_1(s,t)&\\
&= \left(\frac{1}{s}I_b'\left(g\left(s,t\right)\right)g^+\left(s,t\right), \frac{1}{t}I_b'\left(g\left(s,t\right)\right)g^-\left(s,t\right)\right).
\end{aligned} \tag{58}
$$

Lemma 3 and the degree theory yield $\deg(\Phi_0, D, 0) = 1$. From (56), we know that $g = \psi$ on ∂D. Consequently, we have $\deg(\Phi_0, D, 0) = \deg(\Phi_1, D, 0) = 1$. Therefore, $\Phi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$; that is, $\eta(1, \psi(s_0, t_0)) = g(s_0, t_0) \in \mathscr{M}_b$, which is a contradiction. From this point, u_b is a sign-changing critical point of I_b and $I_b'(u_b) = 0$. □

3. The Existence of the Sign-Changing Solutions

In this part, we are devoted to proving Theorem 1.

Proof of Theorem 1. In view of Lemma 7, we know that for $0 < b < 1/2S^2$ and $\lambda < a\lambda_1$, there exists a $u_b \in \mathscr{M}_b$ such that $m_b = I_b(u_b)$ and $I_b'(u_b) = 0$; that is, u_b is a ground state sign-changing solution for problem (1). Then by Lemma 4, we have that

$$
\begin{aligned}
b\left\|\nabla u_b^+\right\|_p^{2p} + b\left\|\nabla u_b^+\right\|_p^p\left\|\nabla u_b^-\right\|_p^p &< \int_\Omega \left|u_b^+\right|^{2p} dx\\
b\left\|\nabla u_b^-\right\|_p^{2p} + b\left\|\nabla u_b^+\right\|_p^p\left\|\nabla u_b^-\right\|_p^p &< \int_\Omega \left|u_b^-\right|^{2p} dx,
\end{aligned} \tag{59}
$$

which implies

$$
\begin{aligned}
b\left\|\nabla u_b^+\right\|_p^{2p} &< \int_\Omega \left|u_b^+\right|^{2p} dx\\
b\left\|\nabla u_b^-\right\|_p^{2p} &< \int_\Omega \left|u_b^-\right|^{2p} dx.
\end{aligned} \tag{60}
$$

Then from Lemma 6, there exist $s_1, t_1 > 0$ such that $s_1u_b^+, t_1u_b^- \in \mathscr{N}_b$. Therefore, we have

$$
\begin{aligned}
m_b &= I_b\left(u_b\right) \ge I_b\left(s_1u_b^+ + t_1u_b^-\right)\\
&= I_b\left(s_1u_b^+\right) + I_b\left(t_1u_b^-\right) + \frac{bs_1^pt_1^p}{p}\left\|\nabla u_b^+\right\|_p^p\left\|\nabla u_b^-\right\|_p^p\\
&> I_b\left(s_1u_b^+\right) + I_b\left(t_1u_b^-\right) \ge 2c_b.
\end{aligned} \tag{61}
$$

Therefore, the energy doubling property is proved.

Next, we prove that u_b changes sign only once; that is, u_b has exactly two nodal domains. We assume by contradiction that $u_b = u_1 + u_2 + u_3$ with $u_i \neq 0, u_1 \ge 0, u_2 \le 0, u_3 \ge 0$ and $\mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset$ for $i \neq j, (i,j = 1,2,3)$.

Since $I_b'(u_b) = 0$, we can get

$$
\begin{aligned}
\left\langle I_b'\left(u_1 + u_2\right), u_1\right\rangle &= \left\langle I_b'\left(u_b\right), u_1\right\rangle\\
&\quad - b\left\|\nabla u_1\right\|_p^p\left\|\nabla u_3\right\|_p^p < 0,\\
\left\langle I_b'\left(u_1 + u_2\right), u_2\right\rangle &= \left\langle I_b'\left(u_b\right), u_2\right\rangle\\
&\quad - b\left\|\nabla u_2\right\|_p^p\left\|\nabla u_3\right\|_p^p < 0.
\end{aligned} \tag{62}
$$

Then, by Lemma 5, there exists a pair $(s', t') \in (0,1]\times(0,1]$ such that $s'u_1 + t'u_2 \in \mathscr{M}_b$ and $I_b(s'u_1 + t'u_2) \ge m_b$.

