Hindawi
Discrete Dynamics in Nature and Society
Volume 2019, Article ID 4902718, 9 pages
https://doi.org/10.1155/2019/4902718

Research Article

Oscillation for a Class of Right Fractional Differential Equations on the Right Half Line with Damping

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Received 24 January 2019; Revised 15 March 2019; Accepted 17 March 2019; Published 1 April 2019

Academic Editor: Chris Goodrich

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In this paper, we discuss a class of fractional differential equations of the form $D_-^{\alpha+1}y(t)\cdot D_-^{\alpha}y(t)-p(t)f(D_-^{\alpha}y(t))+q(t)h(\int_t^{\infty}(s-t)^{-\alpha}y(s)ds)=0.D_-^{\alpha}y(t)$ is the Liouville right-sided fractional derivative of order $\alpha\in(0,1)$. We obtain some oscillation criteria for the equation by employing a generalized Riccati transformation technique. Some examples are given to illustrate the significance of our results.

1. Introduction

The theory of fractional derivatives was originated from G.W. Leibniz's conjecture. To this day, the theory about fractional calculus and fractional differential equation have been well developed; see [1–7]. In the beginning, the theory of fractional derivatives developed mainly as a pure theoretical filed of mathematics, which can be used only for mathematicians. However, in the past few decades, fractional differential equations were widely used in many fields, such as fluid flow, rheology, electrical networks, and many other branches of science. Great attention was paid to study the properties of solutions of fractional differential equations.

Because only few differential equations can be solved, many researches focus on the analysis of qualitative theory for fractional differential equations, such as the existence, uniqueness of solutions, numerical solutions, stability, and oscillation of solutions; see [8–34] and the references therein. Among them, there have been many results for the oscillation of solutions for fractional differential equations.

In 2013, Chen [16] studied oscillatory behavior of the fractional differential equation in the form of

$$\begin{aligned}&D_-^{\alpha+1}y(t)-p(t)D_-^{\alpha}y(t)\\&\quad+q(t)f\left(\int_t^{\infty}(v-t)^{-\alpha}y(v)\,dv\right)=0,\end{aligned}\tag{1}$$

for $t>0$, where $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of order $\alpha\in(0,1)$.

In 2013, Han [17] brought up the oscillation of fractional differential equations

$$\begin{aligned}&\left[r(t)g\left(\left(D_-^{\alpha}y\right)(t)\right)\right]'-p(t)f\left(\int_t^{\infty}(s-t)^{-\alpha}y(s)\right)\\&\quad=0,\end{aligned}\tag{2}$$

for $t>0$, where $0<\alpha<1$ is a real number, and $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of y.

In 2013, Xu [18] studied the oscillation of nonlinear fractional differential equations of the form

$$\begin{aligned}&\left\{a(t)\left[\left(r(t)D_-^{\alpha}x(t)\right)'\right]^{\eta}\right\}'\\&\quad-F\left(t,\int_t^{\infty}(v-t)^{-\alpha}x(v)\,dv\right)=0,\quad t\ge t_0>0,\end{aligned}\tag{3}$$

where $\alpha\in(0,1)$ is a constant, and η is a ratio of two odd positive integers.

In 2013, based on the modified Riemann-Liouville derivative, Qin and Zheng [19] discussed the oscillation of a class of fractional differential equations with damping term as follows:

$$D_t^{\alpha}\left[a(t)\,D_t^{\alpha}\left(r(t)\,D_t^{\alpha}x(t)\right)\right] + p(t)\,D_t^{\alpha}\left(r(t)\,D_t^{\alpha}x(t)\right) + q(t)\,x(t) = 0, \tag{4}$$

for $t \geq t_0 > 0, 0 < \alpha < 1$, where $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative regarding the variable t, the function $a \in C^{\alpha}([t_0,\infty), R_+), r \in C^{2\alpha}([t_0,\infty), R_+), p, q \in C^{\alpha}([t_0,\infty), R_+)$, and C^{α} denotes continuous derivative of order.

In 2014, Jehad Alzabut and Thabet Abdeljawad [20] studied the oscillatory theory of fractional difference equations in the form

$$\begin{gathered}\nabla_{a(q)-1}^{q}x(t) + f_1(t, x(t) = r(t) + f_2(t, x(t)), \\ t \in \mathbb{N}_{a(q)}, \\ \nabla_{a(q)-1}^{-(1-q)}x(t)\Big|_{t=a(q)} = x(a(q)) = c, \quad c \in \mathbb{R},\end{gathered} \tag{5}$$

where $m - 1 < q < m, m \in \mathbb{N}, \nabla_{a(q)}^{q}$ is the Riemann-Liouville difference operator of order q and $\nabla_{a(q)}^{-q}$ is the Riemann-Liouville sum operator where $\mathbb{N}_{a(q)} = \{a(q)+1, a(q)+2, \ldots\}, a(q) = a + m - 1, m = [q] + 1$ and $[q]$ is the greatest integer less than or equal to q.

In 2017, B. Abdalla, K. Abodayeh, T. Abdeljawad, J. Alzabut [21] studied the oscillation of solutions of nonlinear forced fractional difference equations in the form

$$\begin{gathered}\nabla_{a(q)-1}^{q}x(t) + f_1(t, x(t) = r(t) + f_2(t, x(t)), \\ t \in \mathbb{N}_{a(q)}, \\ \nabla_{a(q)-1}^{-(m-q)}x(t)\Big|_{t=a(q)} = x(a(q)) = c, \quad c \in \mathbb{R},\end{gathered} \tag{6}$$

where $q > 0, m = [q] + 1, m \in \mathbb{N}, [q]$ is the greatest integer less than or equal to q, $\mathbb{N}_{a(q)} = \{a(q)+1, a(q)+2, \ldots\}, a(q) = a + m - 1, f_i : \mathbb{N}_{a(q)} \times \mathbb{R} \longrightarrow \mathbb{R} (i = 1, 2)$, and $\nabla_{a(q)}^{-q}$ and $\nabla_{a(q)}^{q}$ are the Riemann-Liouville sum and difference operators.

In 2018, Bai and Xu [22] discussed the oscillation problem of a class of nonlinear fractional difference equations with the damping term in the from

$$\begin{gathered}\Delta\left(c(t)\left[\Delta\left(r(t)\,\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) + p(t)\left[\Delta\left(r(t)\,\Delta^{\alpha}x(t)\right)\right]^{\gamma} \\ + q(t)\,f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right) = 0, \\ t \in N_{t_0},\end{gathered} \tag{7}$$

where $\gamma \geq 1$ is a quotient of two odd positive integers, $0 < \alpha \leq 1$ is a constant, Δ^{α} denotes the Riemann-Liouville fractional difference operator of order α, and $N_{t_0} = \{t_0, t_0+1, t_0+2, \ldots\}$.

