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Research Article

Asymptotic Behavior of Discrete Time Fuzzy Single Species Model

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This work is concerned with the qualitative behavior of discrete time single species model with fuzzy environment $x_{n+1} = x_n \exp(A - Bx_n)$, $n = 0, 1, 2, \ldots$, where x_n denotes the number of individuals of generation n, A is the intrinsic growth rate, and B is interpreted as the carrying capacity of the surrounding environment. $\{x_n\}$ is a sequence of positive fuzzy number. A, B and the initial value x_0 are positive fuzzy numbers. Applying difference of Hukuhara (H-difference), the existence, uniqueness of the positive solution, and global asymptotic behavior of all positive solution with the model are obtained. Moreover a numerical example is presented to show the effectiveness of theoretic results obtained.

1. Introduction

Difference equations or discrete time dynamical systems have many applications in economics, biology, computer science, control engineering, etc. (see, for example, [1–10] and the references therein). In the recent years, many researchers pay a close attention to the study on qualitative behavior of difference equation in mathematical biology and population dynamics [11–13]. In theoretical ecology, the difference equation models are often described the interactions of species with nonoverlapping generations. For example, in 1954, Ricker [14] forecasted fish stock recruitment using the following discrete time single species model

$$x_{n+1} = x_n \exp\left(r - \frac{x_n}{K}\right), \quad n = 0, 1, 2, \ldots. \tag{1}$$

where x_n denotes the number of individuals of generation n, r is the intrinsic growth rate, and K is interpreted as the carrying capacity of the surround environment. In fact, from a biological point of view, researchers focus on whether or not all species in a multispecies community can be permanent or bounded.

As is all known, the parameters of the model are usually based on statistical method or on the choice of some method adapted to the identification. Therefore, these models are subjected to inaccuracies (fuzzy uncertainty) that can be caused by the nature of the state variables, by coefficients of the model and by initial conditions. In our real life, scientists are concerned with uncertainty and accept the fact that uncertainty is very important influencing factor of dynamical behavior of dynamical system. Fuzzy set introduced by Zadeh [15] and its development have been growing rapidly to various situation of theory and application including the theory of differential and difference equations with uncertainty. It is well known that a fuzzy difference equation is a difference equation where parameters or the initial values of systems are fuzzy numbers, and its solutions are sequences of fuzzy numbers.

In fact, the dynamical behavior of fuzzy difference equation is different from the behavior of corresponding parametric ordinary difference equation. In recent decades, researchers have an increasing interest in studying fuzzy difference equation. Some results on fuzzy difference equations have been reported (see, for example, [16–31]). Barros, Bassanezi, and Tonelli [32] have investigated the dynamical behavior of population model with fuzzy uncertainty. However, to the best of our knowledge, few authors study discrete time single species model under fuzzy environment. This paper is the first to study the dynamical behavior of discrete time single species model using fuzzy sets theory.

The main aim of this paper is to study the dynamical behaviors of the following discrete time single species model

$$x_{n+1} = x_n \exp(A - Bx_n), \quad n = 0, 1, \ldots, \tag{2}$$

where x_n denotes the number of individuals of generation n and $\{x_n\}$ is a sequence of positive fuzzy numbers. A, B are the intrinsic growth rate and the carrying capacity of the surround environment, respectively. Parameters A, B and the initial condition x_0 are positive fuzzy numbers. This paper is, to some extent, a generalization of classic discrete time single species model, using the subjectivity which comes from "fuzziness" of the biological phenomenon.

The rest of this paper is organized as follows. In the next section, we introduce some definitions and preliminaries. In Section 3, the dynamical behaviors on the existence, uniqueness, and global asymptotic behaviors of the positive fuzzy solution to system (2) are studied. A numerical example is given to show effectiveness of results obtained in Section 4. Finally, a general conclusion is drawn in Section 5.

2. Mathematical Preliminaries

For convenience, we give some definitions used in the sequel.

Definition 1 (see [33]). A is said to be a fuzzy number if $A : R \longrightarrow [0,1]$ satisfies the below (i)-(iv)

(i) A is normal; i.e., there exists an $x \in R$ such that $A(x) = 1$.

(ii) A is fuzzy convex, i.e., for all $t \in [0,1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}; \tag{3}$$

(iii) A is upper semicontinuous.

(iv) The support of A, $\operatorname{supp} A = \overline{\bigcup_{\alpha\in(0,1]}[A]^\alpha} = \overline{\{x : A(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Let E^1 be the set of all real fuzzy numbers which are normal, upper semicontinuous, convex, and compactly supported fuzzy sets.

Definition 2 (fuzzy number (parametric form) [34]). A fuzzy number u in a parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function.

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function.

(3) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$. A crisp (real) number x is simply represented by $(\underline{u}(r), u_2(r)) = (x, x), 0 \leq r \leq 1$. The fuzzy number space $\{(\underline{u}(r), \overline{u}(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space. [26]

Definition 3. Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1, 0 \leq r \leq 1$, and arbitrary $k \in R$. Then

(i) $u = v$ iff $\underline{u}(r) = \underline{v}(r), \overline{u}(r) = \overline{v}(r)$,

(ii) $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

(iii)

$$ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \geq 0; \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0, \end{cases} \tag{4}$$

(iv) $uv = (\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\})$.

