Research Article

Sensitivities in Models with Backward Dynamics

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In this paper, we study some properties of economic model with backward dynamics. We mainly introduce the concept of P -sensitivity and P -Li-Yorke sensitivity especially for the multivalued forward dynamics of such models and characterize P -sensitivity and P -Li-Yorke sensitivity of the multivalued forward dynamics in terms of sensitivity and Li-Yorke sensitivity of an induced single-valued dynamics by using the theory of inverse limits. Similarly, we also characterize the sensitivity and Li-Yorke sensitivity of the single-valued backward dynamics of such models in terms of the corresponding sensitivities of an induced single-valued dynamics.

1. Introduction

In applications, trajectories generated by a dynamic system are often used to characterize the equilibrium of a dynamic economic model. In economics, there are lots of nonlinear dynamical systems with the property of single-valued moving backward, but multivalued going forward in time. In this paper, we call them models with backward dynamics the same as Medio and Raines called. Among models with backward dynamics, some of them are very popular, such as cash-in-advance (CIA) model, overlapping generations (OG) model, model of credit with limited commitment, and so forth. Just as Stockman characterized in [1] those models were often investigated by a local analysis method or by analyzing the models using their well-defined backward maps. But local analysis method may ignore some potentially interesting equilibria. In [2] Medio notes that the backward map solution is not entirely satisfactory because trajectories for the backward map and the equilibria for the models run in the opposite direction. In [1], the author also characterized that certain properties of the models with backward dynamics can be obtained by using the backward maps of the systems and introduced the concept of chaos for a multivalued dynamical systems where a concept of sensitivity of the multivalued map was raised as well. In [3] it was proved that Li-Yorke sensitivity does not imply Li-Yorke chaos. Inspired by the work of the authors mentioned above, in this paper our main purpose is

to consummate the theoretical framework for analysis models with backward dynamics.

2. Preliminaries

2.1. Model with Backward Dynamics. In this section, we introduce one model with backward dynamics mainly cited from [1, 4].

Mathematically, as the references mentioned an equilibrium in model with backward dynamics can be characterized by an implicitly-defined difference equation $G(x_{t+1}, x_t) = 0$. There are more than one solution for the difference equation when x_t was given, but given x_{t+1} there is only one solution x_t for $G(x_{t+1}, x_t) = 0$; i.e., the implicitly-defined difference equation can be simplified as $x_t = F(x_{t+1})$. In our paper we just take CIA model as an example for models with backward dynamics. Much more models with backward dynamics can be found in ([1, 5–7]).

After Lucas and Stokey introduced CIA model in [6], it was studied by many economists such as Michener and Ravikumar [8], David R. Stockman [1], Kennedy et al. [4], and others. The same as the references characterized CIA model, there is a representative agent and a government, but government do nothing except setting monetary policy using a money growth rule. It is an endowment economy with credit goods and cash goods, the preferences of household were represented by choosing a sequence of $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$,

and it was clearly represented by a utility function of the form

$$\sum_{t=0}^{\infty} \beta^t U(c_{1t}, c_{2t}), \quad (1)$$

in which the discount factor $0 < \beta < 1$, c_{1t}, c_{2t} represent cash goods and credit goods at time t , respectively. We take the following form as utility function as Kennedy did in [4]:

$$U(c_1, c_2) = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\gamma}}{1-\gamma}, \quad (2)$$

with $\sigma > 0$ and $\gamma > 0$. The household needs cash m_t carried from $t-1$ to pay for the cash good c_{1t} at time t . The credit good c_{2t} can be bought on credit. In each period the household has an endowment y which can be transformed into the cash and credit goods such that $c_{1t} + c_{2t} = y$. Technology allows the credit good to be substituted for the cash good one-for-one. Credit good, cash good, and endowment (per unit) sell at the same price p_t in equilibrium.

The aim of household is to maximize (1) by choosing sequence $\{c_{1t}, c_{2t}, m_{t+1}\}_{t=0}^{\infty}$ constrained by $c_{1t}, c_{2t}, m_{t+1} \geq 0$,

$$p_t c_{1t} \leq m_t, \quad (3)$$

$$m_{t+1} \leq p_t y + (m_t - p_t c_{1t}) + \theta M_t - p_t c_{2t}, \quad (4)$$

taking as given m_0 and $\{p_t, m_{t+1}\}_{t=0}^{\infty}$. The government supply money $\{M_t\}$ at a constant growth path $M_{t+1} = (1 + \theta)M_t$, where θ is the growth rate and $M_0 > 0$ is given. In each period the government transfer cash to the household in the amount θM_t . A perfect foresight equilibrium is defined in the usual manner.