In 2018, Bahaaeldin Abdalla and Thabet Abdeljawad [23] studied the oscillation of Hadamard fractional differential equation of the form

$$\begin{gathered}D_a^{\alpha}x(t) + f_1(t, x) = r(t) + f_2(t, x), \quad t > a, \\ \lim_{t\longrightarrow a^+} D_a^{\alpha-j}x(t) = b_j \quad (j = 1, 2, \ldots, n)\end{gathered} \tag{8}$$

where $n = \lceil\alpha\rceil$, D_a^{α} is the left-fractional Hadamard derivative of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) \geq 0$ in the Riemann-Liouville setting.

In 2018, J. Alzabut, T. Abdeljawad, H. Alrabaiah [24] considered the following forced and damped nabla fractional difference equation

$$\begin{gathered}\left(1 - p(n)\right)\nabla\nabla_0^{\alpha}y(n) + p(n)\,\nabla_0^{\alpha}y(n) + q(n)\,f\left(y(n)\right) \\ = g(n), \quad n \in \mathbb{N}_1, \\ \nabla_0^{-(1-\alpha)}y(1) = y(1) = c,\end{gathered} \tag{9}$$

where $\nabla_0^{\alpha}y$ and $\nabla_0^{-\alpha}y$ are the Riemann-Liouville fractional difference and sum operators of y of order α, respectively, α is a real number, c is constant, $\mathbb{N}_1 = 1, 2, \ldots$ and p, q are real sequences from $\mathbb{N}_1 \longrightarrow \mathbb{R}, p(n) < 1, q$ is a positive real sequence from $\mathbb{N}_1 \longrightarrow \mathbb{R}^+$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(s)/s > 0$ for all $s \neq 0$.

In 2018, B. Abdalla, J. Alzabut, T. Abdeljawad [25] investigated the oscillation of solutions for fractional difference equations with mixed nonlinearities in forms

$$\begin{gathered}\nabla_{a(\alpha)-1}^{\alpha}x(t) - p(t)\,x(t) + \sum_{i=1}^{n}q_i(t)\,|x(t)|^{\lambda_i-1}\,x(t) \\ = v(t), \quad t \in \mathbb{N}_{a(\alpha)+1}, \\ \nabla_{a(\alpha-1)}^{-(m-a)}x(t)\Big|_{t=a(\alpha)} = x(a(\alpha)) = c, \quad c \in \mathbb{R},\end{gathered} \tag{10}$$

and

$$\begin{gathered}{}^{c}\nabla_{a(\alpha)}^{\alpha}x(t) - p(t)\,x(t) + \sum_{i=1}^{n}q_i(t)\,|x(t)|^{\lambda_i-1}\,x(t) \\ = v(t), \quad t \in \mathbb{N}_{a(\alpha)}, \\ \nabla^{k}x(a(\alpha)) = b_k, \quad k \in \mathbb{R},\ k = 0, 1, 2, \ldots, m-1,\end{gathered} \tag{11}$$

where $m = [\alpha] + 1, \alpha > 0, p(t), v(t)$ and $q_i(t)(1 \leq i \leq n)$ are functions defined from $\mathbb{N}_{a(\alpha)}$ to R, and $\lambda_i(1 \leq i \leq n)$ are ratios of odd positive integers with $\lambda_1 > \ldots > \lambda_l > 1 > \lambda_{l+1} > \ldots > \lambda_n$.

Inspired by the above results, in this paper, we discuss the oscillatory behavior of the fractional differential equational with damping

$$\begin{gathered}D_-^{\alpha+1}y(t) \cdot D_-^{\alpha}y(t) - p(t)\,f\left(D_-^{\alpha}y(t)\right) \\ + q(t)\,h\left(\int_t^{\infty}(s-t)^{-\alpha}\,y(s)\,ds\right) = 0, \quad t > 0,\end{gathered} \tag{12}$$

where $0 < \alpha < 1$ is a real number. $D_-^{\alpha}y$ is the Liouville right-sided fractional derivative of y. We always assume that the following conditions are valid.

(A_1) $p(t) \geq 0$ and $q(t) \geq 0$ are continuous functions on $t \in [t_0, \infty)$, $t_0 > 0$.

(A_2) $h, f : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions with $xh(x) > 0$, $xf(x) > 0$ for $x \neq 0$, and there exist positive constants k_1, k_2 such that $h(x)/x \geq k_1$, $x/f(x) \geq k_2$ for all $x \neq 0$.

(A_3) $f'(u) \leq u$, $f^{-1}(u) \in C(\mathbb{R}, \mathbb{R})$ are continuous functions with $f^{-1}(u) > 0$ for $u \neq 0$, and there exists some positive constant α_1 such that $f^{-1}(uv) \geq \alpha_1 f^{-1}(u) f^{-1}(v)$ for $uv \neq 0$.

2. Preliminaries

For convenience, some background materials from fractional calculus are given.

From [4], we can get the definition for Liouville right-side fractional integral and Liouville right-side fractional derivative on the whole axis $\mathbb{R}$ of order β for a function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as follows,

$$\left(I_-^{\beta} g\right)(t) := \frac{1}{\Gamma(\beta)} \int_t^{\infty} (v - t)^{\beta - 1} g(v)\, dv, \quad t > 0, \tag{13}$$

$$\begin{aligned} \left(D_-^{\beta} g\right)(t) &:= (-1)^{\lceil \beta \rceil} \frac{d^{\lceil \beta \rceil}}{dt^{\lceil \beta \rceil}} \left(I_-^{\lceil \beta \rceil - \beta} g\right)(t) \\ &= (-1)^{\lceil \beta \rceil} \frac{1}{\Gamma(\lceil \beta \rceil - \beta)} \int_t^{\infty} (v - t)^{\lceil \beta \rceil - \beta - 1} g(v)\, dv, \\ &\qquad t > 0, \end{aligned} \tag{14}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $\Gamma(\cdot)$ is the gamma function defined by $\Gamma(t) := \int_t^{\infty} s^{t-1} e^{-s} ds$, and $\triangle := \min\{z \in \mathbb{Z} : z \geq \beta\}$ is the ceiling function.

