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Research Article

Global Dynamics of Higher-Order Exponential Systems of Difference Equations

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In this paper, we study the global dynamics of three higher-order exponential systems of rational difference equations. This suggested work considerably extends and improves some existing results in the literature.

1. Introduction

Global dynamical properties of difference equations or systems of difference equations involving exponential term have been widely investigated in recent years. For instance, Ozturk et al. [1] have explored the dynamical properties of the following exponential difference equation:

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-1}}, \tag{1}$$

where α, β, γ, and x_p $(p = 0, -1)$ are positive real numbers. Cömert et al. [2] have explored the dynamical properties of the following higher-order exponential difference equation:

$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + x_{n-k}}, \tag{2}$$

where α, β, γ, and x_p $(p = 0, -1, \cdots, -k)$ are positive real numbers. Bozkurt [3] has explored the dynamical properties of the following exponential difference equation:

$$x_{n+1} = \frac{\alpha e^{-x_n} + \beta e^{-x_{n-1}}}{\gamma + \alpha x_n + \beta x_{n-1}}, \tag{3}$$

where α, β, γ, and x_p $(p = 0, -1)$ are positive real numbers. In 2009, Ozturk et al. [4] have explored the dynamical properties of the following higher-order exponential difference equation:

$$x_{n+1} = \frac{\alpha e^{-(nx_n + (n-k)x_{n-k})}}{\beta + nx_n + (n-k)\, x_{n-k}}, \tag{4}$$

where α, β, and x_p $(p = 0, -1, \cdots, -k)$ are positive real numbers. Papaschinopoulos et al. [5] have explored the dynamical properties of the following exponential systems of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \\ y_{n+1} &= \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}} \\ x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \\ y_{n+1} &= \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}} \\ x_{n+1} &= \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \\ y_{n+1} &= \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}}, \end{aligned} \tag{5}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ and x_p, y_p $(p = 0, -1)$ are positive real numbers. Khan and Sharif [6] have explored the dynamical properties of the following 3×6 exponential systems of difference equations:

$$
\begin{aligned}
x_{n+1} &= \frac{\alpha_4 + \beta_4 e^{-y_n}}{\gamma_4 + y_{n-1}}, \\
y_{n+1} &= \frac{\alpha_5 + \beta_5 e^{-z_n}}{\gamma_5 + z_{n-1}}, \\
z_{n+1} &= \frac{\alpha_6 + \beta_6 e^{-x_n}}{\gamma_6 + x_{n-1}} \\
x_{n+1} &= \frac{\alpha_7 + \beta_7 e^{-z_n}}{\gamma_7 + z_{n-1}}, \\
y_{n+1} &= \frac{\alpha_8 + \beta_8 e^{-x_n}}{\gamma_8 + x_{n-1}}, \\
z_{n+1} &= \frac{\alpha_9 + \beta_9 e^{-y_n}}{\gamma_9 + y_{n-1}} \\
x_{n+1} &= \frac{\alpha_{10} + \beta_{10} e^{-x_n}}{\gamma_{10} + x_{n-1}}, \\
y_{n+1} &= \frac{\alpha_{11} + \beta_{11} e^{-y_n}}{\gamma_{11} + z_{n-1}}, \\
z_{n+1} &= \frac{\alpha_{12} + \beta_{12} e^{-z_n}}{\gamma_{12} + y_{n-1}} \\
x_{n+1} &= \frac{\alpha_{13} + \beta_{13} e^{-x_n}}{\gamma_{13} + z_{n-1}}, \\
y_{n+1} &= \frac{\alpha_{14} + \beta_{14} e^{-y_n}}{\gamma_{14} + y_{n-1}}, \\
z_{n+1} &= \frac{\alpha_{15} + \beta_{15} e^{-z_n}}{\gamma_{15} + x_{n-1}} \\
x_{n+1} &= \frac{\alpha_{16} + \beta_{16} e^{-x_n}}{\gamma_{16} + y_{n-1}}, \\
y_{n+1} &= \frac{\alpha_{17} + \beta_{17} e^{-y_n}}{\gamma_{17} + x_{n-1}}, \\
z_{n+1} &= \frac{\alpha_{18} + \beta_{18} e^{-z_n}}{\gamma_{18} + z_{n-1}},
\end{aligned}
\tag{6}
$$

where $\alpha_p, \beta_p, \gamma_p$ $(p = 4, 5, \cdots, 18)$ and x_p, y_p, z_p $(p = 0, -1)$ are positive real numbers. Khan and Qureshi [7] have explored the dynamical properties of the following exponential system of difference equations:

$$
\begin{aligned}
x_{n+1} &= \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha x_n + \beta x_{n-1}}, \\
y_{n+1} &= \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 y_n + \beta_1 y_{n-1}},
\end{aligned}
\tag{7}
$$

where $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and x_p, y_p $(p = 0, -1)$ are positive real numbers. Psarros et al. [8] have explored the dynamical properties of the following six exponential systems of difference equations:

$$
\begin{aligned}
x_{n+1} &= \frac{a_1 y_n}{b_1 + y_n} + \frac{c_1 x_n e^{k_1 - d_1 x_n}}{1 + e^{k_1 - d_1 x_n}}, \\
y_{n+1} &= \frac{a_2 x_n}{b_2 + x_n} + \frac{c_2 y_n e^{k_2 - d_2 y_n}}{1 + e^{k_2 - d_2 y_n}} \\
x_{n+1} &= \frac{a_1 x_n}{b_1 + x_n} + \frac{c_1 y_n e^{k_1 - d_1 y_n}}{1 + e^{k_1 - d_1 y_n}}, \\
y_{n+1} &= \frac{a_2 y_n}{b_2 + y_n} + \frac{c_2 x_n e^{k_2 - d_2 x_n}}{1 + e^{k_2 - d_2 x_n}} \\
x_{n+1} &= \frac{a_1 y_n^2}{b_1 + y_n} + \frac{c_1 e^{d_1 - k_1 x_n}}{1 + e^{d_1 - k_1 x_n}}, \\
y_{n+1} &= \frac{a_2 e^{b_2 - k_2 x_n}}{1 + e^{b_2 - k_2 x_n}} + \frac{c_2 e^{d_2 - k_3 y_n}}{1 + e^{d_2 - k_3 y_n}} \\
x_{n+1} &= \frac{a_1 y_n^2}{b_1 + y_n} + \frac{c_1 e^{d_1 - k_1 x_n}}{1 + e^{d_1 - k_1 x_n}}, \\
y_{n+1} &= \frac{a_2 x_n^2}{b_2 + x_n} + \frac{c_2 e^{d_2 - k_2 y_n}}{1 + e^{d_2 - k_2 y_n}} \\
x_{n+1} &= \frac{a_1 x_n^2}{b_1 + x_n} + \frac{c_1 e^{d_1 - k_1 y_n}}{1 + e^{d_1 - k_1 y_n}}, \\
y_{n+1} &= \frac{a_2 e^{b_2 - k_2 x_n}}{1 + e^{b_2 - k_2 x_n}} + \frac{c_2 e^{d_2 - k_3 y_n}}{1 + e^{d_2 - k_3 y_n}} \\
x_{n+1} &= \frac{a_1 x_n^2}{b_1 + x_n} + \frac{c_1 e^{d_1 - k_1 y_n}}{1 + e^{d_1 - k_1 y_n}}, \\
y_{n+1} &= \frac{a_2 y_n^2}{b_2 + y_n} + \frac{c_2 e^{d_2 - k_2 x_n}}{1 + e^{d_2 - k_2 x_n}},
\end{aligned}
\tag{8}
$$

where a_p, b_p, c_p, d_p, k_q $(p = 1, 2,\ q = 1, 2, 3)$ and x_0, y_0 are positive real numbers. Mylona et al. [9] have explored the dynamical properties of the following two 3×3 close-to-cyclic systems of exponential difference equations:

$$
\begin{aligned}
x_{n+1} &= a_1 x_n + b_1 y_n e^{-x_n}, \\
y_{n+1} &= a_2 y_n + b_2 z_n e^{-y_n}, \\
z_{n+1} &= a_3 z_n + b_3 x_n e^{-z_n} \\
x_{n+1} &= a_1 y_n + b_1 x_n e^{-y_n}, \\
y_{n+1} &= a_2 z_n + b_2 y_n e^{-z_n}, \\
z_{n+1} &= a_3 x_n + b_3 z_n e^{-x_n},
\end{aligned}
\tag{9}
$$

where a_p, b_p $(p = 1, 2, 3)$ and x_0, y_0, z_0 are positive real numbers. Mylona et al. [10] have explored the dynamical

properties of the following systems of difference equations with exponential terms:

$$\begin{aligned} x_{n+1} &= a_1 x_n + a_2 y_n e^{-x_n}, \\ y_{n+1} &= a_3 y_n + a_4 x_n e^{-y_n} \\ x_{n+1} &= a_1 y_n + a_2 x_n e^{-y_n}, \\ y_{n+1} &= a_3 x_n + a_4 y_n e^{-x_n}, \end{aligned} \tag{10}$$

where a_p $(p = 1, \cdots, 4)$ and x_0, y_0 are positive real numbers, and for more other interesting results on difference equations as well as systems of difference equation, we refer the reader to [11–13] and the references cited therein. Motivated by the above systemic studies, in this paper we aim to explore the dynamical properties of the following higher-order exponential systems of difference equations, which are natural extension of the work studied by Ozturk et al. [4]:

$$\begin{aligned} x_{n+1} &= \frac{\alpha e^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)\, y_{n-k}}, \\ y_{n+1} &= \frac{\alpha_1 e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}} \end{aligned} \tag{11}$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}}{\beta + nx_n + (n-k)\, x_{n-k}}, \\ y_{n+1} &= \frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1 + ny_n + (n-k)\, y_{n-k}} \end{aligned} \tag{12}$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}}{\beta + ny_n + (n-k)\, y_{n-k}}, \\ y_{n+1} &= \frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}}, \end{aligned} \tag{13}$$

where $\alpha, \beta, \alpha_1, \beta_1$ and x_p, y_p $(p = 0, -1, \cdots, -k)$ are positive real numbers.

The rest of this paper is organized as follows: Section 2 is about the study of boundedness and persistence of systems (11), (12), and (13). This also includes construction of invariant rectangle of these systems. Section 3 is about the uniqueness and existence of *+ve* fixed point of systems (11), (12), and (13). Section 4 deals with the study of local stability, whereas in Section 5 we study the global dynamics of the unique *+ve* fixed point of these systems. Section 6 deals with the study of rate of convergence of systems (11), (12), and (13). Brief conclusion is given in Section 7.