From $\lambda < a\lambda_1$, $\langle I_b'(u_b), u_b\rangle = 0$, and $\langle I_b'(s'u_1 + t'u_2), s'u_1 + t'u_2\rangle = 0$, we have

$$
\begin{aligned}
m_b &= I_b\left(u_b\right) - \frac{1}{2p}\left\langle I_b'\left(u_b\right), u_b\right\rangle = \frac{1}{2p}\left(a\left\|\nabla u_b\right\|_p^p\right.\\
&\quad \left. - \lambda\int_\Omega \left|u_b\right|^p dx\right)\\
&= \frac{1}{2p}\left\{\left(a\left\|\nabla u_1\right\|_p^p - \lambda\int_\Omega \left|u_1\right|^p dx\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(a\left\|\nabla u_2\right\|_p^p-\lambda \int_\Omega\left|u_2\right|^p dx\right) \\
&+\left.\left(a\left\|\nabla u_3\right\|_p^p-\lambda \int_\Omega\left|u_3\right|^p dx\right)\right\} \\
&>\frac{1}{2p}\left\{\left(a\left\|\nabla u_1\right\|_p^p-\lambda \int_\Omega\left|u_1\right|^p dx\right)\right. \\
&+\left.\left(a\left\|\nabla u_2\right\|_p^p-\lambda \int_\Omega\left|u_2\right|^p dx\right)\right\} \\
&\geq \frac{1}{2p}\left\{\left(s'\right)^p\left(a\left\|\nabla u_1\right\|_p^p-\lambda \int_\Omega\left|u_1\right|^p dx\right)\right. \\
&+\left.\left(t'\right)^p\left(a\left\|\nabla u_2\right\|_p^p-\lambda \int_\Omega\left|u_2\right|^p dx\right)\right\}=I_b\left(s'u_1\right. \\
&\left.+t'u_2\right)-\frac{1}{2p}\left\langle I_b'\left(s'u_1+t'u_2\right), s'u_1+t'u_2\right\rangle \\
&=I_b\left(s'u_1+t'u_2\right) \geq m_b,
\end{aligned} \tag{63}
$$

which leads to a contradiction; thus, u_b has exactly two nodal domains. □

4. The Convergence Property of u_b as $b \searrow 0$

In this part, we regard $b > 0 (b \in (0, 1/2S^2))$ as a small parameter in (1) and discuss the convergence property of the least energy sign-changing solution u_b, where $u_b \in \mathscr{M}_b$ and u_b changes sign only once.

Proof of Theorem 2. We choose a nonzero function $w_0 \in C_0^\infty(\Omega)$ and $\gamma > 0$ such that $w_0^\pm \neq 0$ and

$$
\begin{aligned}
&a\left\|\nabla w_0^+\right\|_p^p+\gamma\left\|\nabla w_0\right\|_p^p\left\|\nabla w_0^+\right\|_p^p \\
&\quad \leq \lambda \int_\Omega\left|w_0^+\right|^p dx+\int_\Omega\left|w_0^+\right|^{2p} dx \\
&a\left\|\nabla w_0^-\right\|_p^p+\gamma\left\|\nabla w_0\right\|_p^p\left\|\nabla w_0^-\right\|_p^p \\
&\quad \leq \lambda \int_\Omega\left|w_0^-\right|^p dx+\int_\Omega\left|w_0^-\right|^{2p} dx.
\end{aligned} \tag{64}
$$

Thus, for any $b \in [0, \gamma]$, we have $\langle I_b'(w_0), w_0^\pm\rangle \leq 0$. It follows from Lemma 5 that there is a unique pair $(s_b, t_b) \in (0,1] \times (0,1]$ such that $s_b w_0^+ + t_b w_0^- \in \mathscr{M}_b$. Thus, we have

$$
\begin{aligned}
&I_b\left(s_b w_0^++t_b w_0^-\right)=I_b\left(s_b w_0^++t_b w_0^-\right) \\
&-\frac{1}{2p}\left\langle I_b'\left(s_b w_0^++t_b w_0^-\right), s_b w_0^++t_b w_0^-\right\rangle \\
&=\frac{1}{2p}\left(a\left\|\nabla\left(s_b w_0^++t_b w_0^-\right)\right\|_p^p\right. \\
&\left.-\lambda \int_\Omega\left|s_b w_0^++t_b w_0^-\right|^p dx\right) \\
&<\frac{a}{2p}\left\|\nabla\left(s_b w_0^++t_b w_0^-\right)\right\|_p^p \leq \frac{a}{2p}\left\|w_0\right\|^p=\Theta .
\end{aligned} \tag{65}
$$