If $\beta \in (0, 1)$, we have

$$D_-^{\alpha} y(t) := -\frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_t^{\infty} (s - t)^{-\alpha} y(s)\, ds, \tag{15}$$

for $t \in \mathbb{R}^+ := (0, \infty)$.

The following relations also existed:

$$\left(D_-^{(1+\alpha)} y\right)(t) = -\left(D_-^{\alpha} y\right)'(t), \quad \alpha \in (0, 1), \ t > 0. \tag{16}$$

Set

$$G(t) := \int_t^{\infty} (v - t)^{-\alpha} y(v)\, dv, \quad \alpha \in (0, 1), \tag{17}$$

and then

$$G'(t) = -\Gamma(1 - \alpha)\left(D_-^{\alpha} y\right)(t), \quad \alpha \in (0, 1). \tag{18}$$

3. Main Results

First, we study the oscillation of (12) under the following condition:

$$\int_{t_0}^{\infty} f^{-1}\left(\exp\left(-\int_{t_0}^{s} p(v)\, dv\right)\right) ds = \infty. \tag{19}$$

Theorem 1. *Suppose that $(A_1)-(A_3)$ and (19) hold; furthermore, assume that there exists a positive function $r(t) \in C^1[t_0, \infty]$ such that*

$$\limsup_{t \longrightarrow \infty} \int_{t_0}^{t} v(s) \cdot \left[k_1 r(s) q(s) - \frac{\left(r'(s)\right)^2}{4k_2 \Gamma(1 - \alpha) r(s)}\right] ds = \infty, \tag{20}$$

where k_1, k_2 are defined as in (A_2), and

$$v(s) := \exp\left(\int_{t_0}^{s} p(v)\, dv\right), \quad s \geq t_0. \tag{21}$$

Then every solution of (12) is oscillatory.

Proof. Suppose that $y(t)$ is a nonoscillation solution of (12); without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (12). Then there exists $t_1 \in [t_0, \infty]$ such that $y(t) > 0$ and $G(t) > 0$ for $t \in [t_1, \infty]$, where G is defined in (16). From (A_3), (12), and (16) we have

$$\begin{aligned} \left[f\left(D_-^{\alpha} y(t)\right) v(t)\right]' &= -f'\left(D_-^{\alpha} y(t)\right) v(t) D_-^{\alpha+1} y(t) \\ &\quad + f\left(D_-^{\alpha} y(t)\right) v(t) p(t) \\ &= f\left(D_-^{\alpha} y(t)\right) v(t) p(t) \\ &\quad - f'\left(D_-^{\alpha} y(t)\right) D_-^{\alpha+1} y(t) v(t) \\ &\geq f\left(D_-^{\alpha} y(t)\right) v(t) p(t) \\ &\quad - D_-^{\alpha+1} y(t) D_-^{\alpha} y(t) v(t) \\ &= q(t) h(G(t)) v(t) > 0, \\ &\qquad t \in [t_0, \infty]. \end{aligned} \tag{22}$$

Thus $f(D_-^{\alpha} y(t)) v(t)$ is strictly increasing on $[t_0, \infty]$. Since $v(t) > 0$ for $t \in [t_0, \infty]$, and from (A_3), we see that $D_-^{\alpha} y(t)$ is eventually of one sign. Now we can claim

$$D_-^{\alpha} y(t) < 0, \quad t \in [t_1, \infty]. \tag{23}$$

If not, then there exists $t_2 \in [t_1, \infty]$ such that $D_-^{\alpha} y(t_2) > 0$. Since $f(D_-^{\alpha} y(t)) v(t)$ is strictly increasing on $[t_1, \infty]$, it is clear that $f(D_-^{\alpha} y(t)) v(t) \geq f(D_-^{\alpha} y(t_2)) v(t_2) := c > 0$ for $t \in [t_2, \infty]$. Therefore, from (18), we have

$$\begin{aligned} -\frac{G'(t)}{\Gamma(1 - \alpha)} &= D_-^{\alpha} y(t) \geq f^{-1}\left(\frac{c}{v(t)}\right) \\ &= f^{-1}\left(c \cdot \exp\left(-\int_{t_0}^{t} p(v)\, dv\right)\right) \\ &\geq \alpha_1 f^{-1}(c) f^{-1}\left(\exp\left(-\int_{t_0}^{t} p(v)\, dv\right)\right), \\ &\qquad t \in [t_2, \infty]. \end{aligned} \tag{24}$$

Then, we get

$$f^{-1}\left(\exp\left(-\int_{t_0}^{t} p(v)\,dv\right)\right) \le -\frac{G'(t)}{\alpha_1 f^{-1}(c)\,\Gamma(1-\alpha)}, \qquad t \in [t_2,\infty]. \tag{25}$$

Integrating the above inequality from t_2 to t, we have

$$\begin{aligned}
&\int_{t_2}^{t} f^{-1}\left(\exp\left(-\int_{t_0}^{s} p(v)\,dv\right)\right) ds \\
&\quad \le -\int_{t_2}^{t} \frac{G'(s)}{\alpha_1 f^{-1}(c)\,\Gamma(1-\alpha)}\,ds \\
&\quad = -\frac{G(t)-G(t_2)}{\alpha_1 f^{-1}(c)\,\Gamma(1-\alpha)} < \frac{G(t_2)}{\alpha_1 f^{-1}(c)\,\Gamma(1-\alpha)}, \\
&\qquad t \in [t_2,\infty].
\end{aligned} \tag{26}$$

Letting $t \longrightarrow \infty$, we see

$$\begin{aligned}
&\int_{t_2}^{\infty} f^{-1}\left(\exp\left(-\int_{t_0}^{s} p(v)\,dv\right)\right) ds \\
&\quad \le \frac{G(t_2)}{\alpha_1 f^{-1}(c)\,\Gamma(1-\alpha)} < \infty.
\end{aligned} \tag{27}$$

This is in contradiction with (19). Hence, (23) holds.

