Research Article

New Safe Approximation of Ambiguous Probabilistic Constraints for Financial Optimization Problem

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In financial optimization problem, the optimal portfolios usually depend heavily on the distributions of uncertain return rates. When the distributional information about uncertain return rates is partially available, it is important for investors to find a robust solution for immunization against the distribution uncertainty. The main contribution of this paper is to develop an ambiguous value-at-risk (VaR) optimization framework for portfolio selection problems, where the distributions of uncertain return rates are partially available. For tractability consideration, we deal with new safe approximations of ambiguous probabilistic constraints under two types of random perturbation sets and obtain two equivalent tractable formulations of the ambiguous probabilistic constraints. Finally, to demonstrate the potential for solving portfolio optimization problems, we provide a practical example about the Chinese stock market. The advantage of the proposed robust optimization method is also illustrated by comparing it with the existing optimization approach via numerical experiments.

1. Introduction

The core of modern portfolio theory is how to formulate a tradeoff between the return and risk, specifically to determine an effective investment strategy for allocating capital over a number of available assets so as to maximize the return while minimizing the risk. The first systematic approach to the problem of asset allocation, called mean-variance (MV) one, was developed by Markowitz [1]. To overcome the limitation of variance as a risk measure, some useful realistic risk measures were proposed to separate undesirable downside movements from desirable upside movements. In this respect, the interested reader may also refer to the absolute semi-deviation [2, 3], moment [4, 5], value-at-risk (VaR) [6–8], and conditional value-at-risk (CVaR) [9, 10].

The researches mentioned above need the precise estimates of the return and risk based on the explicit distribution extracted from considerable current market data. Due to the noise and outliers in the data, or the imperfect data of newly listed stocks, it is often impossible to determine

the exact distributions of future returns [11]. As a result, distributionally robust optimization is a useful approach for decision under uncertainty, where uncertain model data are governed by a family of probability distributions. Using this approach, decision makers can find a robust solution for immunization against the distribution uncertainty.

The first research of distributionally robust optimization was proposed by Scarf [12] to solve a linear inventory problem with a min–max solution. Since then, the approach has been well developed in the literature [13–17]. Among them, Sun et al. [16] developed a distributionally robust joint chance constrained optimization model for a dynamic network design problem, and Wiesemann et al. [17] proposed a unifying framework for modeling and solving a distributionally robust multiproduct newsvendor problem.

In financial field, distributionally robust optimization method has proved useful in constructing the portfolio selection model. Shen and Zhang [18] proposed a robust optimization resolution to stochastic programming based on a scenario tree, with the system parameters being modeled to

reside in ellipsoidal ambiguity sets. Natarajan et al. [19] proposed a computationally tractable approximation method for minimizing the VaR of a portfolio based on robust optimization techniques, where the uncertainty set ranged from polytopes to more advanced conic-representable uncertainty sets. Huang et al. [20] considered the relative robust conditional value-at-risk portfolio selection problem, where the underlying probability distribution of portfolio return belonged to a certain set. Pinar and Paç [21] derived the distributionally robust portfolio rules, addressed potential uncertainty in the mean return vector, and derived a closed-form portfolio rule when the uncertainty in the return vector was modeled via an ellipsoidal uncertainty set. Liu et al. [22] adopted an uncertainty set in distributionally robust portfolio selection problems, where the considered uncertainties are in terms of both the distribution and the first two order moments.

The choice of uncertainty set is important for the tractability of distributionally robust portfolio selection model. Box uncertainty and ellipsoidal uncertainty are the most often used uncertainty structures in robust optimization formulations [23]. The obtained solution based on single uncertainty set like box is usually too conservative. In addition, when chance constraints exist in distributionally robust optimization model, the feasible region is usually nonconvex; thus it is difficult to solve such hard optimization model. In this case, a common technique is to find the safe approximation of ambiguous chance constraint under a given uncertainty set [13, 24].

A set of constraints is called a safe approximation of the chance constraint if the feasible set of the safe approximation is a subset of the feasible set of the chance constraint [25]. For the time being, there are several approximation methods for solving chance constraint problems, such as sample approximation [26–28], scenario approximation [29, 30], Bernstein approximation [31, 32], and robust counterpart approximation [33, 34]. To get a further understanding about the approximation method of chance constraint, the interested readers may refer to the following references. Ben-Tal and Nemirovski [35] developed a computationally tractable approximation of chance constrained linear matrix inequalities based on measure concentration results. Ding et al. [36] presented a tractable convex programming approximation for distributionally robust individual chance constrained problem under interval sets of mean and covariance information. Yang and Xu [37] derived sufficient conditions to guarantee tractability and investigated tractable approximations of joint probabilistic envelope constraints. Postek et al. [38] presented three new safe tractable approximations of chance constraints under partial information of increasing computational complexity and quality.