2. Boundedness, Persistence, and Construction of Invariant Rectangle of Systems (11), (12), and (13)

Theorem 1. *If $2n \geq k$, then every +ve solution $\{(x_n, y_n)\}_{n=-k}^{\infty}$ of* (i) *system (11),* (ii) *system (12), and* (iii) *system (13) is bounded and persists.*

Proof. (i) If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a *+ve* solution of (11), then

$$\begin{aligned} x_n &\leq \frac{\alpha}{\beta} = U_1, \\ y_n &\leq \frac{\alpha_1}{\beta_1} = U_2, \\ & n = 0, 1, \cdots. \end{aligned} \tag{14}$$

From (11) and (14) we get

$$\begin{aligned} x_n &\geq \frac{\alpha e^{-(-k+2n)(\alpha/\beta)}}{\beta + (-k+2n)\left(\alpha_1/\beta_1\right)} = L_1, \\ y_n &\geq \frac{\alpha_1 e^{-(-k+2n)(\alpha_1/\beta_1)}}{\beta_1 + (-k+2n)\left(\alpha/\beta\right)} = L_2, \\ & n = 2, 3, \cdots. \end{aligned} \tag{15}$$

Finally from (14) and (15) we get

$$\begin{aligned} L_1 &\leq x_n \leq U_1, \\ L_2 &\leq y_n \leq U_2, \\ & n = 3, 4, \cdots. \end{aligned} \tag{16}$$

(ii) If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a *+ve* solution of (12), then

$$\begin{aligned} x_n &\leq \frac{\alpha}{\beta} = U_3, \\ y_n &\leq \frac{\alpha_1}{\beta_1} = U_4, \\ & n = 0, 1, \cdots. \end{aligned} \tag{17}$$

From (12) and (17) we have

$$\begin{aligned} x_n &\geq \frac{\alpha e^{-(-k+2n)(\alpha_1/\beta_1)}}{\beta + (-k+2n)\left(\alpha/\beta\right)} = L_3, \\ y_n &\geq \frac{\alpha_1 e^{-(-k+2n)(\alpha/\beta)}}{\beta_1 + (-k+2n)\left(\alpha_1/\beta_1\right)} = L_4, \\ & n = 2, 3, \cdots. \end{aligned} \tag{18}$$

Finally from (17) and (18) we get

$$\begin{aligned} L_3 &\leq x_n \leq U_3, \\ L_4 &\leq y_n \leq U_4, \\ & n = 3, 4, \cdots. \end{aligned} \tag{19}$$

(iii) If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a *+ve* solution of (13), then

$$\begin{aligned} x_n &\leq \frac{\alpha}{\beta} = U_5, \\ y_n &\leq \frac{\alpha_1}{\beta_1} = U_6, \\ & n = 0, 1, \cdots. \end{aligned} \tag{20}$$

From (13) and (20) we get

$$x_n \geq \frac{\alpha e^{-(-k+2n)(\alpha_1/\beta_1)}}{\beta + (-k+2n)(\alpha_1/\beta_1)} = L_5,$$
$$y_n \geq \frac{\alpha_1 e^{-(-k+2n)(\alpha/\beta)}}{\beta_1 + (-k+2n)(\alpha/\beta)} = L_6, \quad (21)$$
$$n = 2, 3, \cdots.$$

Finally from (20) and (21) we get

$$L_5 \leq x_n \leq U_5,$$
$$L_6 \leq y_n \leq U_6, \quad (22)$$
$$n = 3, 4, \cdots.$$

□

Theorem 2. *If $2n \geq k$, then following holds:*

(i) *If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a +ve solution of (11), then $[L_1, U_1] \times [L_2, U_2]$ is invariant rectangle for (11).*

(ii) *If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a +ve solution of (12), then $[L_3, U_3] \times [L_4, U_4]$ is invariant rectangle for (12).*

(iii) *If $\{(x_n, y_n)\}_{n=-k}^{\infty}$ is a +ve solution of (13), then $[L_5, U_5] \times [L_6, U_6]$ is invariant rectangle for (13).*

Proof. The proof follows from mathematical induction. □

3. Existence and Uniqueness of +ve Equilibrium of Systems (11), (12), and (13)

Theorem 3. *Assume that $2n \geq k$, then the following statements hold:*

(i) *If $0 < \beta < 1$ and*

$$\alpha e^{-(-k+2n)L_1} \left(\beta_1 + (-k+2n)U_1\right)\left((-k+2n)U_1 + 1\right) \cdot \left(\alpha e^{-(-k+2n)L_1} + (1-\beta)U_1\right) < (-k+2n) \cdot L_1^2 \left(\alpha e^{-(-k+2n)U_1} - \beta U_1\right), \quad (23)$$

then (11) has a unique +ve equilibrium $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$.

(ii) *If*

$$e^{(-k+2n)U_3}\left(\beta + 2(-k+2n)U_3\right) \cdot \left(\beta_1 + 2\ln\left(\frac{\alpha}{(\beta + (-k+2n)L_3)L_3}\right)\right) < \alpha_1 (-k+2n)^2 \left(\beta + (-k+2n)L_3\right)L_3, \quad (24)$$

then (12) has a unique +ve equilibrium $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$.

(iii) *If*

$$\alpha_1 (2n-k)^2 e^{-(2n-k)L_5}\left(\beta\left(\beta_1 + (2n-k)U_5\right) + (2n-k)\alpha_1 e^{-(2n-k)L_5}\right)U_5\Omega_1 < \left(\beta_1 + (2n-k)L_5\right)^2 \Omega_2^2, \quad (25)$$

then (13) has a unique +ve equilibrium $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ where Ω_1 and Ω_2 are defined in (53) and (54).

Proof. (i) Consider

$$x = \frac{\alpha e^{-(2n-k)x}}{\beta + (2n-k)y},$$
$$y = \frac{\alpha_1 e^{-(2n-k)y}}{\beta_1 + (2n-k)x}. \quad (26)$$

From (26), we have

$$y = \frac{\alpha e^{-(-k+2n)x}}{(-k+2n)x} - \frac{\beta}{-k+2n},$$
$$x = \frac{\alpha_1 e^{-(-k+2n)y}}{(-k+2n)y} - \frac{\beta_1}{-k+2n}. \quad (27)$$

From (27), define

$$Q(x) = \frac{\alpha_1 e^{-(-k+2n)q(x)}}{(-k+2n)q(x)} - \frac{\beta_1}{-k+2n} - x, \quad (28)$$

where

$$q(x) = \frac{\alpha e^{-(-k+2n)x}}{(-k+2n)x} - \frac{\beta}{-k+2n}, \quad (29)$$

and $x \in [L_1, U_1]$. Now our claim is $Q(x) = 0$ has a single solution $x \in [L_1, U_1]$. From (28) and (29) one gets

$$Q'(x) = -\frac{\alpha_1 e^{-(-k+2n)q(x)} q'(x)\left((-k+2n)q(x) + 1\right)}{(-k+2n)\left(q(x)\right)^2} - 1, \quad (30)$$

where

$$q'(x) = -\frac{\alpha e^{-(-k+2n)x}\left((-k+2n)x + 1\right)}{(-k+2n)x^2}. \quad (31)$$

If $\overline{x} \in [L_1, U_1]$ is a solution of $Q(x) = 0$, then from (28) and (29) one gets

$$\frac{\alpha_1 e^{-(-k+2n)q(\overline{x})}}{q(\overline{x})} = \beta_1 + (-k+2n)\overline{x}, \quad (32)$$

where

$$q(\overline{x}) = \frac{\alpha e^{-(-k+2n)\overline{x}}}{(-k+2n)\overline{x}} - \frac{\beta}{-k+2n}. \quad (33)$$

In view of (31), (32), and (33), (30) becomes

$$\begin{aligned} Q'(x) &= \frac{\alpha e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x}\right)\left((-k+2n)\,\overline{x}+1\right)\left(\alpha e^{-(-k+2n)\overline{x}} + (1-\beta)\,\overline{x}\right)}{(-k+2n)\,\overline{x}^2\left(\alpha e^{-(-k+2n)\overline{x}} - \beta\overline{x}\right)} - 1, \\ &\le \frac{\alpha e^{-(-k+2n)L_1}\left(\beta_1 + (-k+2n)\,U_1\right)\left((-k+2n)\,U_1+1\right)\left(\alpha e^{-(-k+2n)L_1} + (1-\beta)\,U_1\right)}{(-k+2n)\,L_1^2\left(\alpha e^{-(-k+2n)U_1} - \beta U_1\right)} - 1. \end{aligned} \tag{34}$$

Now assume that (23) holds, then from (34) one gets $Q'(x) < 0$.

(ii). From (12) one has

$$\begin{aligned} x &= \frac{\alpha e^{-(-k+2n)y}}{\beta + (-k+2n)\,x}, \\ y &= \frac{\alpha_1 e^{-(-k+2n)x}}{\beta_1 + (-k+2n)\,y}. \end{aligned} \tag{35}$$

From (35) one gets

$$\begin{aligned} y &= \frac{1}{-k+2n}\ln\left(\frac{\alpha}{\left(\beta + (-k+2n)\,x\right)x}\right), \\ x &= \frac{1}{-k+2n}\ln\left(\frac{\alpha_1}{\left(\beta_1 + (-k+2n)\,y\right)y}\right). \end{aligned} \tag{36}$$

From (36), define

$$\begin{aligned} G(x) = {} & \frac{1}{-k+2n}\ln\left(\frac{\alpha_1}{\left(\beta_1 + (-k+2n)\,g(x)\right)g(x)}\right) \\ & - x, \end{aligned} \tag{37}$$

where

$$g(x) = \frac{1}{-k+2n}\ln\left(\frac{\alpha}{\left(\beta + (-k+2n)\,x\right)x}\right), \tag{38}$$

and $x \in [L_3, U_3]$. We claim that $G(x) = 0$ has a single solution $x \in [L_3, U_3]$. From (37) and (38) one gets

$$G'(x) = -\frac{g'(x)\left(\beta_1 + 2\,(-k+2n)\,g(x)\right)}{(-k+2n)\left(\beta_1 + (-k+2n)\,g(x)\right)g(x)} - 1, \tag{39}$$

where

$$g'(x) = -\frac{\beta + 2\,(2n-k)\,x}{(2n-k)\left(\beta + (2n-k)\,x\right)x}. \tag{40}$$

If $\overline{x} \in [L_3, U_3]$ is a solution of $G(x) = 0$, then from (37) and (38) one gets

$$\left(\beta_1 + (-k+2n)\,g(\overline{x})\right)g(\overline{x}) = \alpha_1 e^{-(-k+2n)\overline{x}}, \tag{41}$$

where

$$g(\overline{x}) = \frac{1}{2n-k}\ln\left(\frac{\alpha}{\left(\beta + (2n-k)\,\overline{x}\right)\overline{x}}\right). \tag{42}$$

In view of (40), (41), and (42), (39) becomes

$$\begin{aligned} G'(x) &= \frac{e^{(2n-k)\overline{x}}\left(\beta + 2\,(-k+2n)\,\overline{x}\right)\left(\beta_1 + 2\ln\left(\alpha/\left(\beta + (-k+2n)\,\overline{x}\right)\overline{x}\right)\right)}{\alpha_1\,(-k+2n)^2\left(\beta + (-k+2n)\,\overline{x}\right)\overline{x}} - 1, \\ &\le \frac{e^{(-k+2n)U_3}\left(\beta + 2\,(-k+2n)\,U_3\right)\left(\beta_1 + 2\ln\left(\alpha/\left(\beta + (-k+2n)\,L_3\right)L_3\right)\right)}{\alpha_1\,(-k+2n)^2\left(\beta + (-k+2n)\,L_3\right)L_3} - 1. \end{aligned} \tag{43}$$

Finally assume that (24) holds, then from (43) one gets $G'(x) < 0$.