Define the function w as generalized Riccati substitution

$$w(t) = r(t)\,\frac{-v(t)\,f\left(D_{-}^{\alpha} y(t)\right)}{G(t)}, \qquad t \in [t_1,\infty]. \tag{28}$$

Then we have $w(t) > 0$ for $t \in [t_1,\infty]$. From (18), (22), (28), and $(A_1)-(A_3)$, it follows that

$$\begin{aligned}
w'(t) &= r'(t)\,\frac{-f\left(D_{-}^{\alpha} y(t)\right)v(t)}{G(t)} + r(t) \\
&\quad \cdot \frac{-\left[f\left(D_{-}^{\alpha} y(t)\right)v(t)\right]' G(t) + f\left(D_{-}^{\alpha} y(t)\right)v(t)\,G'(t)}{G^2(t)} \\
&\le r'(t)\,\frac{-f\left(D_{-}^{\alpha} y(t)\right)v(t)}{G(t)} + r(t) \\
&\quad \cdot \left[\frac{-q(t)\,h(G(t))\,v(t)}{G(t)} + \frac{f\left(D_{-}^{\alpha} y(t)\right)v(t)\,G'(t)}{G^2(t)}\right] \\
&= \frac{r'(t)}{r(t)}w(t) + r(t)\,\frac{-q(t)\,h(G(t))\,v(t)}{G(t)} + r(t) \\
&\quad \cdot \frac{f\left(D_{-}^{\alpha} y(t)\right)v(t)\left(-\Gamma(1-\alpha)\,D_{-}^{\alpha} y(t)\right)}{G^2(t)} \le \frac{r'(t)}{r(t)}w(t) \\
&\quad - k_1 r(t)\,q(t)\,v(t) + \frac{w^2(t)\left(-\Gamma(1-\alpha)\,D_{-}^{\alpha} y(t)\right)}{r(t)\,f\left(D_{-}^{\alpha} y(t)\right)v(t)} \\
&\le \frac{r'(t)}{r(t)}w(t) - k_1 r(t)\,q(t)\,v(t) - \frac{k_2\Gamma(1-\alpha)}{r(t)\,v(t)}w^2(t).
\end{aligned} \tag{29}$$

That is,

$$\begin{aligned}
w'(t) &\le -k_1 r(t)\,q(t)\,v(t) + \frac{r'(t)}{r(t)}w(t) \\
&\quad - \frac{k_2\Gamma(1-\alpha)}{r(t)\,v(t)}w^2(t).
\end{aligned} \tag{30}$$

Taking $x = w(t)$, $a = k_2\Gamma(1-\alpha)/r(t)v(t)$ $(a \neq 0)$, and $b = r'(t)/r(t)$, from $ax^2+bx \le -b^2/4a$ and (30) we could conclude that

$$w'(t) \le -k_1 r(t)\,q(t)\,v(t) + \frac{v(t)\left(r'(t)\right)^2}{4k_2\Gamma(1-\alpha)\,r(t)}. \tag{31}$$

Integrating both sides of inequality (31) from t_0 to t, we obtain

$$\begin{aligned}
\infty &> w(t_0) > w(t_0) - w(t) \\
&\ge \int_{t_0}^{t}\left[-k_1 r(s)\,q(s)\,v(s) + \frac{v(s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\,r(t)}\right] ds.
\end{aligned} \tag{32}$$

Taking the limit supremum of both sides of the above inequality as $t \longrightarrow \infty$, we get

$$\begin{aligned}
\limsup_{t\longrightarrow\infty} \int_{t_0}^{t} &\left[-k_1 r(s)\,q(s)\,v(s)\right. \\
&\left. + \frac{v(s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\,r(s)}\right] ds < w(t_0) < \infty,
\end{aligned} \tag{33}$$

which is in contradiction with (20).

If $y(t)$ is an eventually negative solution of (12), the proof is similar; hence we omit it.

The proof is complete. □

Theorem 2. *Suppose that $(A_1)-(A_3)$ and (19) hold. Furthermore, assume that there exists a positive function $r(t) \in C^1[t_0,\infty]$ such that*

$$\begin{aligned}
\limsup_{t\longrightarrow\infty} \int_{t_0}^{t} \left[k_1 r(s)\,q(s) - \frac{\left[r'(s) - r(s)\,p(s)\right]^2}{4k_2\Gamma(1-\alpha)\,r(s)}\right] ds \\
= \infty,
\end{aligned} \tag{34}$$

where $v(s)$ are defined in Theorem 1. Then every solution of (12) is oscillatory.

Proof. Suppose that $y(t)$ is a nonoscillation solution of (12); without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (12). Proceeding the same as in the proof of Theorem 1, we get (23). Define the function $w(t)$ as follows

$$w(t) = r(t)\,\frac{-f\left(D_{-}^{\alpha} y(t)\right)}{G(t)}, \qquad t \in [t_1,\infty]. \tag{35}$$

Then we have $w(t) > 0$ for $t \in [t_1, \infty]$. From (16), (18), (22), (35), and $(A_1) - (A_3)$, it follows that

$$\begin{aligned} w'(t) &= r'(t)\frac{-f(D_-^\alpha y(t))}{G(t)} + r(t)\left[\frac{-f(D_-^\alpha y(t))}{G(t)}\right]' \\ &= r'(t)\frac{-f(D_-^\alpha y(t))}{G(t)} + r(t) \\ &\quad \cdot \frac{f'(D_-^\alpha y(t))\, D_-^{\alpha+1} y(t)\, G(t) + f(D_-^\alpha y(t))\, G'(t)}{G^2(t)} \\ &= \frac{r'(t)}{r(t)} w(t) + r(t)\frac{f'(D_-^\alpha y(t))\, D_-^{\alpha+1} y(t)}{G(t)} + r(t) \\ &\quad \cdot \frac{f(D_-^\alpha y(t))\,(-\Gamma(1-\alpha)\, D_-^\alpha y(t))}{G^2(t)} \le \frac{r'(t)}{r(t)} w(t) \\ &\quad + r(t)\frac{D_-^\alpha y(t)\, D_-^{\alpha+1} y(t)}{G(t)} - \frac{k_2\Gamma(1-\alpha)}{r(t)} w^2(t) \\ &= \frac{r'(t)}{r(t)} w(t) + r(t) \\ &\quad \cdot \frac{p(t) f(D_-^\alpha y(t)) - q(t) h(G(t))}{G(t)} - \frac{k_2\Gamma(1-\alpha)}{r(t)} \\ &\quad \cdot w^2(t) \le \frac{r'(t)}{r(t)} w(t) - p(t) w(t) - k_1 r(t) q(t) \\ &\quad - \frac{k_2\Gamma(1-\alpha)}{r(t)} w^2(t). \end{aligned} \tag{36}$$

That is,

$$\begin{aligned} w'(t) \le{}& -k_1 r(t) q(t) + \frac{r'(t) - r(t)p(t)}{r(t)} w(t) \\ & - \frac{k_2\Gamma(1-\alpha)}{r(t)} w^2(t). \end{aligned} \tag{37}$$