Among all the approximation methods of chance constraint, the method on approximating the ambiguous chance constraint with its robust counterpart has received an increasing attention recently [39]. Various robust counterpart optimization formulations are derived based on different types of uncertainty set. Ben-Tal et al. [13] discussed some robust counterpart approximations of chance constraints under box, ellipsoid, and the intersection of box and ellipsoid uncertainty sets. Yuan et al. [39] proposed a two-layer algorithm to

investigate the tractable robust optimization approximation framework for solving the joint chance constrained problem under the intersection of interval & polyhedral and interval & ellipsoidal type uncertainty sets. Li et al. [40] investigated different types of uncertainty set for robust optimization, including box, ellipsoidal, polyhedral, interval & ellipsoidal, and interval & polyhedral. To the best of our knowledge, no discussion of the robust counterpart approximation is under ball & budget and ellipsoid & generalized budget uncertainty sets.

In this paper, we will study the distributionally robust portfolio selection problem from a new perspective, where the uncertain model parameters are described by two new combinations of perturbation sets of ball & budget and ellipsoid & generalized budget. The proposed safe approximation method incorporated the robust counterpart, can refine the approximation set, and further improve approximation capacity via changing the intersection mode of perturbation sets. For tractability consideration, the safe approximations of ambiguous chance constraint are also derived under the new combined perturbation sets.

Our contributions in this paper include the following several aspects:

(1) We develop an ambiguous VaR optimization framework for distributionally robust linear optimization problem, where we optimize the worst case VaR over an ambiguity set. The obtained robust solution is capable of providing immunization against the distribution uncertainty.

(2) To obtain tractable formulation, we approximate the ambiguous probabilistic constraints under two new combinations of perturbation sets: one is the intersection of ball with budget, and the other is the intersection of ellipsoid with generalized budget. Depending on the choice of two new ambiguity sets, the resulting framework of safe approximation includes a system of efficiently computable convex constraints, which can be solved efficiently by general-purpose commercial-grade solvers.

(3) We demonstrate our approach for addressing a practical case about the Chinese stock market. To show the advantage of the proposed approach, we compare the proposed robust optimization approach with nominal distribution optimization approach via numerical experiments.

The paper is organized as follows: In Section 2, we develop an ambiguous VaR optimization framework for distributionally robust linear optimization problem. In Section 3, we approximate the ambiguous probabilistic constraints under two new combinations of perturbation sets. In Section 4, we demonstrate our approach for addressing a practical case study. Section 5 briefly concludes the paper. All proofs are given in the appendix.

2. Stochastic Optimization with Ambiguous Probabilistic Constraints

In this section, we consider a class of uncertain linear optimization problems of the form

$$\max_x c^T x + d$$

$$\begin{aligned} \text{s.t. } & a_i^T x \leq b_i, \quad i = 1, \dots, m \\ & x \in D, \end{aligned} \quad (1)$$

where $x \in R^n$ is the vector of decision variables, D is a collection of deterministic constraints, $c \in R^n$ and $d \in R$ in the objective are uncertain, and the problem data $a_i \in R^n, b_i \in R, i = 1, \dots, m$ are also uncertain.

In stochastic optimization, the uncertain model data are assumed to be random. In the simplest case, these random data follow known in advance probability distributions [41, 42], while in more advanced settings, these distributions are only partially known. In this case, all we know is that P belongs to a prescribed family $\mathcal{P}$ of distributions on the space of the data [43, 44].

In this section, we deal with uncertain model (1) by introducing risk levels $\epsilon \in (0, 1), \epsilon_i \in (0, 1), i = 1, \dots, m$, and enforcing the constraints in the probability sense. To make model (1) meaningful, we introduce an artificial variable t , and we build the following VaR optimization problem of the form

$$\begin{aligned} \max_x \quad & t \\ \text{s.t. } \quad & \Pr \{c^T x + d \geq t\} \geq 1 - \epsilon \\ & \Pr \{a_i^T x \leq b_i\} \geq 1 - \epsilon_i, \quad i = 1, \dots, m \\ & x \in D. \end{aligned} \quad (2)$$

When the distribution of random data is only partially known, we replace problem (2) with the following ambiguous VaR optimization problem

$$\begin{aligned} \max_x \quad & t \\ \text{s.t. } \quad & \Pr_{(c,d) \sim P_0} \{c^T x + d \geq t\} \geq 1 - \epsilon, \quad \forall P_0 \in \mathcal{P}_0 \\ & \Pr_{(a_i, b_i) \sim P_i} \{a_i^T x \leq b_i\} \geq 1 - \epsilon_i, \quad (3) \\ & \forall P_i \in \mathcal{P}_i, \quad i = 1, \dots, m \\ & x \in D, \end{aligned}$$

where the distribution P_0 of c, d belongs to the distribution family $\mathcal{P}_0$, and the distribution P_i of a_i, b_i belongs to the family $\mathcal{P}_i, i = 1, \dots, m$. Model (3) is called an ambiguous probabilistic constraint model because we do not have any information about the probability distribution P except for the fact that it belongs to a family $\mathcal{P}$.