The same as the references [1, 4] did, let $x_t = m_t/p_t$ denote the level of real money balance, and let c be the unique solution to $U_1(x, y - x) = U_2(x, y - x)$. When the cash-in-advance constraint (3) binds, let $c_{1t} = x_t$; otherwise let $c_{1t} = c$. In [8] the authors use this relationship to get a difference equation in x alone that characterizes equilibria in the model:

$$\begin{aligned} & x_t U_2(\min[x_t, c], y - \min[x_t, c]) \\ &= \frac{\beta}{1 + \theta} x_{t+1} U_1(\min[x_{t+1}, c], y - \min[x_{t+1}, c]) \end{aligned} \quad (5)$$

or

$$B(x_t) = A(x_{t+1}), \quad (6)$$

where

$$\begin{aligned} B(x) &= x U_2(\min[x, c], y - \min[x, c]), \\ A(x) &= \frac{\beta}{1 + \theta} x U_1(\min[x, c], y - \min[x, c]). \end{aligned} \quad (7)$$

One can always solve for the backward map $f = B^{-1} \circ A$; when the function B is invertible, let $x_t = f(x_{t+1})$. The dynamic going forward are multivalued when $A(\cdot)$ is not invertible. In [8], the authors set $\beta = 0.98, \gamma = 0.5, \sigma = 4, \gamma = 2$, and θ equal to 0, 0.5, 1.0. Then the function A is not invertible and there exists an invariant set $[x_l, x_h]$, such that the backward map exhibits sensitivity. The backward map with $\theta = 0$ is shown in Figure 1 reproduced from Kennedy et al. [4]. In this case, the CIA model has the property with backward dynamics.

FIGURE 1: Backward map $f : [x_l, x_h] \rightarrow [x_l, x_h]$ from the cash-in-advance model. It is reproduced from Kennedy et al. [4] [under the Creative Commons Attribution License/public domain].

2.2. Inverse Limits. In this section, we give a brief introduction of the concept of inverse limit and the method on how to use inverse limits to study systems with backward dynamics (for more details see [1, 9–11]).

Suppose $\{(X_i, d_i)\}_{i=1}^{\infty}$ is a sequence of compact metric spaces and $f_i : X_{i+1} \rightarrow X_i$ is a continuous map. Let $\varprojlim (X_i, f_i)_{i=1}^{\infty} = \{x = (x_1, x_2, \dots) : x_i \in X_i, x_i = f_i(x_{i+1}) \text{ for } i \in \mathbb{N}\}$; then $\varprojlim (X_i, f_i)_{i=1}^{\infty}$ is called the inverse limit space of the maps $\{f_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ are called bonding maps of the space. Note that $\varprojlim (X_i, f_i)_{i=1}^{\infty}$ is a compact metric subspace of the product space $\prod_{i=1}^{\infty} X_i$, where a metric $\tilde{d}$ on $\prod_{i=1}^{\infty} X_i$ is defined as the following:

$$\tilde{d}(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^{i-1}}. \quad (8)$$

If $g_i : X_i \rightarrow X_i$ is continuous and satisfies that $g_i \circ f_i = f_i \circ g_{i+1}$ for each $i \in \mathbb{N}$, then $\{g_i\}_{i=1}^{\infty}$ induces the map g_{∞} on the inverse limit space $\varprojlim (X_i, f_i)_{i=1}^{\infty}$ in the following way: $g_{\infty}(x) = (g_1(x_1), g_2(x_2), \dots)$ for each $x = (x_1, x_2, \dots) \in \varprojlim (X_i, f_i)$. We say $\{g_i\}_{i=1}^{\infty}$ original maps. In particular, if $X_i = X, d_i = d$ and $f_i = g_i = f$ for each $i \in \mathbb{N}$, then the space

$$\begin{aligned} & \varprojlim (X, f) \\ &= \{(x_1, x_2, \dots, x_n, \dots) : f(x_{i+1}) = x_i, i \in \mathbb{N}\} \end{aligned} \quad (9)$$

is called the inverse limit space of f . Let $m \in \mathbb{N}$; the map $\pi_m : \varprojlim (X, f) \rightarrow X$ defined by $\pi_m(x) = x_m$ is called the m^{th} projection map. Let X^{∞} be the infinite product of X endowed with usual product topology. Recall that the product topology is generated by the following basic open sets. Let $\{u_1, \dots, u_n\}$ be a finite collection of open sets in X . Define

$$\begin{aligned} & \langle u_1, u_2, \dots, u_n \rangle = \{x = (x_1, x_2, \dots, x_n, \dots) \\ & \in X^{\infty}, x_i \in u_i \text{ for } 1 \leq i \leq n\}. \end{aligned} \quad (10)$$

The collection $\{\langle u_1, u_2, \dots, u_n \rangle : \{u_1, u_2, \dots, u_n\} \text{ is a finite collection of open sets in } X\}$ is the collection of basic open sets of X^∞ with product topology. Since (X, d) is a compact metric space, $(X^\infty, \tilde{d})$ is a compact metric space and the product topology on X^∞ is same as the topology generated by the metric $\tilde{d}$ on X^∞ .

In the context of the economic model with backward dynamics, X, f denote the state space and backward map, respectively. The pair (X, f) is called inverse system. It is obvious that $\varprojlim (X, f) \subset X^\infty$ and points in the inverse limit space have special structure which forms backward solutions to the dynamical system $f : X \rightarrow X$. For f is the backward map of model, the points in the limit space also correspond to forward solutions of the implicit difference equation characterizing equilibrium in the model; i.e., the set of equilibrium in the model is an inverse limit space.