Definition 4. Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1, 0 \leq r \leq 1$, and if there exists $w = (\underline{w}(r), \overline{w}(r)) \in E^1$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $w = u - v = (\underline{u}(r) - \underline{v}(r), \overline{u}(r) - \overline{v}(r))$

In this paper the "−" sign stands always for H-difference and let us remark that $u - v \neq u + (-1)v$.

Definition 5 (triangular fuzzy number [34]). A triangular fuzzy number (TFN) denoted by A is defined as (a, b, c) where the membership function

$$A(x) = \begin{cases} 0, & x \leq a; \\ \dfrac{x-a}{b-a}, & a \leq x \leq b; \\ 1, & x = b; \\ \dfrac{c-x}{c-b}, & b \leq x \leq c; \\ 0, & x \geq c. \end{cases} \tag{5}$$

The α−cuts of $A = (a, b, c)$ are denoted by $[A]^\alpha = \{x \in R : A(x) \geq \alpha\} = [a + \alpha(b-a), c - \alpha(c-b)] = [A_{l,\alpha}, A_{r,\alpha}]$, $\alpha \in [0,1]$, and it is clear that the $[A]^\alpha$ are closed interval. A fuzzy number is positive if $\operatorname{supp} A \subset (0, \infty)$.

The following proposition is fundamental since it characterizes a fuzzy set through the α-levels.

Proposition 6 (see [33]). *If $\{A^\alpha : \alpha \in [0,1]\}$ is a compact, convex and not empty subset family of R^n such that*

(i) $\overline{\bigcup A^\alpha} \subset A^0$,

(ii) $A^{\alpha_2} \subset A^{\alpha_1}$ if $\alpha_1 \leq \alpha_2$,

(iii) $A^\alpha = \bigcap_{k\geq1} A^{\alpha_k}$ if $\alpha_k \uparrow \alpha > 0$,

then there is $u \in E^n$ such that $[u]^\alpha = A^\alpha$ for all $\alpha \in (0,1]$ and $[u]^0 = \overline{\bigcup_{0<\alpha\leq1} A^\alpha} \subset A^0$.

Definition 7 (see [27]). A sequence of positive fuzzy numbers $\{x_n\}$ persists (resp., is bounded) if there exists a positive real number M (resp., N) such that

$$\operatorname{supp} x_n \subset [M, \infty) \left(\text{resp. } \operatorname{supp} x_n \subset (0, N]\right), \quad n = 1, 2, \ldots. \tag{6}$$

A sequence of positive fuzzy numbers $\{x_n\}$ is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$\operatorname{supp} x_n \subset [M, N], \quad n = 1, 2, \ldots. \tag{7}$$

Definition 8. x_n is a positive solution of (2), if $\{x_n\}$ is a sequence of positive fuzzy numbers which satisfies (2). The equilibrium of (2) is the solution of the equation $x = x\exp(A - Bx)$.

Definition 9 (see [20]). Let A, B be fuzzy numbers with $[A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}]$. Then the metric of A and B is defined as

$$D(A,B) = \sup_{\alpha\in(0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}, \tag{8}$$

and (E^1, D) is a complete metric space.

Definition 10 (see [23]). Let $\{x_n\}$ be a sequence of positive fuzzy numbers and x is a positive fuzzy number. Suppose that

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0,1], \; n = 0, 1, \ldots, \tag{9}$$

and

$$[x]^\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1]. \tag{10}$$

The sequence $\{x_n\}$ converges to x with respect to D as $n \longrightarrow \infty$ if $\lim_{n\longrightarrow\infty} D(x_n, x) = 0$.

Definition 11. Let x be a positive equilibrium of (2). The positive equilibrium x is stable, if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for every positive solution x_n of (2), which satisfies $D(x_0, x) \le \delta$, we have $D(x_n, x) \le \varepsilon$ for all $n \ge 0$. The positive equilibrium x is asymptotically stable, if it is stable and every positive solution of (2) converges to the positive equilibrium of (2) with respect to D as $n \longrightarrow \infty$.

3. Main Results

3.1. Existence and Uniqueness of Positive Solution. First we study the existence and uniqueness of the positive solutions of (2). We need the following lemma.

Lemma 12 (see [33]). *Let $f : R^+ \times R^+ \longrightarrow R^+$ be continuous; A, B are fuzzy numbers. Then*

$$[f(A,B)]^\alpha = f([A]^\alpha, [B]^\alpha), \quad \alpha \in (0,1] \tag{11}$$

Theorem 13. *Consider (2) where A, B are positive fuzzy numbers, if there exists fuzzy number C such that $A = Bx_n + C$ for $n \ge 0$. Then for any positive fuzzy numbers x_0, there exists a unique positive solution x_n of (2).*

Proof. The proof is similar to Theorem 2.3 [24]. Suppose that there exists a sequence of fuzzy numbers x_n satisfying (2) with initial conditions x_0. Consider α-cuts, $\alpha \in (0,1]$, $n = 0, 1, 2, \ldots$, and applying Lemma 12, and $A = Bx_n + C$ for $n \ge 0$, we have

$$\begin{aligned}
[x_{n+1}]^\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = [x_n \exp(A - Bx_n)]^\alpha \\
&= [x_n]^\alpha \exp\left([A]^\alpha - [B]^\alpha [x_n]^\alpha\right) = [L_{n,\alpha}, R_{n,\alpha}] \\
&\quad \times \exp([A_{l,\alpha}, A_{r,\alpha}] - [B_{l,\alpha}, B_{r,\alpha}] \times [L_{n,\alpha}, R_{n,\alpha}]) \\
&= [L_{n,\alpha}, R_{n,\alpha}] \\
&\quad \times \exp([A_{l,\alpha} - B_{l,\alpha}L_{n,\alpha}, A_{r,\alpha} - B_{r,\alpha}R_{n,\alpha}]) = [L_{n,\alpha} \\
&\quad \cdot \exp(A_{l,\alpha} - B_{l,\alpha}L_{n,\alpha}), R_{n,\alpha}\exp(A_{r,\alpha} - B_{r,\alpha}R_{n,\alpha})]
\end{aligned} \tag{12}$$