(iii) From system (13)

$$\begin{aligned} x &= \frac{\alpha e^{-(2n-k)y}}{\beta + (2n-k)\,y}, \\ y &= \frac{\alpha_1 e^{-(2n-k)x}}{\beta_1 + (2n-k)\,x}. \end{aligned} \tag{44}$$

From (44), define

$$J(x) = \frac{\alpha e^{-(-k+2n)\,j(x)}}{\beta + (-k+2n)\,j(x)} - x, \tag{45}$$

where

$$j(x) = \frac{\alpha_1 e^{-(-k+2n)x}}{\beta_1 + (-k+2n)\,x}, \tag{46}$$

and $x \in [L_5, U_5]$. We claim that $J(x) = 0$ has a single solution $x \in [L_5, U_5]$. From (45) and (46) one gets

$$J'(x) = -\frac{\alpha(-k+2n)e^{-(-k+2n)j(x)}j'(x)(\beta+(-k+2n)j(x)+1)}{(\beta+(-k+2n)j(x))^2} - 1, \quad (47)$$

where

$$j'(x) = -\frac{\alpha_1(2n-k)e^{-(2n-k)x}(\beta_1+(2n-k)x+1)}{(\beta_1+(2n-k)x)^2}. \quad (48)$$

If $\overline{x} \in [L_5, U_5]$ is a solution of $J(x) = 0$, then from (45) and (46) one gets

$$e^{-(-k+2n)j(\overline{x})} = \frac{(\beta+(-k+2n)j(\overline{x}))\overline{x}}{\alpha}, \quad (49)$$

where

$$j(\overline{x}) = \frac{\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1+(-k+2n)\overline{x}}. \quad (50)$$

In view of (48), (49), and (50), (47) becomes

$$\begin{aligned} J'(x) &= \frac{\alpha_1(-k+2n)^2 e^{-(-k+2n)\overline{x}}\left(\beta(\beta_1+(2n-k)\overline{x})+(-k+2n)\alpha_1 e^{-(-k+2n)\overline{x}}\right)\overline{x}\Omega}{(\beta_1+(-k+2n)\overline{x})^2(\beta_1(\beta_1+(-k+2n)\overline{x})+(-k+2n)\alpha_1 e^{-(-k+2n)\overline{x}})^2} - 1, \\ &\le \frac{\alpha_1(-k+2n)^2 e^{-(-k+2n)L_5}\left(\beta(\beta_1+(-k+2n)U_5)+(-k+2n)\alpha_1 e^{-(-k+2n)L_5}\right)U_5\Omega_1}{(\beta_1+(-k+2n)L_5)^2\Omega_2^2} - 1, \end{aligned} \quad (51)$$

where

$$\Omega = (\beta+1)(\beta_1+(2n-k)\overline{x}) + (-k+2n)\alpha_1 e^{-(2n-k)\overline{x}}, \quad (52)$$

$$\Omega_1 = (\beta+1)(\beta_1+(2n-k)U_5) + (-k+2n)\alpha_1 e^{-(2n-k)L_5}, \quad (53)$$

and

$$\Omega_2 = \beta_1(\beta_1+(2n-k)L_5) + (2n-k)\alpha_1 e^{-(2n-k)U_5}. \quad (54)$$

Finally assume that (25) along with (53) and (54) holds, then from (51) one gets $J'(x) < 0$. □

4. Local Asymptotic Stability about Equilibrium of Systems (11), (12), and (13)

Theorem 4. *Assume that $2n \ge k$, then the following statements hold:*

(i) *If*

$$(-k+n)\alpha e^{-(2n-k)L_1}(\beta+(2n-k)U_2+1) + n\alpha e^{-(2n-k)U_1}(\beta+(2n-k)L_2+1) < (\beta+(2n-k)L_2)^2, \quad (55)$$

and

$$(-k+n)\alpha_1 e^{-(-k+2n)L_2}(\beta_1+(-k+2n)U_1+1) + n\alpha_1 e^{-(-k+2n)U_2}(\beta_1+(-k+2n)L_1+1) < (\beta_1+(-k+2n)L_1)^2, \quad (56)$$

then $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ of (11) is a sink.

(ii) *If*

$$(-k+n)\alpha e^{-(-k+2n)L_4}(\beta+(-k+2n)U_3+1) + n\alpha e^{-(-k+2n)U_4}(\beta+(-k+2n)L_3+1) < (\beta+(-k+2n)L_3)^2, \quad (57)$$

and

$$(-k+n)\alpha_1 e^{-(2n-k)L_3}(\beta_1+(2n-k)U_4+1) + n\alpha_1 e^{-(2n-k)U_3}(\beta_1+(2n-k)L_4+1) < (\beta_1+(2n-k)L_4)^2, \quad (58)$$

then $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ of (12) is a sink.

(iii) *If*

$$(-k+n)\alpha e^{-(-k+2n)L_6}(\beta+(-k+2n)U_6+1) + n\alpha e^{-(-k+2n)U_6}(\beta+(-k+2n)L_6+1) < (\beta+(-k+2n)L_6)^2, \quad (59)$$

and

$$(-k+n)\alpha_1 e^{-(-k+2n)L_5}(\beta_1+(-k+2n)U_5+1) + n\alpha_1 e^{-(-k+2n)U_5}(\beta_1+(-k+2n)L_5+1) < (\beta_1+(-k+2n)L_5)^2, \quad (60)$$

then $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ of (13) is a sink.

Proof. (i) If $(\overline{x}, \overline{y})$ is equilibrium of (11), then

$$\begin{aligned} \overline{x} &= \frac{\alpha e^{-(-k+2n)\overline{x}}}{\beta + (-k+2n)\,\overline{y}}, \\ \overline{y} &= \frac{\alpha_1 e^{-(-k+2n))\overline{y}}}{\beta_1 + (-k+2n)\,\overline{x}}. \end{aligned} \tag{61}$$

Moreover, linearized equation of (11) about $(\overline{x}, \overline{y})$ is

$$\Phi_{n+1} = E\Phi_n, \tag{62}$$

where

$$\Phi_n = \left(x_n, \cdots, x_{n-k}, y_n, \cdots, y_{n-k}\right)^T, \tag{63}$$

$$E = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & \mu_2 & \mu_3 & 0 & \dots & 0 & \mu_4 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_5 & 0 & \dots & 0 & \mu_6 & \mu_7 & 0 & \dots & 0 & \mu_8 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{64}$$

and

$$\begin{aligned} \mu_1 &= -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{\beta + (-k+2n)\,\overline{y}}, \\ \mu_2 &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{\beta + (2n-k)\,\overline{y}}, \\ \mu_3 &= -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{\left(\beta + (2n-k)\,\overline{y}\right)^2}, \\ \mu_4 &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{\left(\beta + (2n-k)\,\overline{y}\right)^2}, \\ \mu_5 &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}, \\ \mu_6 &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}, \\ \mu_7 &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1 + (-k+2n)\,\overline{x}}, \\ \mu_8 &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1 + (-k+2n)\,\overline{x}}. \end{aligned} \tag{65}$$

Let eigenvalues of E be $\lambda_i, i = 1, 2, \cdots, 2k+2$. Let $D = \mathrm{diag}(\tau_1, \tau_2, \dots, \tau_{2k+2})$ be a diagonal matrix with $\tau_1 = \tau_{k+2} = 1$, $\tau_{1+m} = \tau_{k+2+m} = 1 - m\epsilon$ $(1 \le m \le k)$, and

$$\begin{aligned} 0 < \epsilon < \min \Bigg\{ & \frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta + (-k+2n)\,\overline{y}\right)^2 - n\alpha e^{-(-k+2n)\overline{x}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}\right), \\ & \frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2 - n\alpha_1 e^{-(-k+2n)\overline{y}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}\right)\Bigg\}. \end{aligned} \tag{66}$$

Since D is invertible. So,

$$DED^{-1} = \begin{pmatrix} \tau_1\tau_1^{-1}\mu_1 & 0 & \dots & 0 & \tau_1\tau_{k+1}^{-1}\mu_2 & \tau_1\tau_{k+2}^{-1}\mu_3 & 0 & \dots & 0 & \tau_1\tau_{2k+2}^{-1}\mu_4 \\ \tau_2\tau_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \tau_{k+1}\tau_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \tau_{k+2}\tau_1^{-1}\mu_5 & 0 & \dots & 0 & \tau_{k+2}\tau_{k+1}^{-1}\mu_6 & \tau_{k+2}\tau_{k+2}^{-1}\mu_7 & 0 & \dots & 0 & \tau_{k+2}\tau_{2k+2}^{-1}\mu_8 \\ 0 & 0 & \dots & 0 & 0 & \tau_{k+3}\tau_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \tau_{2k+2}\tau_{2k+1}^{-1} & 0 \end{pmatrix}. \tag{67}$$

The following two inequalities,

$$\begin{aligned} &\tau_1 > \tau_2 > \cdots > \tau_{k+1} > 0, \\ &\tau_{k+2} > \tau_{k+3} > \cdots > \tau_{2k+2} > 0, \end{aligned} \tag{68}$$

imply that

$$\begin{aligned} \tau_2\tau_1^{-1} &< 1, \\ \tau_3\tau_2^{-1} &< 1, \\ &\vdots \\ \tau_{k+1}\tau_k^{-1} &< 1, \\ \tau_{k+3}\tau_{k+2}^{-1} &< 1, \\ \tau_{k+4}\tau_{k+3}^{-1} &< 1, \\ &\vdots \\ \tau_{2k+2}\tau_{2k+1}^{-1} &< 1. \end{aligned} \tag{69}$$

Furthermore,

$$\begin{aligned} &\tau_1\tau_1^{-1}\mu_1 + \tau_1\tau_{k+1}^{-1}\mu_2 + \tau_1\tau_{k+2}^{-1}\mu_3 + \tau_1\tau_{2k+2}^{-1}\mu_4 \\ &= \frac{\alpha e^{-(-k+2n)\overline{x}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta + (-k+2n)\,\overline{y}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \\ &< 1. \end{aligned} \tag{70}$$

$$\begin{aligned} &\tau_{k+2}\tau_1^{-1}\mu_5 + \tau_{k+2}\tau_{k+1}^{-1}\mu_6 + \mu_7 + \tau_{k+2}\tau_{2k+2}^{-1}\mu_8 \\ &= \frac{\alpha_1 e^{-(-k+2n)\overline{y}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \\ &< 1. \end{aligned} \tag{71}$$

It is a well-known fact that E and DED^{-1} have the same eigenvalues. Thus

$$\begin{aligned} \max_{1\le m\le 2k+2}|\lambda_m| \le \left\|DED^{-1}\right\|_\infty = \max\Bigg\{ &\tau_2\tau_1^{-1}, \cdots, \\ &\tau_{k+1}\tau_k^{-1}, \tau_{k+3}\tau_{k+2}^{-1}, \cdots, \tau_{2k+2}\tau_{2k+1}^{-1}, \\ &\frac{\alpha e^{-(-k+2n)\overline{x}}\left(\beta + (2n-k)\,\overline{y} + 1\right)}{\left(\beta + (-k+2n)\,\overline{y}\right)^2}\left(n + \frac{n-k}{1-k\epsilon}\right), \\ &\frac{\alpha_1 e^{-(-k+2n)\overline{y}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (2n-k)\,\overline{x}\right)^2}\left(n + \frac{2n-k}{1-k\epsilon}\right)\Bigg\} < 1. \end{aligned} \tag{72}$$

Hence $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ of (11) is a sink.