Let $M = \varprojlim (X, f)$. Now we induce a natural map σ_f on the inverse limit space M by the bonding map f as the following: for each $\mathbf{x} = (x_1, x_2, \dots) \in M$,

$$\begin{aligned} \sigma_f(\mathbf{x}) &\equiv \sigma_f((x_1, x_2, \dots)) = (f(x_1), f(x_2), \dots) \\ &= (f(x_1), x_1, x_2, \dots). \end{aligned} \quad (11)$$

It means that the induced map σ_f is a homeomorphism from M onto M , the inverse σ_{-f} of σ_f is defined by

$$\sigma_{-f}(\mathbf{x}) \equiv \sigma_{-f}((x_1, x_2, \dots)) = (x_2, x_3, \dots). \quad (12)$$

We call σ_{-f} the shift homeomorphism of M . Thus dynamical system (M, σ_f) was induced by the inverse system (X, f) , which is single-valued going both forward and backward in time. The approach is new to use inverse limits to analyze models with backward dynamics. In [9, 10] Medio and Raines use it to analyze the long-run behavior of an OG model; they show that typical long-run behavior of equilibria in the model corresponds to an attractor of the shift map on the inverse limit space. Kennedy et al. [11] discuss the topological structure for the inverse limit space associated with the CIA model. Recently, Stockman discusses the chaos properties for models with backward dynamics by using inverse limits approach in [1]. The complexity of inverse limit space and the complexity of the dynamical system are closely related. The main advantage of the inverse limit approach is to study models with backward dynamics. Treating an equilibrium in the model as a single point in a larger space can help us to study the sensitivity property of the dynamics of f^{-1} (which is multivalued) on X by studying the sensitivity property of the dynamics of σ_{-f} on M . Similarly, it also helps us to study the sensitivity property of the dynamics of f on X by studying the sensitivity property of the dynamics of σ_f on M .

2.3. Sensitivity of Dynamical Systems. The concept of sensitivity was first introduced by Auslander and Yorke [12] and popularized by Devaney [13]. The following is its definition.

Definition 1. Let (X, d) be metric space; $f : X \rightarrow X$ is a map. Given a positive ϵ , we consider pairs of points $(x, y) \in X \times X$ whose orbits are frequently at least ϵ apart, that is,

$$\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > \epsilon. \quad (13)$$

The system or f is sensitive on X if for some positive ϵ the set of pairs which satisfy this condition is dense in $X \times X$.

Recently many mathematicians generalized the definition of sensitivity. Here we mainly focus on the one generalized by Akin and Kolyada [14]; a system is called Li-Yorke sensitive if there exists $\epsilon > 0$ such that every $x \in X$ is a limit of points $y \in X$ such that the pair (x, y) is proximal but whose orbits are frequently at least ϵ apart. See the following.

Definition 2. Suppose (X, d) is a metric space and $f : X \rightarrow X$ is a map. If there exists $\epsilon > 0$ such that for any $\delta > 0$ and every $x \in X$ there exists $y \in B(x, \delta)$ such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) &> \epsilon \\ \text{and } \liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) &= 0, \end{aligned} \quad (14)$$

we say that f is Li-Yorke sensitive.

It was pointed out by Akin and Kolyada [14] that Li-Yorke sensitivity is strictly stronger than sensitivity.

2.4. Sensitivity of Backward Dynamics. In this section, we extend the definition of sensitivities to models with backward dynamics. What does it mean that the model with backward dynamics is sensitive? In the following definitions, we give our ways to describe the sensitivities of such models by treating the trajectories as points in a larger metric space.

Definition 3. Let (X, d) be a metric space, $f : X \rightarrow X$ is a map, and $S \subset X^\infty$. We say that f generates S if for each $(x_1, x_2, \dots) \in S$ we have $x_{i+1} = f(x_i)$ for $i \in \mathbb{N}$. We say that f^{-1} generates S if for each $(x_1, x_2, \dots) \in S$ we have $x_i = f(x_{i+1})$ for $i \in \mathbb{N}$.

Definition 4. Let (X, d) be a metric space, $f : X \rightarrow X$ is a map, and $S \subset X^\infty$. For a positive ϵ we consider pairs of points $(\mathbf{x}, \mathbf{y}) \in S \times S$ whose components are frequently at least ϵ apart, that is,

$$\limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) > \epsilon. \quad (15)$$

We say the projection map is sensitive on S or S is P -sensitive if for some positive ϵ the set of pairs which satisfy the above condition is dense in $S \times S$. If S is generated by f^{-1} and S is P -sensitive, then we say f^{-1} is P -sensitive on S .

Definition 5. Let (X, d) be a metric space, $f : X \rightarrow X$ is a map, and $S \subset X^\infty$. We say that S is P -Li-Yorke sensitive if

there exists $\epsilon > 0$ such that for any $\delta > 0$ and every $\mathbf{x} \in S$ there exists $\mathbf{y} \in B(\mathbf{x}, \delta)$ such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) &> \epsilon \\ \text{and } \liminf_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) &= 0. \end{aligned} \quad (16)$$

If S is generated by f^{-1} and S is P -Li-Yorke sensitive, then we say f^{-1} is P -Li-Yorke sensitive on S .