Model (3) is a semi-infinite programming problem with infinite probability constraints. It is usually difficult to evaluate with high accuracy the probability in the left hand side of (3), even in the case when P is simple [45]. Due to the severe computational difficulties associated with ambiguous probabilistic constraints, a natural course of action is to replace the constraints with their computationally tractable safe approximations.

In the next section, we will deal with the scheme for new safe approximations of the ambiguous probabilistic constraints.

3. New Safe Approximation Methods

It has been well known that the safe approximations of chance constraints are highly dependent on the choice of the uncertainty set. In this section, we will introduce two kinds of new uncertainty sets.

For the sake of clarity, we assume that the uncertainty set is parameterized in an affine fashion by perturbation vector $\zeta = [\zeta_1, \dots, \zeta_L]$ varying in a given perturbation set Z :

$$\begin{aligned} & c^T x + d \geq t \\ & \forall \left([c; d] = [c^0; d^0] + \sum_{l=1}^L \zeta_l [c^l; d^l] : \zeta \in Z \in R^L \right); \quad (4) \\ & a_i^T x \leq b_i \\ & \forall \left([a_i; b_i] = [a_i^0; b_i^0] + \sum_{l=1}^L \zeta_l [a_i^l; b_i^l] : \zeta \in Z \in R^L \right). \quad (5) \end{aligned}$$

Ben-Tal et al. [13] reported that the values a, b, c, d of uncertain data are obtained from the nominal values a^0, b^0, c^0, d^0 of the data by random perturbation. The affine formula of the uncertainty set has been used in many works [25, 34, 38].

In the following, we approximate the ambiguous probabilistic constraints of (3) under two new combinations of perturbation sets: one is the intersection of ball with budget, and the other is the intersection of ellipsoid with generalized budget.

Case I (safe approximations under ball and budget perturbation set). We first address the perturbation set Z in (4), and (5) is the intersection of $\|\cdot\|_2$ - and $\|\cdot\|_1$ -balls, i.e.,

$$Z = \{\zeta \in R^L : \|\zeta\|_2 \leq \Omega, \|\zeta\|_1 \leq \gamma\}, \quad (6)$$

where $\Omega > 0$ and $\gamma > 0$ are given parameters. In this case, the perturbation set Z has the following conic representation:

$$Z = \{\zeta \in R^L : P_1 \zeta + p_1 \in \mathbf{K}^1, P_2 \zeta + p_2 \in \mathbf{K}^2\}, \quad (7)$$

where

- (i) $P_1 \zeta = [\zeta; 0], p_1 = [0_{L \times 1}; \Omega]$ and $\mathbf{K}^1 = \{[z; t] \in R^L \times R : t \geq \|z\|_2\}$, whence its dual conic $\mathbf{K}_*^1 = \mathbf{K}^1$;
- (ii) $P_2 \zeta = [\zeta; 0], p_2 = [0_{L \times 1}; \gamma]$ and $\mathbf{K}^2 = \{[z; t] \in R^L \times R : t \geq \|z\|_1\}$, whence its dual conic $\mathbf{K}_*^2 = \{[z; t] \in R^L \times R : t \geq \|z\|_\infty\}$.

In the light of Remark 1.3.6 in [13], the semi-infinite constraint (4) can be represented by the system of conic constraints in variables $x, y^s, s = 1, 2$:

$$\begin{aligned} & \sum_{s=1}^2 p_s^T y^s + t + [-c^0]^T x \leq d^0, \\ & \sum_{s=1}^2 (P_s^T y^s)_l = d^l + [c^l]^T x, \quad l = 1, \dots, L \quad (8) \\ & y^s \in \mathbf{K}_*^s, \quad s = 1, 2. \end{aligned}$$

Setting $y^1 = [\eta_1; \tau_1]$, $y^2 = [\eta_2; \tau_2]$ with L -dimensional η_1, η_2 , and one-dimensional τ_1, τ_2 , and then substituting P_s, p_s into system (8), we have the following system of conic quadratic constraints in variables $\tau_1 \in R, \tau_2 \in R, \eta_1 \in R^L, \eta_2 \in R^L, x$:

$$\begin{aligned} \Omega\tau_1 + \gamma\tau_2 + t + [-c^0]^T x &\leq d^0, \\ (\eta_1 + \eta_2)_l &= d^l + [c^l]^T x, \quad l = 1, \dots, L \\ \|\eta_1\|_2 &\leq \tau_1, \quad \|\eta_2\|_\infty \leq \tau_2. \end{aligned} \quad (9)$$

Similarly, in virtue of the conic representation of (5), the following system of conic constraints in variables $\tau_1, \tau_2, \eta_1, \eta_2, x$ can be obtained:

$$\begin{aligned} \Omega\tau_1 + \gamma\tau_2 + [a_i^0]^T x &\leq b_i^0, \quad i = 1, \dots, m \\ (\eta_1 + \eta_2)_l &= b_i^l - [a_i^l]^T x, \\ l = 1, \dots, L, \quad i = 1, \dots, m \\ \|\eta_1\|_2 &\leq \tau_1, \quad \|\eta_2\|_\infty \leq \tau_2. \end{aligned} \quad (10)$$