It follows from (12) and H-difference of fuzzy numbers that, for $n = 0, 1, 2, \ldots, \alpha \in (0,1]$,

$$\begin{aligned}
L_{n+1,\alpha} &= L_{n,\alpha} \exp(A_{l,\alpha} - B_{l,\alpha}L_{n,\alpha}), \\
R_{n+1,\alpha} &= R_{n,\alpha} \exp(A_{r,\alpha} - B_{r,\alpha}R_{n,\alpha}).
\end{aligned} \tag{13}$$

Then it is obvious that, for any initial condition $(L_{0,\alpha}, R_{0,\alpha})$, $\alpha \in (0,1]$, there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$.

Now we prove that $[L_{n,\alpha}, R_{n,\alpha}]$, $\alpha \in (0,1]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (13) with initial conditions $(L_{0,\alpha}, R_{0,\alpha})$, determines the solution of (2) with initial value x_0, such that,

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n = 0, 1, 2, \ldots, \alpha \in (0,1]. \tag{14}$$

From Definition 2 and since A, B, x_0 are positive fuzzy numbers, for any $\alpha_1, \alpha_2 \in (0,1], \alpha_1 \le \alpha_2$, we have

$$\begin{aligned}
&0 < A_{l,\alpha_1} \le A_{l,\alpha_2} \le A_{r,\alpha_2} \le A_{r,\alpha_1} \\
&0 < B_{l,\alpha_1} \le B_{l,\alpha_2} \le B_{r,\alpha_2} \le B_{r,\alpha_1} \\
&0 < L_{0,\alpha_1} \le L_{0,\alpha_2} \le R_{0,\alpha_2} \le R_{0,\alpha_1}
\end{aligned} \tag{15}$$

We claim that, for $n = 0, 1, \ldots$,

$$0 < L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}. \tag{16}$$

We prove it by induction. It is clear that (16) is true for $n = 0$. Suppose that (16) holds true for $n \le k$, $k \in \{1, 2, \ldots, \}$. Then, from (13), (15), and (16) for $n \le k$, it follows that

$$\begin{aligned}
L_{k+1,\alpha_1} &= L_{k,\alpha_1} \exp\left(A_{l,\alpha_1} - B_{l,\alpha_1}L_{k,\alpha_1}\right) \\
&\le L_{k,\alpha_2} \exp\left(A_{l,\alpha_2} - B_{l,\alpha_2}L_{k,\alpha_2}\right) = L_{k+1,\alpha_2} \\
&\le R_{k,\alpha_2} \exp\left(A_{r,\alpha_2} - B_{r,\alpha_2}R_{k,\alpha_2}\right) = R_{k+1,\alpha_2} \\
&\le R_{k,\alpha_1} \exp\left(A_{r,\alpha_1} - B_{r,\alpha_1}R_{k,\alpha_1}\right) = R_{k+1,\alpha_1}.
\end{aligned} \tag{17}$$

Therefore (16) is satisfied. Moreover, for $\forall\alpha \in (0,1]$, it follows from (13) that

$$\begin{aligned}
L_{1,\alpha} &= L_{0,\alpha} \exp(A_{l,\alpha} - B_{l,\alpha}L_{0,\alpha}), \\
R_{1,\alpha} &= R_{0,\alpha} \exp(A_{r,\alpha} - B_{r,\alpha}R_{0,\alpha}).
\end{aligned} \tag{18}$$

Since A, B, x_0 are positive fuzzy numbers, from Definition 2, then $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}, L_{0,\alpha}, R_{0,\alpha}$ are left continuous. From

(18) we have $L_{1,\alpha}$, $R_{1,\alpha}$ are left continuous. By induction we can get that $L_{n,\alpha}$, $R_{n,\alpha}$ are left continuous.

Next we prove that the support of x_n, $\operatorname{supp} x_n = \overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, and since A, B and x_0 are positive fuzzy numbers, there exist constants $M_A > 0$, $N_A > 0$, $M_B > 0$, $N_B > 0$, $M_0 > 0$, $N_0 > 0$ such that, for $\alpha \in (0, 1]$,

$$\begin{aligned}
&[A_{l,\alpha}, A_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]} [A_{l,\alpha}, A_{r,\alpha}]} \subset [M_A, N_A],\\
&[B_{l,\alpha}, B_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]} [B_{l,\alpha}, B_{r,\alpha}]} \subset [M_B, N_B],\\
&[L_{0,\alpha}, L_{0,\alpha}] \subset \overline{\bigcup_{\alpha\in(0,1]} [L_{0,\alpha}, R_{0,\alpha}]} \subset [M_0, N_0],
\end{aligned} \tag{19}$$