Remark 6. Let (X, f) be an inverse system. The direct limit space D of (X, f) was defined as

$$D = \varinjlim (X, f) = \{\mathbf{x} \in X^\infty \mid x_{i+1} = f(x_i), i \in \mathbb{N}\}. \quad (17)$$

Let $M = \varprojlim (X, f)$, then D is generated by f and M is generated by f^{-1} . These are different subsets of X^∞ (and when sensitivity occurs, they are very different spaces topologically). The examples to illustrate the relationship between these sets could be found in Stockman [1]. If M is P -sensitive or P -Li-Yorke sensitive, then we say the inverse system is P -sensitive or P -Li-Yorke sensitive.

Remark 7. Let (X, d) be a metric space and $f : X \rightarrow X$ a map. $(X^\infty, \tilde{d})$ is the infinite product metric space with $\tilde{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} (d_i(x_i, y_i)/2^{i-1})$, and $S \subset X^\infty$, where S is generated by f . Then for any pair of $(\mathbf{x}, \mathbf{y}) \in S \times S$, $\mathbf{x} = (x, f(x), \dots) \in S$ and $\mathbf{y} = (y, f(y), \dots) \in S$. We have $d(x, y) \leq \tilde{d}(\mathbf{x}, \mathbf{y})$. So if $d(x, y) > \epsilon$, then $\tilde{d}(\mathbf{x}, \mathbf{y}) > \epsilon$. Since

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \\ = \limsup_{n \rightarrow \infty} d(f^{n-1}(x), f^{n-1}(y)) \end{aligned} \quad (18)$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \\ = \liminf_{n \rightarrow \infty} d(f^{n-1}(x), f^{n-1}(y)). \end{aligned} \quad (19)$$

That is if f is sensitive or Li-Yorke sensitive then S is P -sensitive or P -Li-Yorke sensitive.

3. The Main Results

In this section, we use the theory of inverse limit to characterize the P -sensitivity and P -Li-Yorke sensitivity of a multi-valued backward dynamical system (X, f^{-1}) in terms of the sensitivity and Li-Yorke sensitivity of the induced dynamical system (M, σ_{-f}) . We also characterize the sensitivity and Li-Yorke sensitivity of the single-valued dynamical system (X, f) in terms of sensitivity and Li-Yorke sensitivity of the induced dynamical system (M, σ_f) . In order to show our results, the following lemmas (details of proof: see ([1]) are needed.

Lemma 8. Let d be the metric on X and let $\tilde{d}$ be the metric on X^∞ induced by d . Then for $\mathbf{x}, \mathbf{y} \in X^\infty$ and $r > 0$, if

$\tilde{d}(\sigma_{-f}^k(\mathbf{x}), \sigma_{-f}^k(\mathbf{y})) > r$ for some $k \geq 0$, then there exists $m > k$ such that $d(x_m, y_m) > r/2$.

Lemma 9. Let d be the metric on X such that (X, d) is a compact space and let $\tilde{d}$ be the metric on X^∞ induced by d . Suppose $f : X \rightarrow X$ is continuous and onto, and $M = \varprojlim (X, f)$. Let $K = \max_{x, y \in X} d(x, y)$. Then for $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots) \in X^\infty$, and given $r > 0$, if $\tilde{d}(\sigma_f^k(\mathbf{x}), \sigma_f^k(\mathbf{y})) > r$ for some $k > 0$ with $K/2^{k-1} < r/2$, then there exists $1 \leq m \leq k$ such that $d(f^m(x_1), f^m(y_1)) > r/4$.

The following result characterizes P -sensitivity of a multi-valued dynamical system (X, f^{-1}) in terms of sensitivity of the induced single-valued dynamical system (M, σ_{-f}) .

Theorem 10. Suppose that (X, d) is a compact metric space, $f : X \rightarrow X$ is continuous, and X^∞ is the infinite product space with the metric

$$\tilde{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}. \quad (20)$$

Let $M = \varprojlim (X, f)$, $\sigma_{-f} : M \rightarrow M$ be the shift homeomorphism. Then f^{-1} is P -sensitive if and only if σ_{-f} is sensitive.

Proof. $\Rightarrow$: Since for any $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots) \in M$ and any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(\pi_{n+1}(\mathbf{x}), \pi_{n+1}(\mathbf{y})) &= d(x_{n+1}, y_{n+1}) \\ &\leq \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})). \end{aligned} \quad (21)$$

The result follows from the definitions of sensitivity and P -sensitivity immediately.