Eliminating from above two systems the variables τ_1, τ_2 for every feasible solution to these two systems, we have $\tau_1 \geq \bar{\tau}_1 \equiv \|\eta_1\|_2, \tau_2 \geq \bar{\tau}_2 \equiv \|\eta_2\|_\infty$, and the solution obtained when replacing τ_1, τ_2 with $\bar{\tau}_1, \bar{\tau}_2$ is still feasible. The reduced system in variables $x, z = \eta_1, \omega = \eta_2$ reads

$$\Omega \sqrt{\sum_{l=1}^L z_l^2} + \gamma \max_l |\omega_l| + t - [c^0]^T x \leq d^0, \quad (11)$$

$$\begin{aligned} z_l + \omega_l &= d^l + [c^l]^T x, \\ l = 1, \dots, L, \end{aligned} \quad (12)$$

which is an equivalent representation of (4) and (6).

On the other hand, the following system

$$\Omega \sqrt{\sum_{l=1}^L z_l^2} + \gamma \max_l |\omega_l| + [a_i^0]^T x \leq b_i^0, \quad (13)$$

$$\begin{aligned} z_l + \omega_l &= b_i^l - [a_i^l]^T x, \\ l = 1, \dots, L, \quad i = 1, \dots, m \end{aligned} \quad (14)$$

is an equivalent representation of (5) and (6).

Next we present a scheme for the safe approximation of ambiguous probabilistic constraints. For this purpose, suppose perturbation vectors $\zeta_1, \dots, \zeta_L$ included in (4) and (5) possess the following properties:

$$\begin{aligned} \zeta_1, \dots, \zeta_L : E\{\zeta_l\} &= 0 \text{ \& } |\zeta_l| \leq 1, \\ l = 1, \dots, L \text{ \& } \{\zeta_l\}_{l=1}^L &\text{ are mutually independent.} \end{aligned} \quad (15)$$

On the basis of the above analysis, we obtain the following observation.

Theorem 1. *The RC of the uncertain linear constraint (4) with the perturbation set (6) is equivalent to the system of conic quadratic constraints (11)-(12).*

In the case of (15), the x component of every feasible solution to this system satisfies the randomly perturbed inequality (4) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

Theorem 2. *The RC of the uncertain linear constraint (5) with the perturbation set (6) is equivalent to the system of conic quadratic constraints (13)-(14).*

In the case of (15), the x component of every feasible solution to this system satisfies the randomly perturbed inequality (5) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

Case II (safe approximations under ellipsoid and generalized budget perturbation set). Next, consider the perturbation set Z in (4) and (5) is the intersection of ellipsoid and generalized budget, i.e.,

$$Z = \left\{ \zeta \in R^L : \sqrt{\sum_{l=1}^L \left(\frac{\zeta_l}{\sigma_l} \right)^2} \leq \Omega, \sum_{l=1}^L \left| \frac{\zeta_l}{\sigma_l} \right| \leq \gamma \right\}, \quad (16)$$

where $\Omega > 0, \gamma > 0$ and $\sigma_l > 0$ are given parameters. In this case, the perturbation set Z has the following conic representation:

$$Z = \{ \zeta \in R^L : P_1 \zeta + p_1 \in K^1, P_2 \zeta + p_2 \in K^2 \}, \quad (17)$$

where

(i) $P_1 \zeta = [\sum^{-1} \zeta; 0]$, with $\sum = \text{diag}\{\sigma_1, \dots, \sigma_L\}$, $p_1 = [0_{L \times 1}; \Omega]$ and $K^1 = \{[z; t] \in R^L \times R : t \geq \|z\|_2\}$, whence its dual conic $K_*^1 = K^1$;

(ii) $P_2 \zeta = [\sum^{-1} \zeta; 0]$, with $\sum = \text{diag}\{\sigma_1, \dots, \sigma_L\}$, $p_2 = [0_{L \times 1}; \gamma]$ and $K^2 = \{[z; t] \in R^L \times R : t \geq \|z\|_1\}$, whence its dual conic $K_*^2 = \{[z; t] \in R^L \times R : t \geq \|z\|_\infty\}$.

Set $y^1 = [\eta_1; \tau_1]$, $y^2 = [\eta_2; \tau_2]$ with L -dimensional η_1, η_2 , and one-dimensional τ_1, τ_2 , where $[\eta_1; \tau_1] \in K_*^1$ and $[\eta_2; \tau_2] \in K_*^2$. In virtue of the conic representation of (4), the semi-infinite constraint (4) can be represented by the following system of conic constraints in variables $\tau_1 \in R, \tau_2 \in R, \eta_1 \in R^L, \eta_2 \in R^L, x$:

$$\Omega\tau_1 + \gamma\tau_2 + t + [-c^0]^T x \leq d^0,$$

$$\left(\sum_{l=1}^{-1} \eta_1 + \sum_{l=1}^{-1} \eta_2 \right)_l = d^l + [c^l]^T x, \quad l = 1, \dots, L \quad (18)$$