Hence, from (18) and (19), for $\alpha \in (0, 1]$, we get

$$\begin{aligned}
&[L_{1,\alpha}, R_{1,\alpha}]\\
&\quad \subset [M_0 \exp(M_A - N_B N_0), N_0 \exp(N_A - M_B M_0)],
\end{aligned} \tag{20}$$

from which it is clear that

$$\begin{aligned}
&\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}]\\
&\quad \subset [M_0 \exp(M_A - N_B N_0), N_0 \exp(N_A - M_B M_0)],
\end{aligned} \tag{21}$$

Therefore (21) implies $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact, and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. Deducing inductively it can follow easily that

$$\begin{aligned}
&\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \text{ is compact, and } \overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]}\\
&\quad \subset (0, \infty).
\end{aligned} \tag{22}$$

Therefore, from (16), (22) and $L_{n,\alpha}$, $R_{n,\alpha}$ are left continuous. It can conclude that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers $\{x_n\}$ such that (14) holds.

We prove that x_n is a solution of (12) with initial value x_0. Since $\forall\alpha \in (0, 1]$,

$$[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = [L_{n,\alpha} \exp(A_{l,\alpha} - B_{l,\alpha} L_{n,\alpha}), R_{n,\alpha} \exp(A_{r,\alpha} - B_{r,\alpha} R_{n,\alpha})] = [x_n \exp(A - Bx_n)]^\alpha. \tag{23}$$

Namely, x_n is a solution of (12) with initial value x_0.

Suppose that there exists another solution $\overline{x}_n$ of (12) with initial value x_0. Then from arguing as above we can easily get that, for $n = 0, 1, 2, \ldots$,

$$[\overline{x}_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1]. \tag{24}$$

Then from (14) and (24), we have $[x_n]^\alpha = [\overline{x}_n]^\alpha$, $\alpha \in (0, 1]$, $n = 0, 1, 2, \ldots$, and hence $x_n = \overline{x}_n$, $n = 0, 1, 2, \ldots$, and this completes the proof of Theorem 13. □

3.2. Dynamical Behaviour of Positive Solution. In order to study the dynamical behavior of the solution x_n to (2), we first consider the following system of difference equations

$$\begin{aligned}
y_{n+1} &= y_n \exp(a - py_n),\\
z_{n+1} &= z_n \exp(b - qz_n),\\
&\qquad n = 0, 1, \ldots,
\end{aligned} \tag{25}$$

It is clear that the equilibrium points $(\overline{y}, \overline{z})$ of (25) include the following four cases:

$$\begin{aligned}
&(i)\ (0, 0),\\
&(ii)\ \left(0, \frac{b}{q}\right),\\
&(iii)\ \left(\frac{a}{p}, 0\right),\\
&(iv)\ \left(\frac{a}{p}, \frac{b}{q}\right).
\end{aligned} \tag{26}$$

Lemma 14. *Consider difference equation*

$$y_{n+1} = y_n \exp(a - py_n), \quad n = 0, 1, 2, \ldots. \tag{27}$$

If $a > 0$, $p > 0$, then $0 < y_n \le (1/p)\exp(a - 1)$.

Proof. Consider function $f(y) = y \exp(a - py)$, and we have $f'(y) = (1 - py)\exp(a - py)$. It follows that

$$\begin{aligned}
f'(y) > 0 \quad \text{if } y < \frac{1}{p},\\
f'(y) < 0 \quad \text{if } y > \frac{1}{p}.
\end{aligned} \tag{28}$$

It is clear that the maximum of f is equal to $(1/p)\exp(a - 1)$. Therefore we have $0 < y_n \le (1/p)\exp(a - 1)$. □

Lemma 15 (see [2]). *Suppose the vector difference equation*

$$X_{n+1} = H(X_n), \quad n = 0, 1, 2, \ldots, \tag{29}$$

where $X_n \in R^{k+1}$, $H \in C^1[R^{k+1}, R^{k+1}]$. Let $\overline{X}$ be an equilibrium point (29), and $DH(\overline{X})$ denote the Jacobian Matrix of function H at $\overline{X}$. Then *(i) $\overline{X}$ is called a hyperbolic equilibrium if $DH(\overline{X})$ has no eigenvalues with absolute value equal to 1.*

(ii) $\overline{X}$ is called a sink or an attracting equilibrium if every eigenvalue of $DH(\overline{X})$ has absolute value less than 1.

(iii) $\overline{X}$ is called a source or a repelling equilibrium if every eigenvalue of $DH(\overline{X})$ has absolute value greater than 1.

(iv) $\overline{X}$ is called a saddle point if some of the eigenvalues of $DH(\overline{X})$ are greater and some are less than 1 in absolute value.

Theorem 16. *Consider the system of difference equations (25), where a, b, p, and q are positive real constants, and the initial values y_0, z_0 are positive real numbers; then the following statements are true.*

(i) The equilibrium $(0, 0)$ is a source (a repelling equilibrium).

(ii) The equilibrium $(0, b/q)$ is a saddle point if $0 < b < 2$; the equilibrium $(0, b/q)$ is a source (a repelling equilibrium) if $b > 2$.

(iii) The equilibrium $(a/p, 0)$ is a saddle point if $0 < a < 2$; the equilibrium $(a/p, 0)$ is a source (a repelling equilibrium) if $a > 2$.