For any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \longrightarrow \infty$, there exists $u_{b_n} \in \mathscr{M}_b$ such that u_{b_n} is a ground state sign-changing critical point of $I_{b_n}(u)$ and

$$
\begin{aligned}
\Theta+1 &\geq I_{b_n}\left(u_{b_n}\right)-\frac{1}{2p}\left\langle I_{b_n}'\left(u_{b_n}\right), u_{b_n}\right\rangle \\
&=\frac{1}{2p}\left(a\left\|\nabla u_{b_n}\right\|_p^p-\lambda \int_\Omega\left|u_{b_n}\right|^p dx\right) \\
&\geq \frac{1}{2p}\left(a-\frac{\lambda}{\lambda_1}\right)\left\|\nabla u_{b_n}\right\|_p^p .
\end{aligned} \tag{66}
$$

The above inequality shows that u_{b_n} is bounded in W; then there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $u_{b_n} \rightharpoonup u_0$ weakly in W. By the compactness of the embedding $W \hookrightarrow L^s(\Omega)$ for $p \leq s < p^*$, using a standard argument, we have that $u_{b_n}^\pm \longrightarrow u_0^\pm$ in W and $u_0^\pm \neq 0$. Moreover, we have that for all $u \in W$,

$$
\begin{aligned}
0&=\lim_{n \longrightarrow \infty}\left\langle I_{b_n}'\left(u_{b_n}\right), u\right\rangle \\
&=\lim_{n \longrightarrow \infty}\left\{\int_\Omega a\left|\nabla u_{b_n}\right|^{p-2} \nabla u_{b_n} \nabla u dx\right. \\
&+b_n \int_\Omega\left|\nabla u_{b_n}\right|^p dx \int_\Omega\left|\nabla u_{b_n}\right|^{p-2} \nabla u_{b_n} \nabla u dx \\
&\left.-\lambda \int_\Omega\left|u_{b_n}\right|^{p-2} u_{b_n} u dx-\int_\Omega\left|u_{b_n}\right|^{2p-2} u_{b_n} u dx\right\} \\
&=\int_\Omega a\left|\nabla u_0\right|^{p-2} \nabla u_0 \nabla u dx-\lambda \int_\Omega\left|u_0\right|^{p-2} u_0 u dx \\
&-\int_\Omega\left|u_0\right|^{2p-2} u_0 u dx=\left\langle I_0'\left(u_0\right), u\right\rangle,
\end{aligned} \tag{67}
$$

which implies that

$$
I_0'\left(u_0\right)=0, \quad u_0 \in \mathscr{M}_0, \; I_0\left(u_0\right) \geq m_0. \tag{68}
$$

Secondly, in the Proof of Theorem 1, $b = 0$ is allowed; then, there exists a $v_0 \in \mathscr{M}_0$ such that

$$
I_0\left(v_0\right)=m_0=\inf_{u \in \mathscr{M}_0} I_0(u), \tag{69}
$$

and v_0 is a sign-changing solution for (11) which changes sign only once. Similarly, we can pick up $\varepsilon > 0$ which is independent of b_n such that

$$
\begin{aligned}
&\varepsilon\left\|\nabla v_0^+\right\|_p^{2p}+\varepsilon\left\|\nabla v_0^+\right\|_p^p\left\|\nabla v_0^-\right\|_p^p<\int_\Omega\left|v_0^+\right|^{2p} dx \\
&\varepsilon\left\|\nabla v_0^-\right\|_p^{2p}+\varepsilon\left\|\nabla v_0^+\right\|_p^p\left\|\nabla v_0^-\right\|_p^p<\int_\Omega\left|v_0^-\right|^{2p} dx.
\end{aligned} \tag{70}
$$

According to Lemma 3, there exists a unique pair (s_0, t_0) of positive numbers such that $s_0 v_0^+ + t_0 v_0^- \in \mathscr{M}_\varepsilon$.