Taking $x = w(t)$, $a = k_2\Gamma(1-\alpha)/r(t)$ $(a \neq 0)$, and $b = (r'(t) - r(t)p(t))/r(t)$, from $ax^2 + bx \le -b^2/4a$ and (37) we conclude

$$w'(t) \le -k_1 r(s) q(s) + \frac{\left[r'(t) - r(t)p(t)\right]^2}{4k_2\Gamma(1-\alpha) r(t)}. \tag{38}$$

Integrating both sides of inequality (38) from t_0 to t, we obtain

$$\begin{aligned} \infty &> w(t_0) > w(t_0) - w(t) \\ &\ge \int_{t_0}^{t}\left[-k_1 r(s) q(s) + \frac{\left[r'(s) - r(s)p(s)\right]^2}{4k_2\Gamma(1-\alpha) r(s)}\right] ds. \end{aligned} \tag{39}$$

Taking the limit supremum of both sides of the above inequality as $t \longrightarrow \infty$, we get

$$\begin{aligned} \limsup_{t\longrightarrow\infty} \int_{t_0}^{t}\left[k_1 r(s) q(s) - \frac{\left[r'(s) - r(s)p(s)\right]^2}{4k_2\Gamma(1-\alpha) r(s)}\right] ds \\ < w(t_0) < \infty, \end{aligned} \tag{40}$$

which is in contradiction with (34).

If $y(t)$ is an eventually negative solution of (12), the proof is similar; here we omit it.

The proof is complete. □

We define a function class G; set $\mathbb{D} := \{(t,s) := t \ge s \ge t_0\}$, $\mathbb{D}_0 := \{(t,s) := t > s \ge t_0\}$. We say $H(t,s) \in G$, if $H(t,s)$ satisfy

$$\begin{aligned} H(t,t) &= 0, \quad t \ge t_0, \\ H(t,s) &> 0, \quad (t,s) \in \mathbb{D}_0, \end{aligned} \tag{41}$$

and H has a nonpositive continuous partial derivative $H_s'(t,s) := \partial H(t,s)/\partial s$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 3. *Suppose that $(A_1) - (A_3)$ and (19) hold. Furthermore, assume that there exists a positive function $r(t) \in C^1[t_0,\infty]$ and a function $H \in G$ satisfies*

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \frac{1}{H(t,t_0)} \\ &\quad \cdot \int_{t_0}^{t} H(t,s)\left[r(s)q(s) - \frac{(r'(s))^2}{4k_1k_2\Gamma(1-\alpha)r(s)}\right] ds \\ &\quad = \infty. \end{aligned} \tag{42}$$

Then every solution of (12) is oscillatory.

Proof. Suppose that y is a nonoscillation solution of (12); without loss of generality, we may assume that y is an eventually positive solution of (12). Proceeding as in the proof of Theorem 2 we get (37).

Multiplying (37) by $H(t,s)$ and integrating from t_1 to t, we get

$$\begin{aligned} &\int_{t_1}^{t} k_1 r(s) q(s) H(t,s)\, ds \\ &\quad \le -\int_{t_1}^{t} H(t,s) w'(s)\, ds \\ &\qquad + -\int_{t_1}^{t} H(t,s) w(s) \frac{r'(s)}{r(s)} ds \\ &\qquad - \int_{t_1}^{t} H(t,s) w^2(s) \frac{k_2\Gamma(1-\alpha)}{r(s)} ds, \\ &\qquad\qquad t \in [t_1,\infty]. \end{aligned} \tag{43}$$

Using the formula integration by parts, we obtain

$$\begin{aligned} -\int_{t_1}^{t} H(t,s) w'(s)\, ds &= -\left[H(t,s)w(s)\right]_{s=t_1}^{s=t} \\ &\quad + \int_{t_1}^{t} H_s'(t,s) w(s)\, ds \\ &= H(t,t_1) w(t_1) \\ &\quad + \int_{t_1}^{t} H_s'(t,s) w(s)\, ds, \\ &\qquad t \in [t_1,\infty). \end{aligned} \tag{44}$$

Substituting (44) with (43), we have

$$\begin{aligned} k_1 \int_{t_1}^{t} r(s)\, q(s)\, H(t,s)\, ds &\le H(t,t_1)\, w(t_1) \\ &+ \int_{t_1}^{t} \left\{ \left[H_s'(t,s) + H(t,s)\frac{r'(s)}{r(s)} \right] w(s) \right. \\ &\left. - k_2 \Gamma(1-\alpha)\frac{H(t,s)}{r(s)} w^2(s) \right\} ds. \end{aligned} \tag{45}$$

Taking $x = w(t)$, $a = -k_2\Gamma(1-\alpha)(H(t,s)/r(s))$ $(a \ne 0)$, and $b = H_s'(t,s) + H(t,s)(r'(s)/r(s))$, from $ax^2 + bx \le -b^2/4a$ we get

$$\begin{aligned} &\int_{t_1}^{t} \left\{ \left[H_s'(t,s) + H(t,s)\frac{r'(s)}{r(s)} \right] w(s) - k_2\Gamma(1-\alpha)\frac{H(t,s)}{r(s)} w^2(s) \right\} ds \\ &\le \int_{t_1}^{t} \frac{\left(H_s'(t,s)\right)^2 + H^2(t,s)\left(r'(s)\right)^2/r^2(s) + 2H_s'(t,s)\, H(t,s)\left(r'(s)/r(s)\right)}{4k_2\Gamma(1-\alpha)\, H(t,s)/r(s)} ds \\ &\le \int_{t_1}^{t} \frac{\left[\left(H_s'(t,s)\right)^2 + H^2(t,s)\left(r'(s)\right)^2/r^2(s)\right] r(s)}{4k_2\Gamma(1-\alpha)\, H(t,s)} ds = \int_{t_1}^{t} \frac{\left[H_s'(t,s)\, r(s)\right]^2 + \left[H(t,s)\, r'(s)\right]^2\big)}{4k_2\Gamma(1-\alpha)\, H(t,s)} ds \\ &\le \int_{t_1}^{t} \frac{\left[H_s'(t,s)\, r(s) + H(t,s)\, r'(s)\right]^2\big)}{4k_2\Gamma(1-\alpha)\, H(t,s)} ds \le \int_{t_1}^{t} \frac{\left[H(t,s)\, r'(s)\right]^2}{4k_2\Gamma(1-\alpha)\, H(t,s)} ds = \int_{t_1}^{t} \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} ds. \end{aligned} \tag{46}$$