Proof. (i) It is clear that $(0, 0)$ is always an equilibrium. We can easily obtain that the linearized system of (25) about the positive equilibrium $(0, 0)$ is

$$X_{n+1} = D_1 X_n, \quad n = 0, 1, \ldots, \tag{30}$$

where

$$X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}, \qquad D_1 = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}. \tag{31}$$

We can easily obtain that the eigenvalue of Jacobian matrix D_1, $\lambda_1 = e^a$, $\lambda_2 = e^b$. It is clear that $|\lambda| > 1$. This implies that the equilibrium $(0, 0)$ is a source (a repelling equilibrium).

(ii) We can obtain that the linearized system of (25) about the equilibrium $(0, a/p)$ is

$$X_{n+1} = D_2 X_n, \tag{32}$$

where

$$X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}, \qquad D_2 = \begin{pmatrix} e^a & 0 \\ 0 & 1-b \end{pmatrix} \tag{33}$$

It is easy to obtain the following: if $0 < b < 2$, the eigenvalue of D_2, $|\lambda_1| > 1$, $|\lambda_2| < 1$, then $(0, a/p)$ is a saddle point. if $b > 2$, all the eigenvalue of D_2, $|\lambda| > 1$, then $(0, a/p)$ is a source (a repelling equilibrium).

(iii) The proof of (iii) is similar to the proof of (ii). So we omit it. □

Theorem 17. *Consider the system of difference equations (25), where the initial values y_0, z_0 are positive real numbers. If $0 < a < 2, 0 < b < 2$, then (25) has a unique positive equilibrium point $(a/p, b/q)$ which is globally asymptotically stable.*

Proof. It follows easily from (25) that $(a/p, b/q)$ is a unique positive equilibrium point.

We can obtain that the linearized system of (25) about the positive equilibrium $(a/p, b/q)$ is

$$X_{n+1} = D_3 X_n, \tag{34}$$

where

$$X_n = \begin{pmatrix} y_n \\ z_n \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 1-a & 0 \\ 0 & 1-b \end{pmatrix} \tag{35}$$

We can easily obtain that the eigenvalue of Jacobian matrix D_3, $\lambda_1 = 1 - a, \lambda_2 = 1 - b$. It is clear that $|\lambda| < 1$. Hence the equilibrium $(a/p, b/q)$ is a sink (an attracting equilibrium).

Noting Lemma 14, we have

$$\begin{aligned} \lim_{n\to\infty} \sup y_n &= L_1, \\ \lim_{n\to\infty} \inf y_n &= l_1, \\ \lim_{n\to\infty} \sup z_n &= L_2, \\ \lim_{n\to\infty} \inf z_n &= l_2. \end{aligned} \tag{36}$$

where $L_i, l_i \in (0, \infty)$, $i = 1, 2$. Then from (25) we have

$$\begin{aligned} L_1 &\le L_1 \exp(a - pl_1), \\ l_1 &\ge l_1 \exp(a - pL_1), \\ L_2 &\le L_2 \exp(b - ql_2), \\ l_2 &\ge l_2 \exp(b - qL_2). \end{aligned} \tag{37}$$

From this, we have

$$\begin{aligned} 1 &\le \exp(a - pl_1), \\ 1 &\ge \exp(a - pL_1), \\ 1 &\le \exp(b - ql_2), \\ 1 &\ge \exp(b - qL_2). \end{aligned} \tag{38}$$

The relation (38) implies $l_1 \le L_1, l_2 \le L_2$. We claim that

$$\begin{aligned} l_1 &= L_1, \\ l_2 &= L_2. \end{aligned} \tag{39}$$

Suppose on the contrary that $l_1 < L_1, l_2 < L_2$. From (38), we have

$$l_1 \le \frac{a}{p} < L_1 \le \frac{1}{p}\exp(a-1),$$
$$l_2 \le \frac{b}{q} < L_2 \le \frac{1}{q}\exp(b-1), \tag{40}$$

from which we have

$$a < \exp(a-1),$$
$$b < \exp(b-1), \tag{41}$$

It follows from (41) that $a \neq 1$ and $b \neq 1$, which is contradicting with $0 < a < 2, 0 < b < 2$. So $L_i = l_i, i = 1, 2..$ Hence we have $\lim_{n\longrightarrow\infty} y_n = a/p$, $\lim_{n\longrightarrow\infty} z_n = b/q$. Therefore the unique positive equilibrium $(a/p, b/q)$ is globally asymptotically stable. □

Theorem 18. *Consider the fuzzy difference equation (2), where A, B are positive fuzzy numbers. Consider that there exists positive constants N_A, for all $\alpha \in (0,1]$ such that*

$$A_{r,\alpha} < N_A < 2. \tag{42}$$

Then the following statements are true.

(i) Every positive solution x_n of (2) is bounded.

(ii) Equation (2) has a unique positive equilibrium x.

(iii) Every positive solution x_n of (2) converges the unique equilibrium x with respect to D as $n \longrightarrow \infty$.

Moreover the unique positive equilibrium x is asymptotically stable.

Proof. (i) Let x_n be a positive solution of (2) with initial conditions x_0. Suppose that (19) holds; from (25) and using Lemma 14, we get that

$$0 < L_{n,\alpha} \le \frac{1}{B_{l,\alpha}}\exp\left(A_{l,\alpha}-1\right),$$
$$0 < R_{n,\alpha} \le \frac{1}{B_{r,\alpha}}\exp\left(A_{r,\alpha}-1\right), \tag{43}$$
$$\alpha \in (0,1],$$

From (19) and (43), we have that for all $\alpha \in (0,1]$

$$\left[L_{n,\alpha}, R_{n,\alpha}\right] \subset (0, N], \quad n \ge 1, \tag{44}$$

where $N = (1/M_B)\exp(N_A - 1)$. From (44), we have for $n \ge 1$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset (0,N]$, so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0,N]$. Thus the proof of Part (i) is completed.