$$\|\eta_1\|_2 \leq \tau_1, \quad \|\eta_2\|_\infty \leq \tau_2.$$

Similarly, according to the conic representation of (5), the following system of conic constraints in variables $\tau_1, \tau_2, \eta_1, \eta_2, x$ can be obtained:

$$\begin{aligned} \Omega \tau_1 + \gamma \tau_2 + [a_i^0]^T x &\leq b_i^0, \quad i = 1, \dots, m \\ \left(\sum_{l=1}^{-1} \eta_1 + \sum_{l=1}^{-1} \eta_2 \right) &= b_i^l - [a_i^l]^T x, \quad l = 1, \dots, L \end{aligned} \quad (19)$$

$$\|\eta_1\|_2 \leq \tau_1, \quad \|\eta_2\|_\infty \leq \tau_2.$$

Eliminating from above two systems the variables τ_1, τ_2 for every feasible solution to these two systems, we have $\tau_1 \geq \bar{\tau}_1 \equiv \|\eta_1\|_2, \tau_2 \geq \bar{\tau}_2 \equiv \|\eta_2\|_\infty$, and the solution obtained when replacing τ_1, τ_2 with $\bar{\tau}_1, \bar{\tau}_2$ is still feasible. The reduced system in variables $x, z = \sum_{l=1}^{-1} \eta_1, \omega = \sum_{l=1}^{-1} \eta_2$ reads

$$\Omega \sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \gamma \max_l |\sigma_l \omega_l| + t - [c^0]^T x \leq d^0, \quad (20)$$

$$z_l + \omega_l = d^l + [c^l]^T x, \quad l = 1, \dots, L, \quad (21)$$

which is an equivalent representation of (4) and (16).

Furthermore, the following system

$$\Omega \sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \gamma \max_l |\sigma_l \omega_l| + [a_i^0]^T x \leq b_i^0, \quad (22)$$

$$\begin{aligned} z_l + \omega_l &= b_i^l - [a_i^l]^T x, \\ l = 1, \dots, L, \quad i &= 1, \dots, m \end{aligned} \quad (23)$$

is an equivalent representation of (5) and (16).

Next we focus on a scheme for the safe approximation of ambiguous probabilistic constraints. For this purpose, we assume the perturbation vector ζ affecting (4) and (5) possesses the following properties:

P.1 $\zeta_l, l = 1, \dots, L$ are mutually independent random variables;

P.2 the distributions P_l of the components ζ_l satisfy the following conditions

$$\int \exp \{ts\} dP_l(s) \leq \exp \left\{ \frac{1}{2} \sigma_l^2 t^2 \right\}, \quad \forall t \in R \quad (24)$$

with known constants $\sigma_l \geq 0$.

In virtue of the above analysis, we obtain the following observations.

Theorem 3. *Let the random perturbations affecting (4) obey P.1-2; the RC of the uncertain linear constraint (4) with the perturbation set (16) is equivalent to the system of conic quadratic constraints (20)-(21).*

In addition, the x component of every feasible solution to this system satisfies the randomly perturbed inequality (4) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

Theorem 4. *Let the random perturbations affecting (5) obey P.1-2; the RC of the uncertain linear constraint (5) with the perturbation set (16) is equivalent to the system of conic quadratic constraints (22)-(23).*

In addition, the x component of every feasible solution to this system satisfies the randomly perturbed inequality (5) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

4. Financial Application

Portfolio selection is the problem of allocating capital over a number of available assets in order to maximize the return on the investment while minimizing the risk. In this section, we consider the situation that the distribution of random return is partially known and model the portfolio optimization problem with an ambiguous VaR method.

4.1. The Equivalent Conic Quadratic Financial Optimization Model. Suppose there exist n stocks that can be chosen by an investor in the financial market. The goal is to determine the capital allocation among the stocks in order to maximize the VaR of the resulting portfolio. Let random vector $\mathbf{r} = (r_1, \dots, r_n)$ denote uncertain return rates of the n stocks, and $\mathbf{x} = (x_1, \dots, x_n)$ denote the amount of the investments in the n stocks. Each component of uncertain return rate vector $\mathbf{r} = (r_1, \dots, r_n)$ is represented as

$$r_j = r_j^0 + \sum_{i=1}^n \zeta_i r_j^i, \quad j = 1, \dots, n, \quad (25)$$

where r_j^0 is the nominal value of r_j , and $r_j^i, i = 1, \dots, n$, are the basic shifts data of r_j . $\zeta_i, i = 1, \dots, n$, are independent random perturbations, and $\zeta = (\zeta_1, \dots, \zeta_n)$ is varying in a given perturbation set Z . Thus an ambiguous VaR portfolio optimization model can be built as

$$\begin{aligned} \max_x \quad & t \\ \text{s.t.} \quad & \Pr_{\zeta \sim P} \left\{ \left[r^0 + \sum_{i=1}^n \zeta_i r^i \right]^T x \geq t \right\} \geq 1 - \epsilon, \\ & \forall P \in \mathcal{P} \end{aligned} \quad (26)$$

$$x \in D = \left\{ \sum_{j=1}^n x_j = 1, x \geq 0 \right\},$$

where $\mathcal{P}$ is the family of potential probability distributions of random perturbation vector ζ .