Substituting (46) in (45), we have

$$\begin{aligned} &k_1 \int_{t_1}^{t} r(s)\, q(s)\, H(t,s)\, ds \\ &\le H(t,t_1)\, w(t_1) + \int_{t_1}^{t} \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} ds. \end{aligned} \tag{47}$$

Since $H_s'(t,s) \le 0$ for $t > s \ge t_0$, we have $0 < H(t,t_1) \le H(t,t_0)$ for $t > t_1 \ge t_0$. Therefore, from the previous inequality, we get

$$\begin{aligned} &\int_{t_1}^{t} \left[r(s)\, q(s)\, H(t,s)\, ds - \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} \right] ds \\ &\le k^{-1} H(t,t_1)\, w(t_1) \le k^{-1} H(t,t_0)\, w(t_1), \\ &\qquad t \in [t_1, \infty). \end{aligned} \tag{48}$$

Since $0 < H(t,t_1) \le H(t,t_0)$ for $t > s \ge t_0$, we have $0 < H(t,s)/H(t,t_0) \le 1$ for $t > s \ge t_0$. Hence, it follows from (48) that we have

$$\begin{aligned} &\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[r(s)\, q(s)\, H(t,s)\, ds \right. \\ &\left. - \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} \right] ds = \frac{1}{H(t,t_1)} \\ &\cdot \int_{t_0}^{t_1} \left[r(s)\, q(s)\, H(t,s)\, ds \right. \\ &\left. - \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} \right] ds + \frac{1}{H(t,t_0)} \\ &\cdot \int_{t_1}^{t} \left[r(s)\, q(s)\, H(t,s)\, ds \right. \\ &\left. - \frac{H(t,s)\left(r'(s)\right)^2}{4k_2\Gamma(1-\alpha)\, r(s)} \right] ds \le \frac{1}{H(t,t_1)} \int_{t_0}^{t_1} r(s) \\ &\cdot q(s)\, H(t,s)\, ds + \frac{1}{H(t,t_1)} k^{-1} H(t,t_0)\, w(t_1) \\ &\le \int_{t_0}^{t_1} r(s)\, q(s)\, ds + k^{-1} w(t_1), \quad t \in [t_1, \infty). \end{aligned} \tag{49}$$

Letting $t \longrightarrow \infty$, we obtain

$$\begin{aligned} &\limsup_{t \longrightarrow \infty} \frac{1}{H(t,t_0)} \\ &\cdot \int_{t_0}^{t} H(t,s) \left[r(s)\, q(s) - \frac{\left(r'(s)\right)^2}{4k_1 k_2 \Gamma(1-\alpha)\, r(s)} \right] ds \\ &\le \int_{t_0}^{t_1} r(s)\, q(s)\, ds + k^{-1} w(t_1) < \infty, \end{aligned} \tag{50}$$

which yields a contradiction to (42). The proof is complete. □

Second, we study the oscillation of (12) under the following condition:

$$\int_{t_0}^{\infty} f^{-1}\left(\exp\left(-\int_{t_0}^{s} p(v)\,dv\right)\right) ds < \infty. \tag{51}$$

Theorem 4. *Suppose that* $(A_1)-(A_3)$ *and (51) hold, and there exists a positive function* $r(t) \in C^1[t_0, \infty]$ *such that (20) holds. Furthermore, assume that, for every constant* $T \geq t_0$,

$$\int_{T}^{\infty} f^{-1}\left[\frac{1}{v(t)}\int_{T}^{t} q(s)\,v(s)\,ds\right] dt = \infty, \tag{52}$$

where $v(s)$ *are defined as in Theorem 1. Then every solution of (12) is oscillatory or satisfies* $\lim_{t\longrightarrow\infty}\int_{t_0}^{\infty}(s-t)^{-\alpha}y(s)ds = 0$.

Proof. Suppose that y is a nonoscillation solution of (12); without loss of generality, we may assume that y is an eventually positive solution of (12). Proceeding as in the proof of Theorem 1, we know that $D_{-}^{\alpha}y(t)$ is eventually one sign; then there are two cases for the sign of $D_{-}^{\alpha}y(t)$.

If $D_{-}^{\alpha}y(t)$ is eventually negative, similar to Theorem 1, we have the oscillation of (12). Next, if $D_{-}^{\alpha}y(t)$ is eventually positive, then there exists $t_2 \geq t_1$ such that $D_{-}^{\alpha}y(t) > 0$ for $t \geq t_2$. From (18), we get $G'(t) < 0$ for $t \geq t_2$. Thus, we get $\lim_{t\longrightarrow\infty}G(t) := L \geq 0$ and $G(t) \geq L$. We now claim that $L = 0$. Assuming not, that is, $L > 0$, then from (23) and (A_3) we get

$$\begin{aligned}\left[f\left(D_{-}^{\alpha}y(t)\right)v(t)\right]' &\geq q(t)\,h(G(t))\,v(t)\\ &\geq k_1 q(t)\,G(t)\,v(t)\\ &\geq k_1 L q(t)\,v(t), \quad t \in [t_2, \infty).\end{aligned} \tag{53}$$

Integrating both sides of the above inequality from t_2 to t, we have

$$\begin{aligned}f\left(D_{-}^{\alpha}y(t)\right)v(t) &\geq f\left(D_{-}^{\alpha}y(t_2)\right)v(t_2)\\ &\quad + k_1 L\int_{t_2}^{t} q(s)\,v(s)\,ds\\ &> k_1 L\int_{t_2}^{t} q(s)\,v(s)\,ds,\\ &\qquad t \in [t_2, \infty).\end{aligned} \tag{54}$$

Hence, from (17), we get

$$\begin{aligned}-\frac{G'(t)}{\Gamma(1-\alpha)} = D_{-}^{\alpha}y(t) &\geq f^{-1}\left(\frac{k_1 L\int_{t_2}^{t} q(s)\,v(s)\,ds}{v(t)}\right)\\ &\geq \alpha_1 f^{-1}(k_1 L)\, f^{-1}\left(\frac{\int_{t_2}^{t} q(s)\,v(s)\,ds}{v(t)}\right),\\ &\qquad t \in [t_2, \infty).\end{aligned} \tag{55}$$