(ii) We consider the system

$$L_\alpha = L_\alpha \exp\left(A_{l,\alpha} - B_{l,\alpha}L_\alpha\right),$$
$$R_\alpha = R_\alpha \exp\left(A_{r,\alpha} - B_{r,\alpha}R_\alpha\right). \tag{45}$$

Then the positive solution (L_α, R_α) of (45) is given by

$$L_\alpha = \frac{A_{l,\alpha}}{B_{l,\alpha}},$$
$$R_\alpha = \frac{A_{r,\alpha}}{B_{r,\alpha}}, \tag{46}$$
$$\alpha \in (0,1].$$

Let x_n be a positive solution of (2) such that $[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}]$, $\alpha \in (0,1]$, $n = 0, 1, 2, \cdots$. Then applying Theorem 17 to system (13) we have

$$\lim_{n\longrightarrow\infty} L_{n,\alpha} = L_\alpha,$$
$$\lim_{n\longrightarrow\infty} R_{n,\alpha} = R_\alpha \tag{47}$$

From (44) and (47) we have for $0 < \alpha_1 < \alpha_2 < 1$,

$$0 < L_{\alpha_1} \le L_{\alpha_2} \le R_{\alpha_2} \le R_{\alpha_1}. \tag{48}$$

Since $A_{l,\alpha}$, $A_{r,\alpha}$, $B_{l,\alpha}$, $B_{r,\alpha}$ are left continuous, from (45), it follows that L_α, R_α are also left continuous. From (46) and (19), we have

$$c = \frac{M_A}{N_B} \le L_\alpha \le R_\alpha \le \frac{N_A}{M_B} = d. \tag{49}$$

Therefore (49) implies that $[L_\alpha, R_\alpha] \subset [c,d]$, and so $\bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \subset [c,d]$. It is clear that

$$\bigcup_{\alpha\in(0,1]}\left[L_\alpha, R_\alpha\right] \text{ is compact and } \bigcup_{\alpha\in(0,1]}\left[L_\alpha, R_\alpha\right] \subset (0,\infty). \tag{50}$$

So from Definition 2, (45), (48), (50) and $L_\alpha, R_\alpha, \alpha \in (0,1]$ determine a fuzzy number x such that

$$x = x\exp(A - Bx), \quad [x]^\alpha = \left[L_\alpha, R_\alpha\right], \ \alpha \in (0,1]. \tag{51}$$

Provided that there exists another positive equilibrium $\overline{x}$ of (2), then there exist functions $\overline{L}_\alpha, \overline{R}_\alpha : (0,1) \longrightarrow (0,\infty)$ such that

$$\overline{x} = \overline{x}\exp(A - B\overline{x}), \quad [\overline{x}]^\alpha = \left[\overline{L}_\alpha, \overline{R}_\alpha\right], \ \alpha \in (0,1]. \tag{52}$$

From this we get

$$\overline{L}_\alpha = \overline{L}_\alpha \exp\left(A_{l,\alpha} - B_{l,\alpha}\overline{L}_\alpha\right),$$
$$\overline{R}_\alpha = \overline{R}_\alpha \exp\left(A_{r,\alpha} - B_{r,\alpha}\overline{R}_\alpha\right). \tag{53}$$

So $\overline{L}_\alpha = L_\alpha, \overline{R}_\alpha = R_\alpha, \alpha \in (0,1]$. Hence $x = \overline{x}$. This completes the proof of Part (ii).

(iii) From (47) we have

$$\lim_{n\longrightarrow\infty} D(x_n, x) = \lim_{n\longrightarrow\infty}\sup_{\alpha\in(0,1]}\left\{\max\left\{\left|L_{n,\alpha} - L_\alpha\right|, \left|R_{n,\alpha} - R_\alpha\right|\right\}\right\} = 0. \tag{54}$$

Namely, every positive solution x_n of (2) converges the unique equilibrium x with respect to D as $n \longrightarrow \infty$.

Let ε be an arbitrary positive real number; we consider the positive real number δ as follows

$$0 < \delta < \frac{\varepsilon}{\left|1 - M_B\gamma\right| \exp\left(N_A - M_B\gamma\right)}. \tag{55}$$

where $0 < \gamma = \min\{L_{0,\alpha}, \inf_{n\geq 1} L_{n,\alpha}\}$, $\alpha \in (0, 1]$.