(ii). If $(\overline{x}, \overline{y})$ is equilibrium of (12), then

$$\begin{aligned} \overline{x} &= \frac{\alpha e^{-(-k+2n)\overline{y}}}{\beta + (2n-k)\,\overline{x}}, \\ \overline{y} &= \frac{\alpha_1 e^{-(2n-k))\overline{x}}}{\beta_1 + (2n-k)\,\overline{y}}, \end{aligned} \tag{73}$$

and linearized equation of (12) about $(\overline{x}, \overline{y})$ is

$$\Phi_{n+1} = E\Phi_n, \tag{74}$$

where Φ_n is defined in (63). Also

$$E = \begin{pmatrix} \mu_9 & 0 & \dots & 0 & \mu_{10} & \mu_{11} & 0 & \dots & 0 & \mu_{12} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_{13} & 0 & \dots & 0 & \mu_{14} & \mu_{15} & 0 & \dots & 0 & \mu_{16} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{75}$$

and

$$\begin{aligned} \mu_9 &= -\frac{n\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta + (-k+2n)\,\overline{x}\right)^2}, \\ \mu_{10} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta + (-k+2n)\,\overline{x}\right)^2}, \\ \mu_{11} &= -\frac{n\alpha e^{-(-k+2n)\overline{y}}}{\beta + (-k+2n)\,\overline{x}}, \\ \mu_{12} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\beta + (2n-k)\,\overline{x}}, \\ \mu_{13} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1 + (-k+2n)\,\overline{y}}, \\ \mu_{14} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1 + (-k+2n)\,\overline{y}}, \\ \mu_{15} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2}, \\ \mu_{16} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2}. \end{aligned} \tag{76}$$

Using arrangements as in the proof of (i) one gets

$$0 < \epsilon < \min\left\{\frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta + (-k+2n)\,\overline{x}\right)^2 - n\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{x} + 1\right)}\right),\right.$$
$$\left.\frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2 - n\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{y} + 1\right)}\right)\right\}, \tag{77}$$

and

$$DED^{-1} = \begin{pmatrix} \tau_1\tau_1^{-1}\mu_9 & 0 \ldots & 0 & -\tau_1\tau_{k+1}^{-1}\mu_{10} & -\tau_1\tau_{k+2}^{-1}\mu_{11} & 0 \ldots & 0 & -\tau_1\tau_{2k+2}^{-1}\mu_{12} \\ \tau_2\tau_1^{-1} & 0 \ldots & 0 & 0 & 0 & 0 \ldots & 0 & 0 \\ \vdots & \vdots \ldots & \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots \\ 0 & 0 \ldots & \tau_{k+1}\tau_k^{-1} & 0 & 0 & 0 \ldots & 0 & 0 \\ -\tau_{k+2}\tau_1^{-1}\mu_{13} & 0 \ldots & 0 & -\tau_{k+2}\tau_{k+1}^{-1}\mu_{14} & -\tau_{k+2}\tau_{k+2}^{-1}\mu_{15} & 0 \ldots & 0 & -\tau_{k+2}\tau_{2k+2}^{-1}\mu_{16} \\ 0 & 0 \ldots & 0 & 0 & \tau_{k+3}\tau_{k+2}^{-1} & 0 \ldots & 0 & 0 \\ \vdots & \vdots \ddots & \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots \\ 0 & 0 \ldots & 0 & 0 & 0 & 0 \ldots & \tau_{2k+2}\tau_{2k+1}^{-1} & 0 \end{pmatrix}. \tag{78}$$

□

Assume that (68) and (69) hold, then

$$\mu_9 + \tau_1\tau_{k+1}^{-1}\mu_{10} + \tau_1\tau_{k+2}^{-1}\mu_{11} + \tau_1\tau_{2k+2}^{-1}\mu_{12}$$
$$= \frac{\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta + (-k+2n)\,\overline{x}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \tag{79}$$
$$< 1.$$

$$\mu_{13} + \tau_{k+2}\tau_{k+1}^{-1}\mu_{14} + \mu_{15} + \tau_{k+2}\tau_{2k+2}^{-1}\mu_{16}$$
$$= \frac{\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \tag{80}$$
$$< 1.$$

Finally,

$$\max_{1\le m\le 2k+2}|\lambda_m| \le \left\|DED^{-1}\right\|_\infty = \max\left\{\tau_2\tau_1^{-1}, \cdots,\right.$$
$$\left.\tau_{k+1}\tau_k^{-1}, \tau_{k+3}\tau_{k+2}^{-1}, \cdots, \tau_{2k+2}\tau_{2k+1}^{-1},\right.$$
$$\frac{\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta + (-k+2n)\,\overline{x}\right)^2}\left(n + \frac{n-k}{1-k\epsilon}\right),$$
$$\left.\frac{\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2}\left(n + \frac{n-k}{1-k\epsilon}\right)\right\} < 1. \tag{81}$$

Hence equilibrium $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ (12) is a sink.

(iii) If $(\overline{x}, \overline{y})$ is a fixed point of (13), then

$$\overline{x} = \frac{\alpha e^{-(-k+2n)\overline{y}}}{\beta + (-k+2n)\,\overline{y}},$$
$$\overline{y} = \frac{\alpha_1 e^{-(-k+2n))\overline{x}}}{\beta_1 + (-k+2n)\,\overline{x}}, \tag{82}$$

and linearized equation of (13) about $(\overline{x}, \overline{y})$ is

$$\Phi_{n+1} = E\Phi_n, \tag{83}$$

where Φ_n is again the same as (63). Moreover

$$E = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \mu_{17} & 0 & \dots & 0 & \mu_{18} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_{19} & 0 & \dots & 0 & \mu_{20} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad (84)$$

where

$$\begin{aligned} \mu_{17} &= -\frac{n\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta + (2n-k)\,\overline{y}\right)^2}, \\ \mu_{18} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (2n-k)\,\overline{y} + 1\right)}{\left(\beta + (2n-k)\,\overline{y}\right)^2}, \\ \mu_{19} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}, \\ \mu_{20} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}. \end{aligned} \quad (85)$$

Again using similar arrangements as in the proof of (*i*) one gets

$$\begin{aligned} 0 < \epsilon < \min\Bigg\{ &\frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta + (-k+2n)\,\overline{y}\right)^2 - n\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}\right), \\ &\frac{1}{k}\left(1 - \frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{y}\right)^2 - n\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}\right)\Bigg\}. \end{aligned} \quad (86)$$

Moreover,

$$DED^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -\tau_1\tau_{k+2}^{-1}\mu_{17} & 0 & \dots & 0 & -\tau_1\tau_{2k+2}^{-1}\mu_{18} \\ \tau_2\tau_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \tau_{k+1}\tau_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ -\tau_{k+2}\tau_1^{-1}\mu_{19} & 0 & \dots & 0 & -\tau_{k+2}\tau_{k+1}^{-1}\mu_{20} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \tau_{k+3}\tau_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \tau_{2k+2}\tau_{2k+1}^{-1} & 0 \end{pmatrix}. \quad (87)$$

Assume that (68) and (69) hold true, then

$$\begin{aligned} &\tau_1\tau_{k+2}^{-1}\mu_{17} + \tau_1\tau_{2k+2}^{-1}\mu_{18} \\ &= \frac{\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (2n-k)\,\overline{y} + 1\right)}{\left(\beta + (2n-k)\,\overline{y}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \quad (88) \\ &< 1. \end{aligned}$$

$$\begin{aligned} &\tau_{k+2}\tau_1^{-1}\mu_{19} + \tau_{k+2}\tau_{k+1}^{-1}\mu_{20} \\ &= \frac{\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}\left(n + \frac{-k+n}{1-k\epsilon}\right) \quad (89) \\ &< 1. \end{aligned}$$

Finally,

$$\begin{aligned} \max_{1\le m\le 2k+2}\left|\lambda_m\right| \le \left\|DED^{-1}\right\|_\infty = \max\Bigg\{ &\tau_2\tau_1^{-1}, \cdots, \\ &\tau_{k+1}\tau_k^{-1}, \tau_{k+3}\tau_{k+2}^{-1}, \cdots, \tau_{2k+2}\tau_{2k+1}^{-1}, \\ &\frac{\alpha e^{-(-k+2n)\overline{y}}\left(\beta + (-k+2n)\,\overline{y} + 1\right)}{\left(\beta + (-k+2n)\,\overline{y}\right)^2}\left(n + \frac{n--k}{1-k\epsilon}\right), \quad (90) \\ &\frac{\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1 + (-k+2n)\,\overline{x} + 1\right)}{\left(\beta_1 + (-k+2n)\,\overline{x}\right)^2}\Bigg(n \\ &+ \frac{n-k}{1-k\epsilon}\Bigg)\Bigg\} < 1. \end{aligned}$$

Hence equilibrium $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ of (13) is a sink.

Hereafter we study the global dynamics about unique $+ve$ equilibrium of systems (11), (12), and (13). Since investigating the global stability of difference equations or systems of difference equation for higher-order is a challenging task in recent year, here we will investigate the global dynamics about unique $+ve$ equilibrium of systems (11), (12), and (13) by utilizing discrete-time Lyapunov function motivated by the work of [14–16].