$\Leftarrow$: Let $K = \max_{x, y \in X} d(x, y)$. Since σ_{-f} is sensitive, there exists $\epsilon > 0$ such that for any $\delta > 0$ and every $\mathbf{x} \in M$ there exists $\mathbf{y} \in B(\mathbf{x}, \delta)$ such that

$$2K \geq \limsup_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) > \epsilon. \quad (22)$$

Let

$$c = \limsup_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) > 0. \quad (23)$$

It means there exists a subsequence such that

$$\lim_{j \rightarrow \infty} \tilde{d}(\sigma_{-f}^{n_j}(\mathbf{x}), \sigma_{-f}^{n_j}(\mathbf{y})) = c > 0. \quad (24)$$

For $0 < c - \epsilon$, there exists an N such that for all $j \geq N$

$$0 < c - \epsilon < \tilde{d}(\sigma_{-f}^{n_j}(\mathbf{x}), \sigma_{-f}^{n_j}(\mathbf{y})). \quad (25)$$

Our aim is to show that there exists a subsequence of $\{d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))\}_{n \in \mathbb{N}}$ converging to something strictly positive. Since X is compact. This subsequence will have a convergent sub-subsequence strictly bounded away from 0,

i.e., $\limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) = c^* > 0$. Choose $t_1 = n_1$ such that

$$0 < c - \epsilon < \tilde{d}(\sigma_{-f}^{t_1}(\mathbf{x}), \sigma_{-f}^{t_1}(\mathbf{y})). \quad (26)$$

By Lemma 8, then there exists an $m_1 \geq t_1$ such that

$$d(\pi_{m_1}(\mathbf{x}), \pi_{m_1}(\mathbf{y})) \geq \frac{(c - \epsilon)}{2}. \quad (27)$$

Then we can choose $n_j > m_1$, calling it t_2 . Then by the same reasoning and repeating this process, we get an sequence $m_i < m_{i+1}$ and $m_i \rightarrow \infty$ with the property that

$$d(\pi_{m_i}(\mathbf{x}), \pi_{m_i}(\mathbf{y})) \geq \frac{(c - \epsilon)}{2}, \quad \forall i \in \mathbb{N}. \quad (28)$$

Since X is compact then

$$d(\pi_{m_i}(\mathbf{x}), \pi_{m_i}(\mathbf{y})) \in \left[\frac{(c - \epsilon)}{2}, K \right], \quad \forall i \in \mathbb{N}. \quad (29)$$

Since $[(c - \epsilon)/2, K]$ is compact, there exists a subsequence of $\{d(\pi_{m_i}(\mathbf{x}), \pi_{m_i}(\mathbf{y}))\}_{i=1}^{\infty}$ converging to a point $c^* \in [(c - \epsilon)/2, K]$; i.e., there exists a subsequence $\{d(\pi_{k_i}(\mathbf{x}), \pi_{k_i}(\mathbf{y}))\}_{i=1}^{\infty}$ with

$$\lim_{i \rightarrow \infty} d(\pi_{k_i}(\mathbf{x}), \pi_{k_i}(\mathbf{y})) = c^* \geq \frac{(c - \epsilon)}{2}, \quad (30)$$

hence

$$\limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) \geq c^* \geq \frac{(c - \epsilon)}{2} > 0. \quad (31)$$

□

The following result characterizes P -Li-Yorke sensitivity of a multivalued dynamical system (X, f^{-1}) in terms of Li-Yorke sensitivity of the induced single-valued dynamical system (M, σ_{-f}) .

Theorem 11. Suppose that (X, d) is a compact metric space, $f : X \rightarrow X$ is continuous, and X^{∞} is the infinite product space with the metric

$$\tilde{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}. \quad (32)$$

Let $M = \varprojlim (X, f)$ and $\sigma_{-f} : M \rightarrow M$ be the shift homeomorphism. Then f^{-1} is P -Li-Yorke sensitive if and only if σ_{-f} is Li-Yorke sensitive.

Proof. $\Rightarrow$: Since f^{-1} is P -Li-Yorke sensitive, there exists $\epsilon > 0$ such that for any $\delta > 0$ and every $\mathbf{x} \in M$ there exists $\mathbf{y} \in B(\mathbf{x}, \delta)$ such that

$$\limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) > \epsilon \text{ and} \quad (33)$$

$$\liminf_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) = 0. \quad (34)$$

Since for any $n \in \mathbb{N}$, we have

$$\sigma_{-f}^n(\mathbf{x}) = (x_{n+1}, x_{n+2}, \dots) = (\pi_{n+1}(\mathbf{x}), \pi_{n+2}(\mathbf{x}), \dots) \quad (35)$$

and

$$\begin{aligned} \sigma_{-f}^n(\mathbf{y}) &= (y_{n+1}, y_{n+2}, \dots) \\ &= (\pi_{n+1}(\mathbf{y}), \pi_{n+2}(\mathbf{y}), \dots). \end{aligned} \quad (36)$$

So

$$\begin{aligned} d(\pi_{n+1}(\mathbf{x}), \pi_{n+1}(\mathbf{y})) &\leq \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) \\ &= \sum_{i=1}^{\infty} \frac{d(\pi_{n+i}(\mathbf{x}), \pi_{n+i}(\mathbf{y}))}{2^{i-1}}, \end{aligned} \quad (37)$$

and it follows that

$$\begin{aligned} \epsilon &< \limsup_{n \rightarrow \infty} d(\pi_{n+1}(\mathbf{x}), \pi_{n+1}(\mathbf{y})) \\ &\leq \limsup_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})). \end{aligned} \quad (38)$$