Let x_n be a positive solution of (2) such that $D(x_0, x) \leq \delta < \varepsilon$. From this we have

$$\begin{gathered} \left|L_{0,\alpha} - L_\alpha\right| \leq \delta, \\ \left|R_{0,\alpha} - R_\alpha\right| \leq \delta, \\ \alpha \in (0, 1]. \end{gathered} \tag{56}$$

From (18), (19), and (45), we have

$$\begin{aligned} \left|L_{1,\alpha} - L_\alpha\right| &= \left|L_{0,\alpha} \exp\left(A_{l,\alpha} - B_{l,\alpha} L_{0,\alpha}\right) \right. \\ &\quad \left. - L_\alpha \exp\left(A_{l,\alpha} - B_{l,\alpha} L_\alpha\right)\right| \\ &\leq \left|\left(1 - B_{l,\alpha}\underline{\xi}\right) \exp\left(A_{l,\alpha} - B_{l,\alpha}\underline{\xi}\right)\right| \left|L_{0,\alpha} - L_\alpha\right| \\ &= \delta \left|\left(1 - B_{l,\alpha}\underline{\xi}\right) \exp\left(A_{l,\alpha} - B_{l,\alpha}\underline{\xi}\right)\right| \\ &< \delta \left|\left(1 - M_B\underline{\xi}\right) \exp\left(N_A - M_B\underline{\xi}\right)\right| \end{aligned} \tag{57}$$

$$\begin{aligned} \left|R_{1,\alpha} - R_\alpha\right| &= \left|R_{0,\alpha} \exp\left(A_{r,\alpha} - B_{r,\alpha} R_{0,\alpha}\right) \right. \\ &\quad \left. - R_\alpha \exp\left(A_{r,\alpha} - B_{r,\alpha} R_\alpha\right)\right| \\ &\leq \left|\left(1 - B_{r,\alpha}\underline{\eta}\right) \exp\left(A_{r,\alpha} - B_{r,\alpha}\underline{\eta}\right)\right| \left|R_{0,\alpha} - R_\alpha\right| \\ &= \delta \left|\left(1 - B_{r,\alpha}\underline{\eta}\right) \exp\left(A_{r,\alpha} - B_{r,\alpha}\underline{\eta}\right)\right| \\ &< \delta \left|\left(1 - M_B\underline{\eta}\right) \exp\left(N_A - M_B\underline{\eta}\right)\right| \end{aligned} \tag{58}$$

where $\underline{\xi} = \min\{L_{0,\alpha}, L_\alpha\}$, $\underline{\eta} = \min\{R_{0,\alpha}, R_\alpha\}$, $\alpha \in (0, 1]$.

From (55), it is obvious that

$$\begin{gathered} \left|L_{1,\alpha} - L_\alpha\right| < \varepsilon, \\ \left|R_{1,\alpha} - R_\alpha\right| < \varepsilon. \end{gathered} \tag{59}$$

From this and working inductively we can easily prove that

$$\begin{gathered} \left|L_{n,\alpha} - L_\alpha\right| < \varepsilon, \\ \left|R_{n,\alpha} - R_\alpha\right| < \varepsilon, \\ \alpha \in (0, 1], \quad n = 0, 1, 2, \cdots. \end{gathered} \tag{60}$$

And so $D(x_n, x) < \varepsilon$. Therefore the positive equilibrium x is stable. Moreover relation (54) holds. So the equilibrium x is asymptotically stable. The proof of Theorem 18 is completed. □

Remark 19. From Theorem 18, we can know that the dynamical behavior of (2) is relevant to the intrinsic growth rate A of population. No matter what the initial population quantity x_0 is small and no matter what the carrying capacity of the surround environment B is large. As long as the intrinsic growth rate of population A satisfies $A_{r,\alpha} < 2, 0 < A_{l,\alpha} - B_{l,\alpha}L_{n,\alpha} \leq A_{r,\alpha} - B_{r,\alpha}R_{n,\alpha}$, for all $\alpha \in (0, 1)$, the solution of fuzzy difference equation (2) is bounded and eventually converges to the unique positive equilibrium.

4. An Illustrative Example

In order to illustrate our obtained results, we give a numerical example to show effectiveness of theoretic results.

Example 1. Consider discrete time fuzzy single species model

$$x_{n+1} = x_n \exp\left(A - Bx_n\right), \quad n = 0, 1, \ldots, \tag{61}$$

where A, B and the initial value x_0 are positive triangular fuzzy numbers such that

$$\begin{gathered} A = (1.2, 1.5, 1.8), \\ B = (0.9, 1, 1.1), \\ x_0 = (1, 1.1, 1.2) \end{gathered} \tag{62}$$

from which, we get, for $\alpha \in (0, 1]$,

$$\begin{gathered} [A]^\alpha = [1.2 + 0.3\alpha, 1.8 - 0.3\alpha], \\ [B]^\alpha = [0.9 + 0.1\alpha, 1.1 - 0.1\alpha], \\ [x_0]^\alpha = [1 + 0.1\alpha, 1.2 - 0.1\alpha]. \end{gathered} \tag{63}$$

From (61) and (63), it results in a coupled system of difference equation with parameter $\alpha \in (0, 1]$,

$$\begin{gathered} L_{n+1,\alpha} = L_{n,\alpha} \exp\left[1.2 + 0.3\alpha - (0.9 + 0.1\alpha) L_{n,\alpha}\right], \\ R_{n+1,\alpha} = R_{n,\alpha} \exp\left[1.8 - 0.3\alpha - (1.1 - 0.1\alpha) R_{n,\alpha}\right]. \end{gathered} \tag{64}$$

It is clear that (42) of Theorem 18 is satisfied. Therefore the solution of (61) converges to the unique positive equilibrium $x = (4/3, 1.5, 18/11)$ (see Figures 1–5).