In model (26), the uncertain return rate of each stock is an affine function of bounded and mutually independent perturbation variables ζ_i . When historical data for stocks are available, some statistical tools such as principal component analysis and linear regression can be used to calibrate the values of ζ_i from the data [46]. The perturbation set Z of ζ provides the support information of $\mathbf{r}$, while the moment information of ζ characterizes the random nature of the return $\mathbf{r}$. In this section, we consider the following two cases in our numerical experiments:

- (i) *Case I.* Perturbation vector ζ fluctuates in a ball and budget perturbation set Z depicted in (6), and the distribution family $\mathcal{P}$ of $\zeta_l, l = 1, \dots, L$ satisfies the case of (15).
- (ii) *Case II.* Perturbation vector ζ fluctuates in an ellipsoid and generalized budget perturbation set Z depicted in (16), and the distribution family $\mathcal{P}$ of $\zeta_l, l = 1, \dots, L$ satisfies the assumptions **P.1-2**.

When ζ fluctuates in a ball and budget perturbation set Z as depicted in *Case I*, it follows from Theorem 1 that ambiguous VaR model (26) is equivalent to the following conic quadratic optimization model:

$$\begin{aligned}
 & \max_x \quad t \\
 & \text{s.t.} \quad \Omega \sqrt{\sum_{i=1}^n z_i^2} + \gamma \max_i |\omega_i| + t - [r^0]^T x \leq 0 \\
 & \quad z_i + \omega_i = [r^i]^T x, \quad i = 1, \dots, n \\
 & \quad \sum_{j=1}^n x_j = 1 \\
 & \quad x \geq 0.
 \end{aligned} \tag{27}$$

On the other hand, when ζ fluctuates in an ellipsoid and generalized budget perturbation set Z as depicted in *Case II*, by Theorem 3, ambiguous VaR model (26) is equivalent to the following conic quadratic optimization model:

$$\begin{aligned}
 & \max_x \quad t \\
 & \text{s.t.} \quad \Omega \sqrt{\sum_{i=1}^n \sigma_i^2 z_i^2} + \gamma \max_i |\sigma_i \omega_i| + t - [r^0]^T x \leq 0 \\
 & \quad z_i + \omega_i = [r^i]^T x, \quad i = 1, \dots, n \\
 & \quad \sum_{j=1}^n x_j = 1 \\
 & \quad x \geq 0.
 \end{aligned} \tag{28}$$

The above conic quadratic optimization models (27) and (28) are established optimization formats supported by commercial solvers that scale well with the number of variables in the problem. In our numerical experiments, we employ the LINGO 11.0.0.20 solver to find the robust optimal portfolio. All the experiments are conducted in a personal computer with Inter(R) Core(TM) i5-4200U CPU 1.60GHz and RAM 4.00GB.

4.2. Data Description. We select ten stocks with different return rates from Chinese stock market on October 24, 2018, as the candidate. Each stock has a daily return rate $r_j, j = 1, \dots, 10$. These data are obtained through stock software, named Straight Flush, which can provide stock market

display, analysis, and trading. Since the stock return rates are affected by various factors, the daily return rate r_j is varying in a segment around its daily expected return rate. The daily expected return rate is viewed as the nominal value of the random return rate r_j . For instance, the return rate of stock 601116 takes its value in $[-0.03\%, 2.32\%]$, where -0.03% and 2.32% represent the minimum value and the maximum value of the return rate, respectively, and its nominal value is 1.20% . Figure 1 shows the fluctuations of the daily return rate of stock 601116, from which it is not difficult to see that daily return rate fluctuates around the nominal value.

The nominal values of ten return rates and the variation ranges of the return rates are summarized in Table 1. In the following, we will conduct some numerical experiments to find the optimal portfolio policy based on these data.

4.3. The Computational Results under Case I. In order to find the optimal portfolio selection, we solve the distributionally robust portfolio optimization problem under *Case I*. First of all, we need to analyze the connections between perturbation parameters Ω, γ and risk level ϵ to ensure that the selected values of parameters Ω, γ are reasonable. From the proofs of Theorems 1 and 3, it is known that the relationship between the risk level and perturbation set parameters is

$$\begin{aligned}
 \epsilon &= \exp \left\{ -\frac{s^2}{2} \right\}, \\
 s &= \min \left\{ \Omega, \frac{\gamma}{\sqrt{n}} \right\}.
 \end{aligned} \tag{29}$$

Note that the values of parameters Ω and γ reflect the relationship between ball and budget perturbation set. When $\Omega = \gamma$, the intersection of ball and budget perturbation set is illustrated in Figure 2(a); when $\Omega = \gamma/\sqrt{2}$, the intersection of ball and budget perturbation set is illustrated in Figure 2(b). Based on this observation, the parametric values should be chosen from the interval $\gamma/\sqrt{2} < \Omega < \gamma$.