Integrating both sides of the last inequality from t_2 to t, we obtain

$$\begin{aligned}G(t) \leq G(t_2) &- \alpha_1\Gamma(1-\alpha)\,f^{-1}(k_1 L)\\ &\cdot\int_{t_2}^{t} f^{-1}\left(\frac{\int_{t_2}^{t} q(s)\,v(s)\,ds}{v(u)}\right) du,\\ &\qquad t \in [t_2, \infty).\end{aligned} \tag{56}$$

Letting $t \longrightarrow \infty$, from (52), we get $\lim_{t\longrightarrow\infty}G(t) = -\infty$; this is in contradiction with $G(t) > 0$. Therefore, we have $L = 0$, that is, $\lim_{t\longrightarrow\infty}G(t) = 0$. The proof is complete. □

Theorem 5. *Suppose that* $(A_1)-(A_3)$ *and (51) hold. Let* $r(t)$ *and* $H(t,s)$ *be defined as in Theorem 3 such that (42) holds. Furthermore, assume that, for every constant* $T \geq t_0$, *(52) holds. Then every solution of (12) is oscillatory or satisfies* $\lim_{t\longrightarrow\infty}\int_{t_0}^{\infty}(s-t)^{-\alpha}y(s)ds = 0$.

From Theorem 3, proceeding as in the proof of Theorem 4, we get the results of the theorem.

4. Examples

Example 1. Consider the fractional differential equation

$$\begin{aligned}\left(D_{-}^{3/2}y\right)(t)\cdot\left(D_{-}^{1/2}y\right)(t) - \frac{1}{t^2}\left(D_{-}^{1/2}y\right)(t)&\\ + \frac{1}{t}\int_{t}^{\infty}(s-t)^{-\alpha}\,y(s)\,ds = 0, \quad t > 0.&\end{aligned} \tag{57}$$

In (57), $\alpha = 1/2$, $p(t) = 1/t^2$, $q(t) = 1/t$, and $f(x) = h(x) = x$. Since

$$\begin{aligned}&\int_{t_0}^{\infty} f^{-1}\left(\exp\left(-\int_{t_0}^{t} p(v)\,dv\right)\right) dt\\ &\quad = \int_{t_0}^{\infty}\left(\exp\left(-\int_{t_0}^{t} p(v)\,dv\right)\right) dt\\ &\quad = \int_{t_0}^{\infty}\exp\left(\frac{1}{t}-\frac{1}{t_0}\right) dt \geq \int_{t_0}^{\infty}\exp\left(-\frac{1}{t_0}\right) dt = \infty,\end{aligned} \tag{58}$$

then (19) holds.

Taking $t_0 = 1$, $k_1 = k_2 = 1$. It is clear that conditions $(A_1)-(A_3)$ hold. Furthermore, taking $r(t) = t^2$, we have

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\int_{1}^{t}\left[k_1 r(s)\,q(s) - \frac{\left[r'(s)-r(s)\,p(s)\right]^2}{4k_2\Gamma(1-\alpha)\,r(s)}\right] ds\\ &\quad = \limsup_{t\longrightarrow\infty}\int_{1}^{t}\left[\frac{1}{s}s^2 - \frac{\left(2s - s^2\left(1/s^2\right)\right)^2}{4\sqrt{\pi}s^2}\right] ds\\ &\quad = \limsup_{t\longrightarrow\infty}\int_{1}^{t}\left(s - \frac{1}{4\sqrt{\pi}}\left(\frac{2s-1}{s}\right)^2\right) ds = \infty,\end{aligned} \tag{59}$$

which shows that (34) holds. Therefore, by Theorem 2, every solution of (12) is oscillatory.

Example 2. Consider the fractional differential equation

$$\begin{aligned} &\left(D_{-}^{5/4} y\right)(t) \cdot \left(D_{-}^{1/4} y\right)(t) - \frac{1}{t^2}\left(D_{-}^{1/4} y\right)(t) \\ &\quad + t \int_t^{\infty} (s-t)^{-\alpha}\, y(s)\, ds = 0, \quad t > 0. \end{aligned} \tag{60}$$

In (60), $\alpha = 1/4$, $p(t) = 1/t^2$, $q(t) = t$, and $f(x) = h(x) = x$.

Proceeding the same process as Example 1, we see that (19) holds. Taking $t_0 = 1$, $k_1 = k_2 = 1$, $r(t) = 1$. It is clear that conditions $(A_1) - (A_3)$ hold. Furthermore, taking $H(t,s) = (t-s)^{1/4}$, it meets $H_s'(t,s) = -(1/4)(t-s) < 0$ for $(t,s) \in \mathbb{D}_0$, $\mathbb{D}_0 := \{(t,s) : t > s \geq t_0\}$.

Since

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} \frac{1}{H(t,t_0)} \\ &\quad \cdot \int_{t_0}^{t} H(t,s)\left[r(s)\,q(s) - \frac{\left(r'(s)\right)^2}{4k_1k_2\Gamma(1-\alpha)\,r(s)}\right] ds \\ &= \limsup_{t\longrightarrow\infty} \frac{1}{(t-t_0)^{1/4}} \int_{t_0}^{t} s\,(t-s)^{1/4}\, ds = \limsup_{t\longrightarrow\infty} \\ &\quad \cdot \frac{1}{(t-t_0)^{1/4}}\left[\frac{4}{5}(t-t_0)^{5/4} t_0 + \frac{16}{45}(t-t_0)^{9/4}\right] \\ &= \limsup_{t\longrightarrow\infty}\left(\frac{4}{5}(t-t_0)\,t_0 + \frac{16}{45}(t-t_0)^2\right) = \infty, \end{aligned} \tag{61}$$

which shows that (42) holds, by Theorem 3, every solution of (12) is oscillatory.

5. Conclusion

In the paper, by using the generalized Riccati transformation and inequality technique, we study a class of $2\alpha + 1$ order fractional differential equations in the form (12), which contains the damping term and has not been studied before. The oscillation criteria of (12) are obtained and some examples are given to reinforce our results.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

Hui Liu carried out the main results and completed the corresponding proof. Run Xu participated in the proof and helped in completing Section 4. All authors read and approved the final manuscript.

Acknowledgments

This research is supported by National Science Foundation of China (11671227).

References

[1] G. W. Leibniz, *Mathematische Schriften*, Grorg Olms Verlagsbuchhandlung, Hildesheim, Germany, 1962.

[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, San Diego, USA, 1974.

[3] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fraction Differential Equations, to Methods of Their Solution and some of Their Applications*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, CA, USA, 1999.

[4] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[5] R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," *Journal of Rheology*, vol. 27, no. 3, pp. 201–210, 1983.