Then we need to show that

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) = 0 \quad (39)$$

To this end, it is sufficient to show that for any $\eta > 0$, there exists a subsequence $n_1, n_2, \dots$, with $n_j \rightarrow \infty$ such that

$$\tilde{d}(\sigma_{-f}^{n_j}(\mathbf{x}), \sigma_{-f}^{n_j}(\mathbf{y})) < \eta. \quad (40)$$

Let $K = \max_{x, y \in X} d(x, y)$. Then

$$\begin{aligned} \tilde{d}(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^T \frac{d(x_i, y_i)}{2^{i-1}} + \sum_{i=T+1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}} \\ &\leq \sum_{i=1}^T \frac{d(x_i, y_i)}{2^{i-1}} + K \sum_{i=T+1}^{\infty} \frac{1}{2^{i-1}} \\ &= \sum_{i=1}^T \frac{d(x_i, y_i)}{2^{i-1}} + \frac{K}{2^{T-1}} \end{aligned} \quad (41)$$

Pick $T \in \mathbb{N}$ large enough such that $K/2^{T-1} < \eta/2$. Then pick $\delta > 0$ such that

$$\sum_{i=1}^T \frac{\delta}{2^{i-1}} = \frac{1 - (1/2)^T}{1 - (1/2)} \delta < \frac{\eta}{2}. \quad (42)$$

Since f is continuous and X is compact, f^n is uniformly continuous on X for any $n \in \mathbb{N}$. For the given $\delta > 0$, there exists $\tau_n > 0$ such that $d(f^n(x), f^n(y)) < \delta$ if $d(x, y) < \tau_n$, $n \in 1, 2, \dots, T$. Let $\tau = \min\{\delta, \tau_1, \tau_2, \dots, \tau_T\}$. Since $\liminf_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) = 0$, there exists $\{m_i\}_{i=1}^{\infty}$ such that $d(\pi_{m_i}(\mathbf{x}), \pi_{m_i}(\mathbf{y})) \leq \tau$, $m_1 > T + 1$ and $m_{i+1} - m_i > T + 1$. Let $n_i = m_i - (T + 1)$. We have

$$\begin{aligned} \sigma_{-f}^n(\mathbf{x}) &= (f^T(x_{m_i}), f^{T-1}(x_{m_i}), \dots, f^2(x_{m_i}), f(x_{m_i}), \\ &\quad x_{m_i}, x_{m_{i+1}}, \dots), \\ \sigma_{-f}^{n_i}(\mathbf{y}) &= (f^T(y_{m_i}), f^{T-1}(y_{m_i}), \dots, f^2(y_{m_i}), f(y_{m_i}), \\ &\quad y_{m_i}, y_{m_{i+1}}, \dots), \end{aligned} \quad (43)$$

Since $d(x_{m_i}, y_{m_i}) < \tau$, we have $d(f^k(x_{m_i}), f^k(y_{m_i})) < \delta$ for $k = 1, 2, \dots, T$. Then

$$\begin{aligned} \tilde{d}(\sigma_{-f}^{n_j}(\mathbf{x}), \sigma_{-f}^{n_j}(\mathbf{y})) &< \frac{\eta}{2} + \frac{\eta}{2} = \eta. \\ \liminf_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) &\leq \eta \end{aligned} \quad (44)$$

Since this is true for any $\eta > 0$, we have

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) = 0. \quad (45)$$

So σ_{-f} is Li-Yorke sensitive.

$\Leftarrow$: Since σ_{-f} is Li-Yorke sensitive, there exists $\epsilon > 0$ such that for any $\delta > 0$ and every $\mathbf{x} \in M$ there exists $\mathbf{y} \in B(\mathbf{x}, \delta)$ such that

$$\limsup_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) > \epsilon \quad (46)$$

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) = 0 \quad (47)$$

Note that by definition of $\tilde{d}$ and the fact that d is a metric, we have

$$\begin{aligned} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) &= \sum_{i=1}^{\infty} \frac{d(x_{n+i}, y_{n+i})}{2^{i-1}} \\ &\geq d(x_{n+1}, y_{n+1}) \\ &= d(\pi_{n+1}(\mathbf{x}), \pi_{n+1}(\mathbf{y})). \end{aligned} \quad (48)$$

This implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} d(\pi_{n+1}(\mathbf{x}), \pi_{n+1}(\mathbf{y})) \\ \leq \liminf_{n \rightarrow \infty} \tilde{d}(\sigma_{-f}^n(\mathbf{x}), \sigma_{-f}^n(\mathbf{y})) = 0. \end{aligned} \quad (49)$$

Hence we have

$$\liminf_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) = 0. \quad (50)$$

The same argument from Theorem 10 gives the need condition for $\limsup_{n \rightarrow \infty} d(\pi_n(\mathbf{x}), \pi_n(\mathbf{y})) > \epsilon$. So f^{-1} is P -Li-Yorke sensitive. $\square$

The following result characterizes Li-Yorke sensitivity of a single-valued dynamical system (X, f) in terms of Li-Yorke sensitivity of the induced single-valued dynamical system (M, σ_f) .