5. Global Stability about Equilibrium of Systems (11), (12), and (13)

Theorem 5. *Assume that $2n \geq k$, then the following statements hold:*

(i) *If*

$$\alpha e^{-(-k+2n)L_1} < \overline{x}\left(\beta + (-k+2n)\, L_2\right), \tag{91}$$

and

$$\alpha_1 e^{-(-k+2n)L_2} < \overline{y}\left(\beta_1 + (-k+2n)\, L_1\right), \tag{92}$$

then $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ of (11) is globally asymptotically stable.

(ii) *If*

$$\alpha e^{-(-k+2n)L_4} < \overline{x}\left(\beta + (2n-k)\, L_3\right), \tag{93}$$

and

$$\alpha_1 e^{-(-k+2n)L_3} < \overline{y}\left(\beta_1 + (-k+2n)\, L_4\right), \tag{94}$$

then $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ of (12) is globally asymptotically stable.

(iii) *If*

$$\alpha e^{-(-k+2n)L_6} < \overline{x}\left(\beta + (-k+2n)\, L_6\right), \tag{95}$$

and

$$\alpha_1 e^{-(-k+2n)L_5} < \overline{y}\left(\beta_1 + (-k+2n)\, L_5\right), \tag{96}$$

then $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ of (13) is globally asymptotically stable.

Proof. (i) Consider

$$\Gamma_n = \overline{x}\left(-1 + \frac{x_n}{\overline{x}} - \ln\frac{x_n}{\overline{x}}\right) + \overline{y}\left(-1 + \frac{y_n}{\overline{y}} - \ln\frac{y_n}{\overline{y}}\right), \tag{97}$$

where its positivity follows from the inequality below:

$$-1 + x - \ln x \geq 0, \quad \forall x > 0. \tag{98}$$

Moreover,

$$\begin{aligned} -\ln\left(\frac{x_{n+1}}{x_n}\right) &= \ln\left(-\left(1 - \frac{x_n}{x_{n+1}}\right) + 1\right) \\ &\leq -\frac{x_{n+1} - x_n}{x_{n+1}}, \\ -\ln\left(\frac{y_{n+1}}{y_n}\right) &= \ln\left(-\left(1 - \frac{y_n}{y_{n+1}}\right) + 1\right) \\ &\leq -\frac{y_{n+1} - y_n}{y_{n+1}}. \end{aligned} \tag{99}$$

Now

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n = {} & (x_{n+1} - x_n) - \overline{x}\ln\frac{x_{n+1}}{x_n} + (y_{n+1} - y_n) \\ & - \overline{y}\ln\frac{y_{n+1}}{y_n}. \end{aligned} \tag{100}$$

In view of (99), (100) takes the form

$$\begin{aligned} \Gamma_{n+1} - \Gamma_n &\leq (x_{n+1} - x_n)\left(1 - \frac{\overline{x}}{x_{n+1}}\right) + (y_{n+1} - y_n) \\ &\quad \cdot \left(1 - \frac{\overline{y}}{y_{n+1}}\right), \\ &= (x_{n+1} - x_n) \\ &\quad \cdot \left(\frac{\alpha e^{-(nx_n + (n-k)x_{n-k})} - \overline{x}\left(\beta + ny_n + (n-k)\, y_{n-k}\right)}{\alpha e^{-(nx_n + (n-k)x_{n-k})}}\right) \\ &\quad + (y_{n+1} - y_n) \\ &\quad \cdot \left(\frac{\alpha_1 e^{-(ny_n + (n-k)y_{n-k})} - \overline{y}\left(\beta_1 + nx_n + (n-k)\, x_{n-k}\right)}{\alpha_1 e^{-(ny_n + (n-k)y_{n-k})}}\right), \\ &\leq (U_1 - L_1)\left(\frac{\alpha e^{-(-k+2n)L_1} - \overline{x}\left(\beta + (-k+2n)\, L_2\right)}{\alpha e^{-(-k+2n)L_1}}\right) \\ &\quad + (U_2 - L_2) \\ &\quad \cdot \left(\frac{\alpha_1 e^{-(-k+2n)L_2} - \overline{y}\left(\beta_1 + (-k+2n)\, L_1\right)}{\alpha_1 e^{-(-k+2n)L_2}}\right). \end{aligned} \tag{101}$$

Assuming that (91) and (92) hold, then from (101) one gets $\Gamma_{n+1} - \Gamma_n < 0$ for all $n \geq 0$. Hence, $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Therefore, $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ of (11) is globally asymptotically stable.

(ii). Using similar construction as in the proof of (*i*) one gets

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n &\le (x_{n+1}-x_n)\\ &\cdot\left(\frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}-\overline{x}\left(\beta+nx_n+(n-k)\,x_{n-k}\right)}{\alpha e^{-(ny_n+(n-k)y_{n-k})}}\right)\\ &+(y_{n+1}-y_n)\\ &\cdot\left(\frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}-\overline{y}\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)}{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}\right),\\ &\le (U_3-L_3)\left(\frac{\alpha e^{-(-k+2n)L_4}-\overline{x}\left(\beta+(-k+2n)\,L_3\right)}{\alpha e^{-(-k+2n)L_4}}\right)\\ &+(U_4-L_4)\\ &\cdot\left(\frac{\alpha_1 e^{-(-k+2n)L_3}-\overline{y}\left(\beta_1+(-k+2n)\,L_4\right)}{\alpha_1 e^{-(-k+2n)L_3}}\right).\end{aligned} \tag{102}$$

Assuming that (93) and (94) hold, then from (102) one gets $\Gamma_{n+1}-\Gamma_n<0$ for all $n\ge 0$. Hence, $\lim_{n\longrightarrow\infty}(\Gamma_{n+1}-\Gamma_n)=0$. Therefore, $(\overline{x},\overline{y})\in[L_3,U_3]\times[L_4,U_4]$ of (12) is globally asymptotically stable.

(iii) Again using similar arrangements, one gets

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n &\le (x_{n+1}-x_n)\\ &\cdot\left(\frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}-\overline{x}\left(\beta+ny_n+(n-k)\,y_{n-k}\right)}{\alpha e^{-(ny_n+(n-k)y_{n-k})}}\right)\\ &+(y_{n+1}-y_n)\\ &\cdot\left(\frac{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}-\overline{y}\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)}{\alpha_1 e^{-(nx_n+(n-k)x_{n-k})}}\right),\\ &\le (U_5-L_5)\left(\frac{\alpha e^{-(-k+2n)L_6}-\overline{x}\left(\beta+(-k+2n)\,L_6\right)}{\alpha e^{-(-k+2n)L_6}}\right)\\ &+(U_6-L_6)\\ &\cdot\left(\frac{\alpha_1 e^{-(-k+2n)L_5}-\overline{y}\left(\beta_1+(-k+2n)\,L_5\right)}{\alpha_1 e^{-(-k+2n)L_5}}\right).\end{aligned} \tag{103}$$

Assuming that (95) and (96) hold, then from (103) one gets $\Gamma_{n+1}-\Gamma_n<0$ for all $n\ge 0$. Hence, $\lim_{n\longrightarrow\infty}(\Gamma_{n+1}-\Gamma_n)=0$. Thus, $(\overline{x},\overline{y})\in[L_5,U_5]\times[L_6,U_6]$ of (13) is globally asymptotically stable. □

Hereafter we will study rate of convergence of systems (11), (12), and (13) motivated by the work of [15, 17, 18].

6. Rate of Convergence of Systems (11), (12), and (13)

6.1. Rate of Convergence of System (11). Let $\{(x_n, y_n)\}$ be any solution of (11) such that the following hold:

$$\begin{aligned}\lim_{n\longrightarrow\infty}x_n&=\overline{x},\\ \lim_{n\longrightarrow\infty}y_n&=\overline{y}.\end{aligned} \tag{104}$$

Now computing error terms one gets

$$\begin{aligned}x_{n+1}-\overline{x} &= \frac{\alpha e^{-(nx_n+(n-k)x_{n-k})}}{\beta+ny_n+(n-k)\,y_{n-k}}\\ &-\frac{\alpha e^{-(-k+2n)\overline{x}}}{\beta+(-k+2n)\,\overline{y}},\\ &=-\frac{\alpha e^{-(nx_n+(n-k)x_{n-k})}\left(e^{nx_n+(n-k)x_{n-k}-(-k+2n)\overline{x}}-1\right)}{\beta+ny_n+(n-k)\,y_{n-k}}\\ &-\frac{\alpha n e^{-(-k+2n)\overline{x}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}(y_n\\ &-\overline{y})\\ &-\frac{\alpha(n-k)\,e^{-(-k+2n)\overline{x}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}(y_{n-k}\\ &-\overline{y}).\end{aligned} \tag{105}$$

After some tedious calculations, from (105) one gets

$$\begin{aligned}x_{n+1}-\overline{x} &= -\frac{\alpha n e^{-(nx_n+(n-k)x_{n-k})}}{\beta+ny_n+(n-k)\,y_{n-k}}(x_n-\overline{x})\\ &-\frac{\alpha(n-k)\,e^{-(nx_n+(n-k)x_{n-k})}}{\beta+ny_n+(n-k)\,y_{n-k}}(x_{n-k}-\overline{x})\\ &-\frac{\alpha n e^{-(-k+2n)\overline{x}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}(y_n\\ &-\overline{y})\\ &-\frac{\alpha(n-k)\,e^{-(-k+2n)\overline{x}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}(y_{n-k}\\ &-\overline{y})+O_1\left(nx_n+(n-k)\,x_{n-k}-(-k+2n)\,\overline{x}\right)^2.\end{aligned} \tag{106}$$

Similarly,

$$\begin{aligned}y_{n+1}-\overline{y} &= -\frac{\alpha_1 n e^{-(-k+2n)\overline{y}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}(x_n\\ &-\overline{x})\\ &-\frac{\alpha_1(n-k)\,e^{-(-k+2n)\overline{y}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}(x_{n-k}\\ &-\overline{x})-\frac{\alpha_1 n e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1+nx_n+(n-k)\,x_{n-k}}(y_n-\overline{x})\\ &-\frac{\alpha_1(n-k)\,e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1+nx_n+(n-k)\,x_{n-k}}(y_{n-k}-\overline{y})\\ &+O_2\left(ny_n+(n\right.\\ &\left.-k)\,y_{n-k}-(-k+2n)\,\overline{y}\right)^2.\end{aligned} \tag{107}$$