According to the discussion above, we give the corresponding values of perturbations set parameters Ω, γ and $1 - \epsilon$ in Table 2. Then, we solve model (27) to find the optimal portfolio strategy under the values of these parameters. Note that the value of perturbation parameter Ω can be different even at the same risk. Taking into account the different values of Ω , we select some of the computational results on portfolio strategy and report them in Table 3. For example, when $1 - \epsilon$ is set to be 0.93 and γ equals 7.293, we take Ω as 5.2, 6.4, and 7.2. In order to better demonstrate the variation of optimal portfolio strategies in Table 3 at different parameter values of ϵ, γ , and Ω , it is plotted in Figures 3(a)–3(j). The table shows the variation of allocation proportion of each stock with respect to $1 - \epsilon$, and the corresponding values of perturbation parameter Ω are marked in Figure 3.

Further, we plot Figure 4 to illustrate the effect of parameter Ω on the value of the objective function. When $1 - \epsilon$ equals 0.92 and Ω takes values in the corresponding interval (5.026, 7.107), the changeable range of optimal value is smaller. However, when $1 - \epsilon$ equals 0.97 and Ω takes values in (5.922, 8.374), the changeable range of optimal value on the

FIGURE 1: The return rate of stock 601116 on October 24, 2018.

FIGURE 2: The relationship between ball and budget perturbation set.

TABLE 1: Descriptive statistics for the return rates of 10 stocks.

Stock code	Nominal value r^0 (%)	Range of r (%)
600408	0.33	[-1.68,4.72]
601116	1.20	[-0.03,2.32]
000750	1.93	[-1.52,10.08]
000750	2.49	[-1.20,7.53]
300444	3.19	[0.39,7.07]
603607	3.82	[-1.68,7.24]
300392	4.53	[1.49,8.11]
600844	5.82	[-1.94,10.11]
600695	6.61	[3.33,10.06]
000153	7.99	[4.47,10.04]

(a) Allocation proportion x_1 (b) Allocation proportion x_2 (c) Allocation proportion x_3 (d) Allocation proportion x_4 (e) Allocation proportion x_5 (f) Allocation proportion x_6 (g) Allocation proportion x_7 (h) Allocation proportion x_8

FIGURE 3: Continued.

FIGURE 3: The variation of optimal portfolio strategies in *Case I* under different risks. FIGURE 4: The comparison of the optimal values under different ball parameters- Ω .

TABLE 2: The corresponding values between risk parameter $1 - \epsilon$ and perturbation parameters Ω , γ .

$1 - \epsilon$	budget parameters γ	ball parameters Ω
0.91	6.940	(4.907, 6.940)
0.92	7.107	(5.026, 7.107)
0.93	7.293	(5.157, 7.293)
0.94	7.501	(5.304, 7.501)
0.95	7.740	(5.473, 7.740)
0.96	8.024	(5.674, 8.024)
0.97	8.374	(5.922, 8.374)
0.98	8.845	(6.255, 8.845)
0.99	9.597	(6.786, 9.597)

FIGURE 5: The optimal values under ball and budget perturbation set.

objective function is larger. Thus the optimal objective value is sensitive to the parameter Ω . The reason why different portfolio strategies and objective values at the same risk level are obtained is that the size of the perturbation sets is different. When the perturbation set is larger, the safe approximation is more conservative. From the computational results, the decentralized investment decisions can be obtained when uncertain return fluctuates under *Case I*. Taking into account the difference of portfolio under different allocations, the relationship between the optimal objective value and perturbation set parameters is given; see Figure 5.

4.4. The Computational Results under Case II. To find the optimal portfolio selection, we also solve the distributionally robust portfolio optimization problem in *Case II*.

The situation of *Case II* considers the fluctuation in the intersection of ellipsoid and generalized budget, and the values on the radius of ellipsoid and generalized budget σ are set as

$$\begin{aligned} \sigma &= (\sigma_1, \dots, \sigma_{10}) \\ &= (0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0). \end{aligned} \quad (30)$$

Here it is required to solve model (28). Similarly, the relationship between risk level and perturbation set parameter Ω , γ is shown in Table 2, and the computational results on portfolio strategies with different values of parameters are reported in Table 4. Further, to illustrate the variation of

optimal portfolio strategies in Table 4, it is depicted in Figures 6(a)–6(j). From these figures, we can know how to change the allocation proportion of each stock at risk parameter $1 - \epsilon$, and the values of perturbation parameter Ω are marked in Figure 6.

From the computational results, the decentralized investment decisions can be obtained when uncertain return fluctuates under *Case II*. Taking into account the difference of portfolio under different allocations, the relationship between optimal objective value and perturbation set parameters is given in Figure 7.

4.5. Comparison Study. In this subsection, the distributionally robust portfolio optimization model is compared with nominal distribution optimization model, in which uncertain return vector has a nominal probability distribution.

According to uncertain portfolio optimization model (26), we have the stochastic portfolio optimization model with a nominal distribution P of random return. To compare the distributionally robust optimization model with nominal distribution optimization model, we analyze the normal distribution information under *Case II*.