5. Conclusion

Difference equation is one of the most important models when it is applicable in various problems in different fields. It is also important if it is applied to study population dynamic in fuzzy environment. When it can be studied in fuzzy environment the behavior of it changes. In this work, we first consider the dynamical behavior of discrete time single species $x_{n+1} = x_n \exp(A - Bx_n)$ with fuzzy parameters and fuzzy initial conditions. Compared with crisp discrete time single species model, the dynamical behavior of system is different. The parameters of system A and B satisfy the condition $A_{r,\alpha} < 2, 0 < A_{l,\alpha} - B_{l,\alpha}L_{n,\alpha} \leq A_{r,\alpha} - B_{r,\alpha}R_{n,\alpha}$, for all $\alpha \in (0, 1], n = 0, 1, 2, \ldots$, and the solution of (2) is bounded and converges to unique fuzzy positive equilibrium.

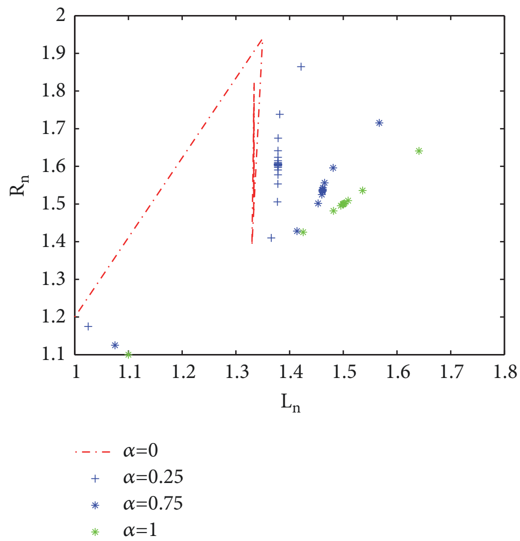

FIGURE 1: The dynamics of system (64).

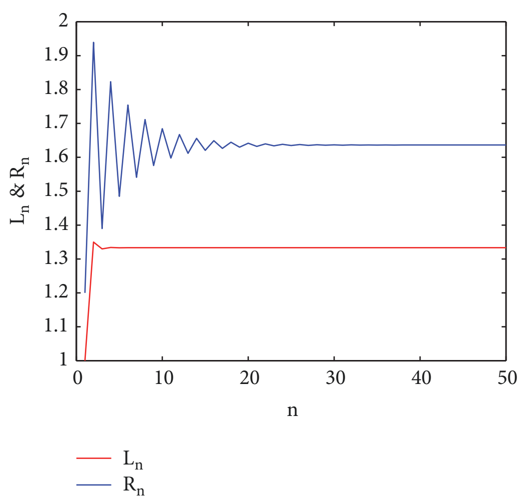

FIGURE 2: The solution of system (64) at $\alpha = 0$.

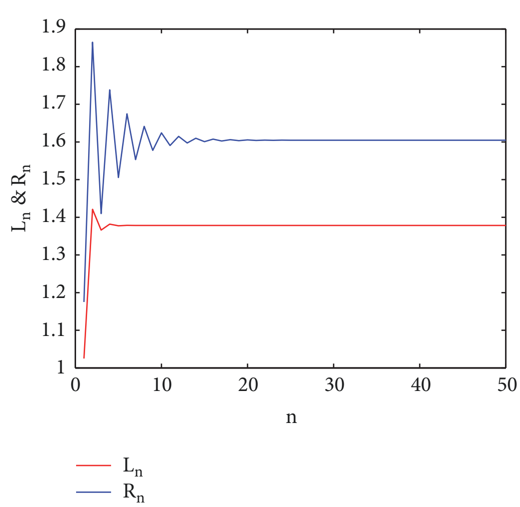

FIGURE 3: The solution of system (64) at $\alpha = 0.25$.

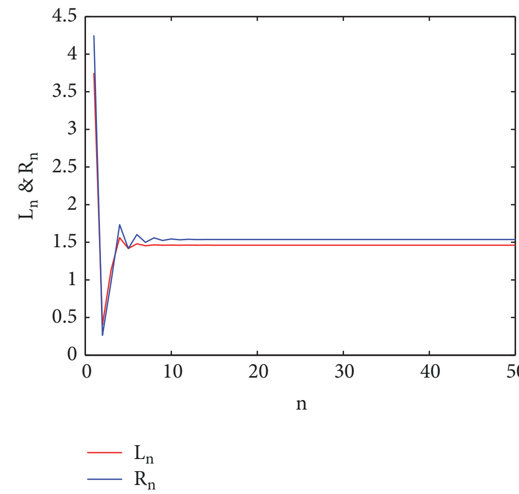

FIGURE 4: The solution of system (64) at $\alpha = 0.75$.

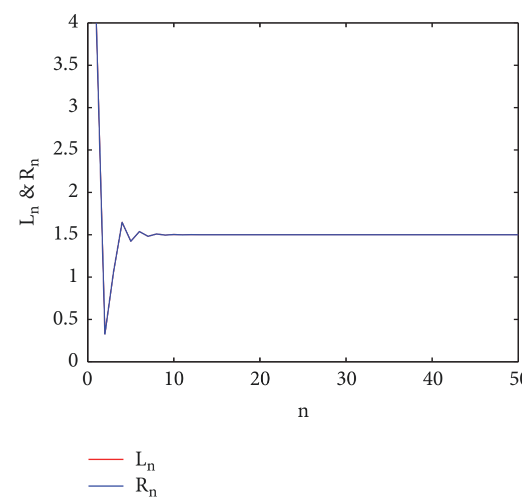

FIGURE 5: The solution of system (64) at $\alpha = 1$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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