From (106) and (107), we have

$$\begin{aligned}
x_{n+1} - \overline{x} &\approx -\frac{\alpha n e^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)\, y_{n-k}} (x_n - \overline{x}) \\
&- \frac{\alpha (n-k)\, e^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)\, y_{n-k}} (x_{n-k} - \overline{x}) \\
&- \frac{\alpha n e^{-(-k+2n)\overline{x}}}{(\beta + ny_n + (n-k)\, y_{n-k})(\beta + (-k+2n)\,\overline{y})} (y_n \\
&- \overline{y}) \\
&- \frac{\alpha (n-k)\, e^{-(-k+2n)\overline{x}}}{(\beta + ny_n + (n-k)\, y_{n-k})(\beta + (-k+2n)\,\overline{y})} (y_{n-k} \\
&- \overline{y}), \\
y_{n+1} - \overline{y} &\approx -\frac{\alpha_1 n e^{-(-k+2n)\overline{y}}}{(\beta_1 + nx_n + (n-k)\, x_{n-k})(\beta_1 + (-k+2n)\,\overline{x})} (x_n \\
&- \overline{x}) \\
&- \frac{\alpha_1 (n-k)\, e^{-(-k+2n)\overline{y}}}{(\beta_1 + nx_n + (n-k)\, x_{n-k})(\beta_1 + (-k+2n)\,\overline{x})} (x_{n-k} \\
&- \overline{x}) - \frac{\alpha_1 n e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}} (y_n - \overline{x}) \\
&- \frac{\alpha_1 (n-k)\, e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}} (y_{n-k} - \overline{y}).
\end{aligned} \tag{108}$$

Let

$$\begin{aligned}
\epsilon_n^1 &= x_n - \overline{x}, \\
\epsilon_n^2 &= y_n - \overline{y},
\end{aligned} \tag{109}$$

and then (108) becomes

$$\begin{aligned}
\epsilon_{n+1}^1 &\approx A_{n_1}\epsilon_n^1 + A_{n_2}\epsilon_{n-k}^1 + A_{n_3}\epsilon_n^2 + A_{n_4}\epsilon_{n-k}^2, \\
\epsilon_{n+1}^2 &\approx A_{n_5}\epsilon_n^1 + A_{n_6}\epsilon_{n-k}^1 + A_{n_7}\epsilon_n^2 + A_{n_8}\epsilon_{n-k}^2,
\end{aligned} \tag{110}$$

where

$$\begin{aligned}
A_{n_1} &= -\frac{\alpha n e^{-(nx_n+(-k+n)x_{n-k})}}{\beta + ny_n + (-k+n)\, y_{n-k}}, \\
A_{n_2} &= -\frac{\alpha (-k+n)\, e^{-(nx_n+(n-k)x_{n-k})}}{\beta + ny_n + (n-k)\, y_{n-k}}, \\
A_{n_3} &= -\frac{\alpha n e^{-(-k+2n)\overline{x}}}{(\beta + ny_n + (n-k)\, y_{-k+n})(\beta + (-k+2n)\,\overline{y})}, \\
A_{n_4} &= -\frac{\alpha (-k+n)\, e^{-(-k+2n)\overline{x}}}{(\beta + ny_n + (n-k)\, y_{n-k})(\beta + (-k+2n)\,\overline{y})}, \\
A_{n_5} &= -\frac{\alpha_1 n e^{-(-k+2n)\overline{y}}}{(\beta_1 + nx_n + (n-k)\, x_{n-k})(\beta_1 + (-k+2n)\,\overline{x})}, \\
A_{n_6} &= -\frac{\alpha_1 (-k+n)\, e^{-(-k+2n)\overline{y}}}{(\beta_1 + nx_n + (n-k)\, x_{n-k})(\beta_1 + (-k+2n)\,\overline{x})}, \\
A_{n_7} &= -\frac{\alpha_1 n e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}}, \\
A_{n_8} &= -\frac{\alpha_1 (-k+n)\, e^{-(ny_n+(n-k)y_{n-k})}}{\beta_1 + nx_n + (n-k)\, x_{n-k}}.
\end{aligned} \tag{111}$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} A_{n_1} &= -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{\beta + (-k+2n)\,\overline{y}}, \\
\lim_{n\to\infty} A_{n_2} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{\beta + (-k+2n)\,\overline{y}}, \\
\lim_{n\to\infty} A_{n_3} &= -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{(\beta + (-k+2n)\,\overline{y})^2}, \\
\lim_{n\to\infty} A_{n_4} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{(\beta + (-k+2n)\,\overline{y})^2}, \\
\lim_{n\to\infty} A_{n_5} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{(\beta_1 + (-k+2n)\,\overline{x})^2}, \\
\lim_{n\to\infty} A_{n_6} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{(\beta_1 + (-k+2n)\,\overline{x})^2}, \\
\lim_{n\to\infty} A_{n_7} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1 + (-k+2n)\,\overline{x}}, \\
\lim_{n\to\infty} A_{n_8} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1 + (-k+2n)\,\overline{x}}.
\end{aligned} \tag{112}$$

Hence limiting system of error terms becomes

$$E_{n+1} = EE_n, \tag{113}$$

where

$$E = \begin{pmatrix} -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{\beta+(-k+2n)\,\overline{y}} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{\beta+(-k+2n)\,\overline{y}} & -\frac{n\alpha e^{-(-k+2n)\overline{x}}}{(\beta+(-k+2n)\,\overline{y})^2} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{x}}}{(\beta+(-k+2n)\,\overline{y})^2} \\ 1 & 0\ \dots\ 0 & 0 & 0 & 0\ \dots\ 0 & 0 \\ \vdots & \vdots\ \dots\ \vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots \\ 0 & 0\ \dots\ 1 & 0 & 0 & 0\ \dots\ 0 & 0 \\ -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{(\beta_1+(-k+2n)\,\overline{x})^2} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{(\beta_1+(-k+2n)\,\overline{x})^2} & -\frac{n\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1+(-k+2n)\,\overline{x}} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{y}}}{\beta_1+(-k+2n)\,\overline{x}} \\ 0 & 0\ \dots\ 0 & 0 & 1 & 0\ \dots\ 0 & 0 \\ \vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots \\ 0 & 0\ \dots\ 0 & 0 & 0 & 0\ \dots\ 1 & 0 \end{pmatrix}, \tag{114}$$

which is similar to linearized system of (11). Finally we have the following theorem by utilizing Perron's Theorems (see [19]).

Theorem 6. *Assume that $\{(x_n, y_n)\}$ is a positive solution of (11) such that (104) along with the following relation holds:*

$$\begin{aligned} \overline{x} &\in [L_1, U_1], \\ \overline{y} &\in [L_2, U_2]. \end{aligned} \tag{115}$$

Then the error vector

$$E_n = \begin{pmatrix} \epsilon_n^1 \\ \epsilon_{n-1}^1 \\ \vdots \\ \epsilon_{n-k}^1 \\ \epsilon_n^2 \\ \epsilon_{n-1}^2 \\ \vdots \\ \epsilon_{n-k}^2 \end{pmatrix} \tag{116}$$

of every solution of (11) satisfies

$$\begin{aligned} \lim_{n\longrightarrow\infty} \left(\left\|E_n\right\|\right)^{1/n} &= |\lambda E|, \\ \lim_{n\longrightarrow\infty} \frac{\left\|E_{n+1}\right\|}{\left\|E_n\right\|} &= |\lambda E|, \end{aligned} \tag{117}$$

where λE are the characteristic roots of E about $(\overline{x}, \overline{y})$.

6.2. Rate of Convergence of System (12). If $\{(x_n, y_n)\}$ is any solution of (12) such that (104) holds, then

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}} \\ &\quad - \frac{\alpha e^{-(-k+2n)\overline{y}}}{\beta+(-k+2n)\,\overline{x}}, \\ &= -\frac{\alpha n e^{-(-k+2n)\overline{y}}}{(\beta+nx_n+(n-k)\,x_{n-k})\,(\beta+(-k+2n)\,\overline{x})}\,(x_n-\overline{x}) \\ &\quad -\frac{\alpha\,(n-k)\,e^{-(-k+2n)\overline{y}}}{(\beta+nx_n+(n-k)\,x_{n-k})\,(\beta+(-k+2n)\,\overline{x})}\,(x_{n-k}-\overline{x}) \\ &\quad -\frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}\left(e^{ny_n+(n-k)y_{n-k}-(-k+2n)\overline{y}}-1\right)}{\beta+nx_n+(n-k)\,x_{n-k}}. \end{aligned} \tag{118}$$

After calculations, from (118) one gets

$$\begin{aligned} &x_{n+1} - \overline{x} \\ &\quad = -\frac{\alpha n e^{-(-k+2n)\overline{y}}}{(\beta+nx_n+(n-k)\,x_{n-k})\,(\beta+(-k+2n)\,\overline{x})}\,(x_n \\ &\quad -\overline{x}) \\ &\quad -\frac{\alpha\,(n-k)\,e^{-(-k+2n)\overline{y}}}{(\beta+nx_n+(n-k)\,x_{n-k})\,(\beta+(-k+2n)\,\overline{x})}\,(x_{n-k} \\ &\quad -\overline{x}) - \frac{\alpha n e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}}\,(y_n-\overline{y}) \\ &\quad -\frac{\alpha\,(n-k)\,e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}}\,(y_{n-k}-\overline{y}) \\ &\quad + O_3\left(ny_n+(n\right. \\ &\quad \left.-k)\,y_{n-k}-(-k+2n)\,\overline{y}\right)^2. \end{aligned} \tag{119}$$

Similarly,

$$
\begin{aligned}
y_{n+1}-\overline{y} &= -\frac{\alpha_1 n e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}}\left(x_n-\overline{x}\right)\\
&-\frac{\alpha_1\,(n-k)\,e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}}\left(x_{n-k}-\overline{x}\right)\\
&-\frac{\alpha_1 n e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)}\left(y_n\right.\\
&\left.-\overline{y}\right)\\
&-\frac{\alpha_1\,(n-k)\,e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)}\left(y_{n-k}\right.\\
&\left.-\overline{y}\right)+O_4\left(nx_n+(n-k)\,x_{n-k}-(-k+2n)\,\overline{x}\right)^2.
\end{aligned}
\tag{120}
$$

From (119) and (120), we have

$$
\begin{aligned}
&x_{n+1}-\overline{x}\\
&\quad\approx -\frac{\alpha n e^{-(-k+2n)\overline{y}}}{\left(\beta+nx_n+(n-k)\,x_{n-k}\right)\left(\beta+(-k+2n)\,\overline{x}\right)}\left(x_n\right.\\
&\quad\left.-\overline{x}\right)\\
&\quad-\frac{\alpha\,(n-k)\,e^{-(-k+2n)\overline{y}}}{\left(\beta+nx_n+(n-k)\,x_{n-k}\right)\left(\beta+(-k+2n)\,\overline{x}\right)}\left(x_{n-k}\right.\\
&\quad\left.-\overline{x}\right)-\frac{\alpha n e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}}\left(y_n-\overline{y}\right)\\
&\quad-\frac{\alpha\,(n-k)\,e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}}\left(y_{n-k}-\overline{y}\right),\\
&y_{n+1}-\overline{y}\approx -\frac{\alpha_1 n e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}}\left(x_n-\overline{x}\right)\\
&\quad-\frac{\alpha_1\,(n-k)\,e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}}\left(x_{n-k}-\overline{x}\right)\\
&\quad-\frac{\alpha_1 n e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)}\left(y_n\right.\\
&\quad\left.-\overline{y}\right)\\
&\quad-\frac{\alpha_1\,(n-k)\,e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)}\left(y_{n-k}\right.\\
&\quad\left.-\overline{y}\right).
\end{aligned}
\tag{121}
$$