Theorem 12. Suppose that (X, d) is a compact metric space, $f : X \rightarrow X$ is continuous, and X^∞ is the infinite product space with the metric

$$\tilde{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}. \quad (51)$$

Let $M = \varprojlim (X, f)$ and $\sigma_f : M \rightarrow M$ be the induced homeomorphism. Then f is Li-Yorke sensitive if and only if σ_f is Li-Yorke sensitive.

Proof. $\Rightarrow$ For f is Li-Yorke sensitive, let $x_1 \in X$ and $\mathbf{x} = (x_1, x_2, \dots) \in M$; there exists $\epsilon > 0$ such that for any $\delta > 0$ there exists $y_1 \in B(x_1, \delta)$ such that

$$\limsup_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) > \epsilon, \quad (52)$$

$$\liminf_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) = 0. \quad (53)$$

Let $\mathbf{y} = (y_1, y_2, \dots) \in M$. Since

$$\tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) \geq d(f^n(x_1), f^n(y_1)) \quad (54)$$

it follows that

$$\limsup_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) > \epsilon. \quad (55)$$

Then we need to prove that

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) = 0. \quad (56)$$

That is, for any $\eta > 0$, there exists a positive integer sequence $\{n_i\}_{i=1}^{\infty}$ with $n_i \rightarrow \infty, i \rightarrow \infty$ such that

$$\tilde{d}(\sigma_f^{n_i}(\mathbf{x}), \sigma_f^{n_i}(\mathbf{y})) < \eta. \quad (57)$$

Let $K = \max_{x, y \in X} d(x, y)$. Then

$$\begin{aligned} \tilde{d}(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^T \frac{d(x_i, y_i)}{2^{i-1}} + \sum_{i=T+1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}} \\ &\leq \sum_{i=1}^T \frac{d(x_i, y_i)}{2^{i-1}} + K \sum_{i=T+1}^{\infty} \frac{1}{2^{i-1}} \\ &= \sum_{i=1}^{T-1} \frac{d(x_i, y_i)}{2^{i-1}} + \frac{K}{2^{T-1}}. \end{aligned} \quad (58)$$

Pick $T \in \mathbb{N}$ large enough so that $(K/2^{T-1}) < \eta/2$. Then pick $\delta > 0$ such that

$$\sum_{i=1}^T \frac{\delta}{2^{i-1}} = \frac{1 - (1/2)^T}{1 - (1/2)} \delta < \frac{\eta}{2}. \quad (59)$$

Since f is continuous and X is compact, f^n is uniformly continuous on X for any $n \in \mathbb{N}$. Given $\delta > 0$, there exists $\tau_n > 0$ such that $d(f^n(x), f^n(y)) < \delta$ if $d(x, y) < \tau_n$, $n \in \{1, 2, \dots, T\}$. Let $\tau = \min\{\delta, \tau_1, \tau_2, \dots, \tau_T\}$. Since $\liminf_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) = 0$, there exists $\{m_i\}_{i=1}^{\infty}$ such that $d(f^{m_i}(x_1), f^{m_i}(y_1)) < \tau$, $m_1 > T + 1$ and $m_{i+1} - m_i > T + 1$. Let $n_i = m_i + T$. Then

$$\begin{aligned} \sigma_f^{n_i}(\mathbf{x}) &= (f^T \circ f^{m_i}(x_1), f^{T-1} \circ f^{m_i}(x_1), \dots, f^2 \\ &\quad \circ f^{m_i}(x_1), f \circ f^{m_i}(x_1), f^{m_i}(x_1), \dots), \\ \sigma_f^{n_i}(\mathbf{y}) &= (f^T \circ f^{m_i}(y_1), f^{T-1} \circ f^{m_i}(y_1), \dots, f^2 \\ &\quad \circ f^{m_i}(y_1), f \circ f^{m_i}(y_1), f^{m_i}(y_1), \dots), \end{aligned} \quad (60)$$

Since $d(f^{m_i}(x_1), f^{m_i}(y_1)) < \tau$, we have $d(f^k \circ f^{m_i}(x_1), f^k \circ f^{m_i}(y_1)) < \delta$ for $k = 1, 2, \dots, T$. For any $\eta > 0$, the following inequality holds

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) \leq \eta. \quad (61)$$

That is,

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) = 0. \quad (62)$$

For any $\delta > 0$, we choose T large enough such that $\sum_{i=T}^{\infty} (K/2^{i-1}) < \delta/2$, with $K = \max_{x,y \in X} d(x, y)$. Since f^i ($i = 1, 2, \dots, T-1$) is uniformly continuous on X , there exists $\delta_0 > 0$ such that $d(f^i(x), f^i(y))/2^{T-i-1} < \delta/2(T-1)$, ($i = 1, 2, \dots, T-1$) whenever $d(x, y) < \delta_0$. For any $\mathbf{x} = (x_1, x_2, \dots, x_T, \dots) \in M$, let $\mathbf{y} = (y_1, y_2, \dots, y_T, \dots) \in M$ such that $d(y_T, x_T) < \delta_0$; then $d(x_1, y_1) = d(f^{T-1}(x_T), f^{T-1}(y_T)) < \delta/2(T-1) < \delta$. By the above arguments, we have $\limsup_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) > \epsilon$ and $\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) = 0$. Since