[6] R. T. Baillie, "Long memory processes and fractional integration in econometrics," *Journal of Econometrics*, vol. 73, no. 1, pp. 5–59, 1996.

[7] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Goron and Breach Science Publishers, Yverdon, Switzerland, 1993.

[8] Y. A. Rossikhin and M. V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids," *Applied Mechanics Reviews*, vol. 50, no. 1, pp. 15–67, 1997.

[9] J. T. Edwards, N. J. Ford, and A. C. Simpson, "The numerical solution of linear multi-term fraction differential equations: systems of equations," *Journal of Computational and Applied Mathematics*, vol. 148, pp. 401–418, 2002.

[10] F. Ghoreishi and S. Yazdani, "An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis," *Computers & Mathematics with Applications*, vol. 61, no. 1, pp. 30–43, 2011.

[11] J. C. Trigeassou, N. Maamri, J. Sabatier, and A. Oustaloup, "A Lyapunov approach to the stability of fractional differential equations," *Signal Processing*, vol. 91, no. 3, pp. 437–445, 2011.

[12] W. Deng, "Smoothness and stability of the solutions for nonlinear fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 72, no. 3-4, pp. 1768–1777, 2010.

[13] A. Saadatmandi and M. Dehghan, "A new operational matrix for solving fractional-order differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1326–1336, 2010.

[14] S. R. Grace, R. P. Agarwal, P. J. Y. Wong, and A. Zafer, "On the oscillation of fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 2, pp. 222–231, 2012.

[15] D.-X. Chen, "Oscillation criteria of fractional differential equations," *Advances in Difference Equations*, vol. 2012, article 33, 18 pages, 2012.

[16] D.-X. Chen, "Oscillatory behavior of a class of fractional differential equations with damping," *"Politehnica" University of Bucharest. Scientific Bulletin. Series A. Applied Mathematics and Physics*, vol. 75, no. 1, pp. 107–118, 2013.

[17] Z. Han, Y. Zhao, Y. Sun, and C. Zhang, "Oscillation for a class of fractional differential equation," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 390282, 6 pages, 2013.

[18] R. Xu, "Oscillation criteria for nonlinear fractional differential equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 971357, 7 pages, 2013.

[19] Q. Huizeng and B. Zheng, "Oscillation of a class of fractional differential equations with damping term," *The Scientific World Journal*, vol. 2013, Article ID 685621, 9 pages, 2013.

[20] J. O. Alzabut and T. Abdeljawad, "Sufficient conditions for the oscillation of nonlinear fractional difference equations," *Journal of Fractional Calculus and Applications*, vol. 5, no. 1, pp. 177–187, 2014.

[21] B. Abdalla, K. Abodayeh, T. Abdeljawad, and J. Alzabut, "New oscillation criteria for forced nonlinear fractional difference equations," *Vietnam Journal of Mathematics*, vol. 45, no. 4, pp. 609–618, 2017.

[22] Z. Bai and R. Xu, "The asymptotic behavior of solutions for a class of nonlinear fractional difference equations with damping term," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 5232147, 11 pages, 2018.

[23] B. Abdalla and T. Abdeljawad, "On the oscillation of Hadamard fractional differential equations," *Advances in Difference Equations*, vol. 2018, no. 409, 2018.

[24] J. Alzabut, T. Abdeljawad, and H. Alrabaiah, "Oscillation criteria for forced and damped nabla fractional difference equations," *Journal of Computational Analysis and Applications*, vol. 24, no. 8, pp. 1387–1394, 2018.

[25] B. Abdalla, J. Alzabut, and T. Abdeljawad, "On the oscillation of higher order fractional difference equations with mixed nonlinearities," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 2, pp. 207–217, 2018.

[26] W. Wusheng, "Analytic invariant curves of a nonlinear second order difference equation," *Acta Mathematica Scientia*, vol. 29, no. 2, pp. 415–426, 2009.

[27] H. Liu and F. Meng, "Some new nonlinear integral inequalities with weakly singular kernel and their applications to FDEs," *Journal of Inequalities and Applications*, vol. 2015, no. 209, 2015.

[28] R. Xu and F. Meng, "Some new weakly singular integral inequalities and their applications to fractional differential equations," *Journal of Inequalities and Applications*, vol. 2016, no. 1, pp. 1–16, 2016.

[29] R. Xu, "Some new nonlinear weakly singular integral inequalities and their applications," *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1007–1018, 2017.

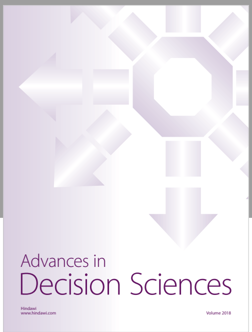
[30] Y. Guan, Z. Zhao, and X. Lin, "On the existence of positive solutions and negative solutions of singular fractional differential equations via global bifurcation techniques," *Boundary Value Problems*, vol. 2016, no. 1, article 141, 2016.

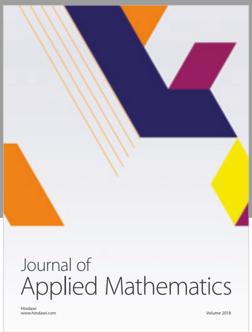
[31] X. Wang and R. Xu, "The existence and uniqueness of solutions and Lyapunov-type inequality for CFR fractional differential equations," *Journal of Function Spaces*, vol. 2018, Article ID 5875108, 7 pages, 2018.

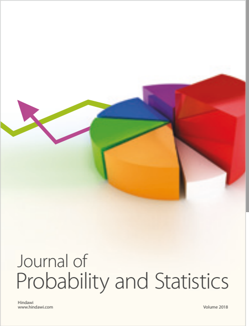
[32] D. Zan and R. Xu, "The existence results of solutions for system of fractional differential equations with integral boundary conditions," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 8534820, 8 pages, 2018.

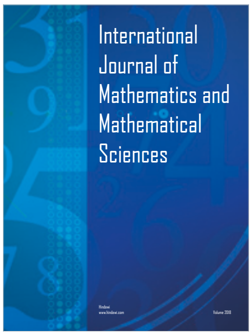
[33] R. Xu and Y. Zhang, "Generalized Gronwall fractional summation inequalities and their applications," *Journal of Inequalities and Applications*, vol. 2015, no. 242, 2015.

[34] M. Fanwei, "A new approach for solving fractional partial differential equation," *Journal of Applied Mathematics*, pp. 1–11, 2013.









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