Let (109) hold, then from (121) one gets

$$
\begin{aligned}
\epsilon_{n+1}^1 &\approx A_{n_9}\epsilon_n^1+A_{n_{10}}\epsilon_{n-k}^1+A_{n_{11}}\epsilon_n^2+A_{n_{12}}\epsilon_{n-k}^2,\\
\epsilon_{n+1}^2 &\approx A_{n_{13}}\epsilon_n^1+A_{n_{14}}\epsilon_{n-k}^1+A_{n_{15}}\epsilon_n^2+A_{n_{10}}\epsilon_{n-k}^2,
\end{aligned}
\tag{122}
$$

where

$$
\begin{aligned}
A_{n_9} &= -\frac{\alpha n e^{-(-k+2n)\overline{y}}}{\left(\beta+nx_n+(n-k)\,x_{n-k}\right)\left(\beta+(-k+2n)\,\overline{x}\right)},\\
A_{n_{10}} &= -\frac{\alpha\,(-k+n)\,e^{-(-k+2n)\overline{y}}}{\left(\beta+nx_n+(n-k)\,x_{n-k}\right)\left(\beta+(-k+2n)\,\overline{x}\right)},\\
A_{n_{11}} &= -\frac{\alpha n e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}},\\
A_{n_{12}} &= -\frac{\alpha\,(-k+n)\,e^{-(ny_n+(n-k)y_{n-k})}}{\beta+nx_n+(n-k)\,x_{n-k}},\\
A_{n_{13}} &= -\frac{\alpha_1 n e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}},\\
A_{n_{14}} &= -\frac{\alpha_1\,(-k+n)\,e^{-(nx_n+(n-k)x_{n-k})}}{\beta_1+ny_n+(n-k)\,y_{n-k}},\\
A_{n_{15}} &= -\frac{\alpha_1 n e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)},\\
A_{n_{16}} &= -\frac{\alpha_1\,(-k+n)\,e^{-(-k+2n)\overline{x}}}{\left(\beta_1+ny_n+(n-k)\,y_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{y}\right)}.
\end{aligned}
\tag{123}
$$

Moreover,

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}A_{n_9} &= -\frac{n\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta+(-k+2n)\,\overline{x}\right)^2},\\
\lim_{n\longrightarrow\infty}A_{n_{10}} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta+(-k+2n)\,\overline{x}\right)^2},\\
\lim_{n\longrightarrow\infty}A_{n_{11}} &= -\frac{n\alpha e^{-(2n-k)\overline{y}}}{\beta+(2n-k)\,\overline{x}},\\
\lim_{n\longrightarrow\infty}A_{n_{12}} &= -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\beta+(2n-k)\,\overline{x}},\\
\lim_{n\longrightarrow\infty}A_{n_{13}} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1+(-k+2n)\,\overline{y}},\\
\lim_{n\longrightarrow\infty}A_{n_{14}} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1+(-k+2n)\,\overline{y}},\\
\lim_{n\longrightarrow\infty}A_{n_{15}} &= -\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1+(-k+2n)\,\overline{y}\right)^2},\\
\lim_{n\longrightarrow\infty}A_{n_{16}} &= -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1+(-k+2n)\,\overline{y}\right)^2}.
\end{aligned}
\tag{124}
$$

Hence limiting system of error terms becomes

$$
E_{n+1}=EE_n,
\tag{125}
$$

where

$$E = \begin{pmatrix} -\dfrac{n\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta+(-k+2n)\,\overline{x}\right)^2} & 0 \ \dots \ 0 & -\dfrac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\left(\beta+(-k+2n)\,\overline{x}\right)^2} & -\dfrac{n\alpha e^{-(-k+2n)\overline{y}}}{\beta+(-k+2n)\,\overline{x}} & 0 \ \dots \ 0 & -\dfrac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}}{\beta+(2n-k)\,\overline{x}} \\ 1 & 0 \ \dots \ 0 & 0 & 0 & 0 \ \dots \ 0 & 0 \\ \vdots & \vdots \ \dots \ \vdots & \vdots & \vdots & \ddots \ \vdots \ \vdots & \vdots \\ 0 & 0 \ \dots \ 1 & 0 & 0 & 0 \ \dots \ 0 & 0 \\ -\dfrac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1+(-k+2n)\,\overline{y}} & 0 \ \dots \ 0 & -\dfrac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\beta_1+(-k+2n)\,\overline{y}} & -\dfrac{n\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1+(-k+2n)\,\overline{y}\right)^2} & 0 \ \dots \ 0 & -\dfrac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}}{\left(\beta_1+(-k+2n)\,\overline{y}\right)^2} \\ 0 & 0 \ \dots \ 0 & 0 & 1 & 0 \ \dots \ 0 & 0 \\ \vdots & \vdots \ \ddots \ \vdots & \vdots & \vdots & \ddots \ \vdots \ \vdots & \vdots \\ 0 & 0 \ \dots \ 0 & 0 & 0 & 0 \ \dots \ 1 & 0 \end{pmatrix}, \tag{126}$$

which is the same as linearized system of (12) about $(\overline{x}, \overline{y})$. Finally we have following theorem.

Theorem 7. *Assume that $\{(x_n, y_n)\}$ is a positive solution of (12) such that (104) along with the following relation holds:*

$$\overline{x} \in [L_3, U_3], \qquad \overline{y} \in [L_4, U_4]. \tag{127}$$

Then the error vector E_n defined in (116) satisfies

$$\lim_{n\longrightarrow\infty} \left(\|E_n\|\right)^{1/n} = |\lambda E|, \qquad \lim_{n\longrightarrow\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda E|, \tag{128}$$

where λE are the characteristic roots of E about $(\overline{x}, \overline{y})$.

6.3. Rate of Convergence of System (13). If $\{(x_n, y_n)\}$ is any solution of (13) such that (104) holds, then

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}}{\beta + ny_n + (n-k)\,y_{n-k}} - \frac{\alpha e^{-(-k+2n)\overline{y}}}{\beta+(-k+2n)\,\overline{y}}, \\ &= -\frac{\alpha e^{-(ny_n+(n-k)y_{n-k})}\left(e^{ny_n+(n-k)y_{n-k}-(-k+2n)\overline{y}}-1\right)}{\beta+ny_n+(n-k)\,y_{n-k}} \\ &\quad - \frac{\alpha n e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\left(y_n - \overline{y}\right) \\ &\quad - \frac{\alpha(-k+n)\,e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\left(y_{n-k} - \overline{y}\right). \end{aligned} \tag{129}$$

After simplifying, one gets

$$\begin{aligned} x_{n+1} - \overline{x} &= -\alpha n\left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right)\left(y_n - \overline{y}\right) \\ &\quad - \alpha(-k+n) \cdot \left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right)\left(y_{n-k} - \overline{y}\right) \\ &\quad + O_5\left(ny_n + (n-k)\,y_{n-k} - (-k+2n)\,\overline{y}\right)^2. \end{aligned} \tag{130}$$

Similarly,

$$\begin{aligned} y_{n+1} - \overline{y} &= -\alpha_1 n\left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right)\left(x_n - \overline{x}\right) \\ &\quad - \alpha_1(n-k) \cdot \left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right)\left(x_{n-k} - \overline{x}\right) \\ &\quad + O_6\left(nx_n + (n-k)\,x_{n-k} - (-k+2n)\,\overline{x}\right)^2. \end{aligned} \tag{131}$$

From (130) and (131), we have

$$\begin{aligned} x_{n+1} - \overline{x} &\approx -\alpha n\left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right)\left(y_n - \overline{y}\right) \\ &\quad - \alpha(-k+n) \cdot \left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right)\left(y_{n-k} - \overline{y}\right), \end{aligned}$$

$$
\begin{aligned}
&y_{n+1}-\overline{y}\\
&\approx -\alpha_1 n\left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right)(x_n-\overline{x})\\
&-\alpha_1\,(n-k)\\
&\cdot\left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right)(x_{n-k}-\overline{x}).
\end{aligned}
\tag{132}
$$

Let (109) hold, then (132) can be represented as

$$
\begin{aligned}
\epsilon^1_{n+1} &\approx A_{n_{17}}\epsilon^2_n + A_{n_{18}}\epsilon^2_{n-k},\\
\epsilon^2_{n+1} &\approx A_{n_{19}}\epsilon^1_n + A_{n_{20}}\epsilon^1_{n-k},
\end{aligned}
\tag{133}
$$

where

$$
\begin{aligned}
&A_{n_{17}}\\
&\quad=-\alpha n\left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right),\\
&A_{n_{18}}\\
&\quad=-\alpha\,(-k+n)\\
&\quad\cdot\left(\frac{e^{-ny_n+(n-k)y_{n-k}}\left(\beta+(-k+2n)\,\overline{y}\right)+e^{-(-k+2n)\overline{y}}}{\left(\beta+ny_n+(n-k)\,y_{n-k}\right)\left(\beta+(-k+2n)\,\overline{y}\right)}\right),\\
&A_{n_{19}}\\
&\quad=-\alpha_1 n\left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&A_{n_{20}}\\
&\quad=-\alpha_1\,(-k+n)\\
&\quad\cdot\left(\frac{e^{-nx_n+(n-k)x_{n-k}}\left(\beta_1+(-k+2n)\,\overline{x}\right)+e^{-(-k+2n)\overline{x}}}{\left(\beta_1+nx_n+(n-k)\,x_{n-k}\right)\left(\beta_1+(-k+2n)\,\overline{x}\right)}\right).
\end{aligned}
\tag{134}
$$

Moreover,

$$
\begin{aligned}
&\lim_{n\longrightarrow\infty}A_{n_{17}}=-\frac{n\alpha e^{-(-k+2n)\overline{y}}\left(\beta+(-k+2n)\,\overline{y}+1\right)}{\left(\beta+(-k+2n)\,\overline{y}\right)^2},\\
&\lim_{n\longrightarrow\infty}A_{n_{18}}\\
&\quad=-\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}\left(\beta+(-k+2n)\,\overline{y}+1\right)}{\left(\beta+(-k+2n)\,\overline{y}\right)^2},\\
&\lim_{n\longrightarrow\infty}A_{n_{19}}=-\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1+(-k+2n)\,\overline{x}+1\right)}{\left(\beta_1+(-k+2n)\,\overline{x}\right)^2},\\
&\lim_{n\longrightarrow\infty}A_{n_{20}}\\
&\quad=-\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1+(-k+2n)\,\overline{x}+1\right)}{\left(\beta_1+(-k+2n)\,\overline{x}\right)^2}.
\end{aligned}
\tag{135}
$$