$$\begin{aligned} \tilde{d}(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} d(x_i, y_i) \\ &= \sum_{i=1}^{T-1} \frac{1}{2^{i-1}} d(x_i, y_i) + \sum_{i=T}^{\infty} \frac{1}{2^{i-1}} d(x_i, y_i) \\ &= \sum_{i=1}^{T-1} \frac{1}{2^{i-1}} d(f^i(x_T), f^i(y_T)) \\ &\quad + \sum_{i=T}^{\infty} \frac{1}{2^{i-1}} d(x_i, y_i) \\ &< (T-1) \cdot \frac{\delta}{2(T-1)} + \sum_{i=T}^{\infty} \frac{K}{2^{i-1}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned} \quad (63)$$

hence $\mathbf{y} \in B(\mathbf{x}, \delta)$. So σ_f is Li-Yorke sensitive.

($\Leftarrow$) Suppose that σ_f is Li-Yorke sensitive. Then there exists $\epsilon > 0$, for any $\mathbf{x} = (x_1, x_2, \dots) \in M$ and for any $\delta > 0$, there exists $\mathbf{y} = (y_1, y_2, \dots) \in B(\mathbf{x}, \delta)$ such that

$$\limsup_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) > \epsilon, \text{ and} \quad (64)$$

$$\liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) = 0. \quad (65)$$

By the definition of $\tilde{d}$, we have that $d(y_1, x_1) < \tilde{d}(\mathbf{x}, \mathbf{y}) < \delta$. And similarly, for any $n \in \mathbb{N}$ we have

$$\tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) \geq d(f^n(x_1), f^n(y_1)), \quad (66)$$

which implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) \\ \leq \liminf_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) = 0. \end{aligned} \quad (67)$$

So

$$\liminf_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) = 0. \quad (68)$$

Let

$$c = \limsup_{n \rightarrow \infty} \tilde{d}(\sigma_f^n(\mathbf{x}), \sigma_f^n(\mathbf{y})) > 0. \quad (69)$$

This implies the existence of a subsequence with property

$$\lim_{j \rightarrow \infty} \tilde{d}(\sigma_f^{n_j}(\mathbf{x}), \sigma_f^{n_j}(\mathbf{y})) = c > 0 \quad (70)$$

For $0 < r < c$, there exists an N such that for all $j \geq N$ we have

$$0 < r < \tilde{d}(\sigma_f^{n_j}(\mathbf{x}), \sigma_f^{n_j}(\mathbf{y})). \quad (71)$$

Let $k > 1$ such that $K/2^{k-1} < r/2$. Without loss of generality assume that $n_1 > k$ and $n_{j+1} - n_j > k$. Let $k_1 = n_1$, and $k_{i+1} = n_{i+1} - n_i$ for $i \in \mathbb{N}$. Note for each k_i , we have $K/2^{k_i-1} < r/2$. Since

$$\tilde{d}(\sigma_f^{n_j}(\mathbf{x}), \sigma_f^{n_j}(\mathbf{y})) > r, \quad (72)$$

by Lemma 9, there exists m_j with $n_{j-1} \leq m_j \leq n_j$ with

$$d(f^{m_j}(x_1), f^{m_j}(y_1)) \geq \frac{r}{4}. \quad (73)$$

This implies that

$$\limsup_{n \rightarrow \infty} d(f^n(x_1), f^n(y_1)) \geq \frac{r}{4} > 0. \quad (74)$$

Hence f is Li-Yorke sensitive. $\square$

Similar to the proof of Theorem 12, we have the following theorem, which characterizes sensitivity of single-valued dynamical system (X, f) in terms of sensitivity of the induced single-valued dynamical system (M, σ_f) .

Theorem 13. Suppose that (X, d) is a compact metric space, $f : X \rightarrow X$ is continuous, and X^∞ is the infinite product space with the metric

$$\tilde{d}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^{i-1}}. \quad (75)$$

Let $M = \varprojlim (X, f)$ and $\sigma_f : M \rightarrow M$ be the induced homeomorphism. Then f is sensitive if and only if σ_f is sensitive.

4. Conclusion

In some economic models, the dynamical system characterizing equilibria in the model has multivalued forward dynamics but single-valued backward dynamics, i.e., models with backward dynamics. In this paper, we offered a definition of P -sensitive and P -Li-Yorke sensitive for such multivalued forward dynamical systems, and by using of theory of inverse limits, we characterize P -sensitive and P -Li-Yorke sensitive

of a multivalued dynamics on X in terms of sensitivity and Li-Yorke sensitivity of a single-valued forward dynamics on a subspace of X^∞ .

In the future research, we will continue to investigate how does P -sensitivity and P -Li-Yorke sensitivity about the backward orbits of f correspond to equilibria in the model. We are also interested in the tracing property for such system with backward dynamics which are motivated by the work of Gu and Sheng [15] and Wu and Chen ([16–18]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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