Let $b_n \in [0, \varepsilon]$; we know that

$$\begin{aligned}
&\left\langle I'_{b_n}\left(s_0 v_0^+ + t_0 v_0^-\right), s_0 v_0^+\right\rangle \\
&\quad = a\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^p + b_n\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^{2p} \\
&\qquad + b_n\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^p\left\|\nabla\left(t_0 v_0^-\right)\right\|_p^p - \lambda\int_\Omega\left|s_0 v_0^+\right|^p dx \\
&\qquad - \int_\Omega\left|s_0 v_0^+\right|^{2p} dx \\
&\quad \le a\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^p + \varepsilon\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^{2p} \\
&\qquad + \varepsilon\left\|\nabla\left(s_0 v_0^+\right)\right\|_p^p\left\|\nabla\left(t_0 v_0^-\right)\right\|_p^p - \lambda\int_\Omega\left|s_0 v_0^+\right|^p dx \\
&\qquad - \int_\Omega\left|s_0 v_0^+\right|^{2p} dx = \left\langle I'_\varepsilon\left(s_0 v_0^+ + t_0 v_0^-\right), s_0 v_0^+\right\rangle \\
&\quad = 0.
\end{aligned} \tag{71}$$

In the same way, we can obtain that

$$\begin{aligned}
&\left\langle I'_{b_n}\left(s_0 v_0^+ + t_0 v_0^-\right), t_0 v_0^-\right\rangle \le \left\langle I'_\varepsilon\left(s_0 v_0^+ + t_0 v_0^-\right), t_0 v_0^-\right\rangle \\
&\quad = 0.
\end{aligned} \tag{72}$$

It follows from Lemma 5 that for all $b_n \in [0, \varepsilon]$, there exists a unique pair $(s_n, t_n) \in (0, s_0] \times (0, t_0]$ such that $s_n v_0^+ + t_n v_0^- \in \mathcal{M}_{b_n}$. Then, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \longrightarrow \infty$, we have as $n \longrightarrow \infty$,

$$\begin{gathered}
b_n s_n^{2p}\left\|\nabla v_0^+\right\|_p^{2p} \longrightarrow 0, \\
b_n s_n^p t_n^p\left\|\nabla v_0^+\right\|_p^p\left\|\nabla v_0^-\right\|_p^p \longrightarrow 0, \\
b_n t_n^{2p}\left\|\nabla v_0^-\right\|_p^{2p} \longrightarrow 0.
\end{gathered} \tag{73}$$

According to $\langle I'_{b_n}(s_n v_0^+ + t_n v_0^-), s_n v_0^+\rangle = \langle I'_{b_n}(s_n v_0^+ + t_n v_0^-), t_n v_0^-\rangle = 0$, we have

$$\begin{aligned}
a\left\|\nabla v_0^+\right\|_p^p + \circ(1) &= \lambda\int_\Omega\left|v_0^+\right|^p dx + s_n^p\int_\Omega\left|v_0^+\right|^{2p} dx \\
a\left\|\nabla v_0^-\right\|_p^p + \circ(1) &= \lambda\int_\Omega\left|v_0^-\right|^p dx + t_n^p\int_\Omega\left|v_0^-\right|^{2p} dx.
\end{aligned} \tag{74}$$

From $\langle I'_0(v_0), v_0^\pm\rangle = 0$, we have

$$\begin{aligned}
a\left\|\nabla v_0^+\right\|_p^p &= \lambda\int_\Omega\left|v_0^+\right|^p dx + \int_\Omega\left|v_0^+\right|^{2p} dx \\
a\left\|\nabla v_0^-\right\|_p^p &= \lambda\int_\Omega\left|v_0^-\right|^p dx + \int_\Omega\left|v_0^-\right|^{2p} dx.
\end{aligned} \tag{75}$$

Combining (74) with (75), we have that as $n \longrightarrow \infty$, $s_n \longrightarrow 1$, $t_n \longrightarrow 1$. Lastly, we only need to show $I_0(u_0) = I_0(v_0)$; then by (68), u_0 is a ground state sign-changing solution for (11), which changes sign only once. In fact,

$$\begin{aligned}
I_0\left(v_0\right) \le I_0\left(u_0\right) &= \lim_{n\longrightarrow\infty} I_{b_n}\left(u_{b_n}\right) \\
&\le \lim_{n\longrightarrow\infty} I_{b_n}\left(s_n v_0^+ + t_n v_0^-\right) = I_0\left(v_0^+ + v_0^-\right) \\
&= I_0\left(v_0\right).
\end{aligned} \tag{76}$$

Then, the proof of Theorem 2 is complete. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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