Hence limiting system of error terms becomes

$$
E_{n+1}=EE_n,
\tag{136}
$$

where

$$
E=\begin{pmatrix}
0 & 0\ \dots\ 0 & 0 & -\frac{n\alpha e^{-(-k+2n)\overline{y}}\left(\beta+(-k+2n)\,\overline{y}+1\right)}{\left(\beta+(-k+2n)\,\overline{y}\right)^2} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha e^{-(-k+2n)\overline{y}}\left(\beta+(-k+2n)\,\overline{y}+1\right)}{\left(\beta+(-k+2n)\,\overline{y}\right)^2}\\
1 & 0\ \dots\ 0 & 0 & 0 & 0\ \dots\ 0 & 0\\
\vdots & \vdots\ \dots\ \vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots\\
0 & 0\ \dots\ 1 & 0 & 0 & 0\ \dots\ 0 & 0\\
-\frac{n\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1+(-k+2n)\,\overline{x}+1\right)}{\left(\beta_1+(-k+2n)\,\overline{x}\right)^2} & 0\ \dots\ 0 & -\frac{(-k+n)\,\alpha_1 e^{-(-k+2n)\overline{x}}\left(\beta_1+(-k+2n)\,\overline{x}+1\right)}{\left(\beta_1+(-k+2n)\,\overline{x}\right)^2} & 0 & 0\ \dots\ 0 & 0\\
0 & 0\ \dots\ 0 & 0 & 1 & 0\ \dots\ 0 & 0\\
\vdots & \vdots\ \ddots\ \vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots\\
0 & 0\ \dots\ 0 & 0 & 0 & 0\ \dots\ 1 & 0
\end{pmatrix},
\tag{137}
$$

which is the same as linearized system of (12) about $(\overline{x},\overline{y})$. Finally we have following theorem.

Theorem 8. *Assume that $\{(x_n, y_n)\}$ is a positive solution of (13) such that (104) along with following relation holds:*

$$
\begin{aligned}
\overline{x}&\in\left[L_5,U_5\right],\\
\overline{y}&\in\left[L_6,U_6\right].
\end{aligned}
\tag{138}
$$

Then the error vector E_n defined in (116) satisfies

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}\left(\left\|E_n\right\|\right)^{1/n}&=\left|\lambda E\right|,\\
\lim_{n\longrightarrow\infty}\frac{\left\|E_{n+1}\right\|}{\left\|E_n\right\|}&=\left|\lambda E\right|,
\end{aligned}
\tag{139}
$$

where λE are the characteristic roots of E about $(\overline{x},\overline{y})$.

7. Conclusion

This work is about the global dynamics of three higher-order exponential systems of difference equations. We proved that every positive solution is bounded and persistent, and further $[L_1,U_1]\times[L_2,U_2]$, $[L_3,U_3]\times[L_4,U_4]$, and $[L_5,U_5]\times[L_6,U_6]$, respectively, are invariant rectangle for systems (11), (12), and (13). We studied existence and uniqueness of $+ve$ equilibrium and global stability, and conclusions are presented in Table 1. Finally rate of convergence for (11), (12), and (13) is also investigated.

Data Availability

All the data utilized are included in this article and their sources are cited accordingly.

Table 1: Systems with corresponding qualitative behavior.

Systems	Corresponding behavior
(11)	Unique positive equilibrium point $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ if $0 < \beta < 1$ and $\alpha e^{-(-k+2n)L_1}(\beta_1 + (-k+2n)U_1)((-k+2n)U_1 + 1)(\alpha e^{-(-k+2n)L_1} + (1-\beta)U_1) < (-k+2n)^2 L_1^2(\alpha e^{-(-k+2n)L_1} - \beta U_1)$; $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ is locally asymptotically stable if $(-k+n)\alpha e^{-(-k+2n)L_1}(\beta + (-k+2n)U_2 + 1) + n\alpha e^{-(-k+2n)U_1}(\beta + (-k+2n)L_2 + 1) < (\beta + (-k+2n)L_2)^2$, and $(-k+n)\alpha_1 e^{-(-k+2n)L_2}(\beta_1 + (-k+2n)U_1 + 1) + n\alpha_1 e^{-(-k+2n)U_2}(\beta_1 + (-k+2n)L_1 + 1) < (\beta_1 + (-k+2n)L_1)^2$; $(\overline{x}, \overline{y}) \in [L_1, U_1] \times [L_2, U_2]$ is globally asymptotically stable if $\alpha e^{-(-k+2n)L_1} < \overline{x}(\beta + (-k+2n)L_2)$ *and* $\alpha_1 e^{-(-k+2n)L_2} < \overline{y}(\beta_1 + (-k+2n)L_1)$.
(12)	Unique positive equilibrium point $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ if $e^{(-k+2n)U_3}(\beta + 2(-k+2n)U_3)(\beta_1 + 2\ln(\alpha/(\beta + (-k+2n)L_3)L_3)) < \alpha_1(-k+2n)^2(\beta + (-k+2n)L_3)L_3$; $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ is locally asymptotically stable if $(-k+n)\alpha e^{-(-k+2n)L_4}(\beta + (-k+2n)U_3 + 1) + n\alpha e^{-(-k+2n)U_4}(\beta + (-k+2n)L_3 + 1) < (\beta + (-k+2n)L_3)^2$, and $(-k+n)\alpha_1 e^{-(-k+2n)L_3}(\beta_1 + (-k+2n)U_4 + 1) + n\alpha_1 e^{-(-k+2n)U_3}(\beta_1 + (-k+2n)L_4 + 1) < (\beta_1 + (-k+2n)L_4)^2$; $(\overline{x}, \overline{y}) \in [L_3, U_3] \times [L_4, U_4]$ is globally asymptotically stable if $\alpha e^{-(-k+2n)L_4} < \overline{x}(\beta + (-k+2n)L_3)$ *and* $\alpha_1 e^{-(-k+2n)L_3} < \overline{y}(\beta_1 + (-k+2n)L_4)$.
(13)	Unique positive equilibrium point $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ if $\alpha_1(-k+2n)^2 e^{-(-k+2n)L_5}(\beta(\beta_1 + (-k+2n)U_5) + (-k+2n)\alpha_1 e^{-(-k+2n)L_5})U_5\Omega_1 < (\beta_1 + (-k+2n)L_5)^2\Omega_2^2$; $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ is locally asymptotically stable if $(-k+n)\alpha e^{-(-k+2n)L_6}(\beta + (-k+2n)U_6 + 1) + n\alpha e^{-(-k+2n)U_6}(\beta + (-k+2n)L_6 + 1) < (\beta + (-k+2n)L_6)^2$, and $(-k+n)\alpha_1 e^{-(-k+2n)L_5}(\beta_1 + (-k+2n)U_5 + 1) + n\alpha_1 e^{-(-k+2n)U_5}(\beta_1 + (-k+2n)L_5 + 1) < (\beta_1 + (-k+2n)L_5)^2$; $(\overline{x}, \overline{y}) \in [L_5, U_5] \times [L_6, U_6]$ is globally asymptotically stable if $\alpha e^{-(-k+2n)L_6} < \overline{x}(\beta + (-k+2n)L_6)$ *and* $\alpha_1 e^{-(-k+2n)L_5} < \overline{y}(\beta_1 + (-k+2n)L_5)$.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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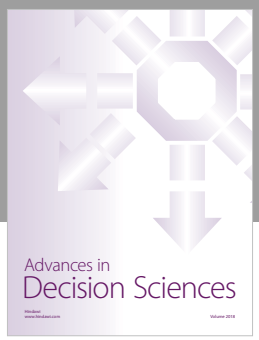

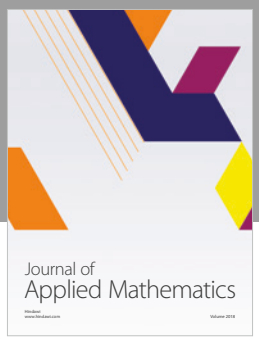

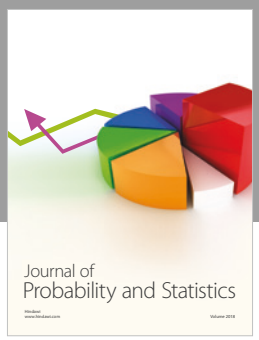

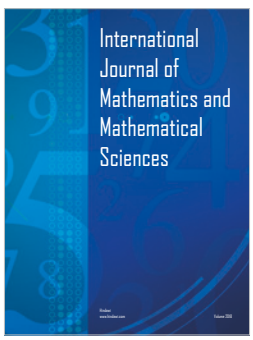

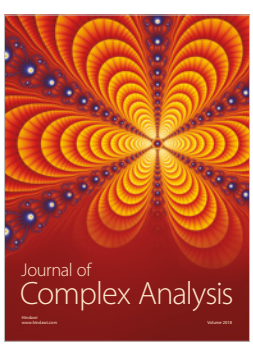

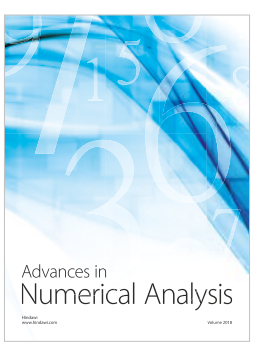

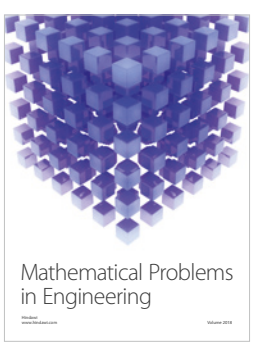

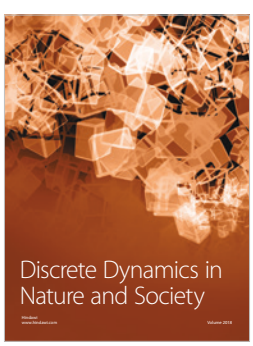

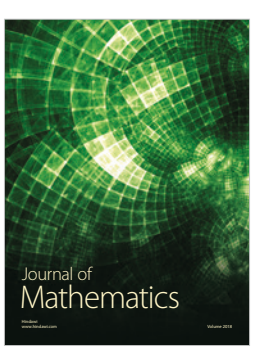

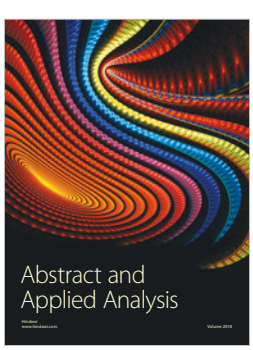

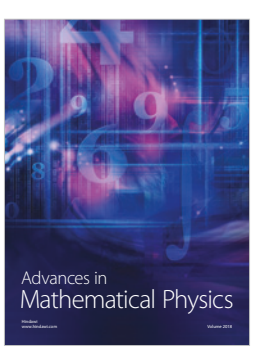

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