In *Case II*, the perturbation vector ζ fluctuates in an ellipsoid and generalized budget perturbation set Z depicted in (16), the distribution family $\mathcal{P}$ in model (26) satisfies assumptions **P.1-2**, and the mutually independent random variables ζ_i are only partially known with expectations $\mu_i = 0$ and variances $s_i^2 \leq \sigma_i^2$. We next consider the scenario of $s_i^2 = \sigma_i^2$, i.e., $\zeta_i \sim N(0, \sigma_i^2)$, $i = 1, \dots, n$. In this situation, the family of normal distributions in *Case II* degenerates into a single normal distribution. As a result, model (26) reduces to the following nominal distribution model:

$$\begin{aligned} \max_x \quad & t \\ \text{s.t.} \quad & \Phi \left(\frac{\sum_{j=1}^n x_j r_j^0 - t}{\sqrt{\sum_{j=1}^n x_j^2 \sum_{i=1}^n \sigma_i^2 (r_j^i)^2}} \right) \geq 1 - \epsilon \\ & \sum_{j=1}^n x_j = 1 \\ & x \geq 0. \end{aligned} \quad (31)$$

(a) Allocation proportion x_1 (b) Allocation proportion x_2 (c) Allocation proportion x_3 (d) Allocation proportion x_4 (e) Allocation proportion x_5 (f) Allocation proportion x_6 (g) Allocation proportion x_7 (h) Allocation proportion x_8

FIGURE 6: Continued.

FIGURE 6: The variation of optimal portfolio strategies in *Case II* under different risks.

FIGURE 7: The optimal values under ellipsoid and generalized budget perturbation set.

To compare the ambiguous VaR model with nominal distribution model, we analyze the computational results at the same risk level. Based on the data in Table 2, we solve model (31) and obtain the computational results under different risks as shown in Table 5. Further, Figure 8 is plotted to show the evolution of the optimal portfolio values of nominal distribution model at risk parameter $1 - \epsilon$.

Under the same risk level, we select the portfolio of ten stocks with positive return, by using the methods of distributionally robust optimization and stochastic optimization with nominal distribution, and obtain the following observations.

First of all, stochastic optimization method with nominal normal distribution is based on the assumption that the full distribution information of future return can be estimated exactly. When the full information on the probability distribution is unavailable, this optimization method will be invalid. In fact, due to the impacts of the economic environment and political factors, it is usually difficult to obtain the exact distribution information of future return. In such case, portfolio strategy under the nominal distribution cannot be recommended to the investors.

When only partial information on the underlying probability distribution is available, the investor should employ the distributionally robust portfolio optimization method. The distributionally robust optimal portfolio policy is the best choice for investors when the uncertain returns are fluctuating in a given ambiguity set. That is, the robust optimal portfolio provides immunization against the distribution uncertainty.

In order to compare the optimal values of distributionally robust optimization model and nominal distribution model, we firstly transform the values of risk parameter ϵ in model (31) to the corresponding values of Ω and γ and then plot, in Figure 9, a comparison between the optimal values of distributionally robust model under ellipsoid and generalized budget perturbation set and nominal distribution model. In terms of optimal solution, we find that the optimal portfolio policy obtained from distributionally robust optimization model is more decentralized than that from nominal distribution model. For example, if $1 - \epsilon$ is 0.98, based on nominal distribution model, the investor should select three stocks with number 8, 9, and 10 as a portfolio. However, solving

TABLE 3: The optimal portfolio strategies under ball and budget perturbation set (%).

$1 - \epsilon$	γ	Ω	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.91	6.940	5.3	2.984554	13.012910	6.135631	3.185021	12.655810	13.610000	17.450490	18.766460	9.504393	2.694748
		6.2	2.984527	13.012800	6.135777	3.184999	12.655910	13.609980	17.450390	18.766480	9.504414	2.694729
		6.8	2.975392	12.991980	6.127129	3.176417	12.680580	13.618360	17.461880	18.778560	9.501712	2.687999
0.92	7.107	5.1	2.984560	13.012930	6.135599	3.185026	12.655790	13.610000	17.450520	18.766450	9.504388	2.694753
		6.2	2.984526	13.012800	6.135777	3.184999	12.655910	13.609980	17.450390	18.766480	9.504414	2.694729
		7.0	2.975386	12.991980	6.127086	3.176410	12.680590	13.618370	17.461920	18.778570	9.501704	2.687995
0.93	7.293	5.2	2.984557	13.012920	6.135615	3.185023	12.655800	13.610000	17.450500	18.766450	9.504390	2.694750
		6.4	2.984520	13.012780	6.135810	3.184994	12.655930	13.609980	17.450370	18.766490	9.504419	2.694724
		7.2	2.975385	12.991980	6.127073	3.176408	12.680590	13.618380	17.461930	18.778580	9.501702	2.687993
0.94	7.501	5.4	2.984551	13.012890	6.135648	3.185018	12.655820	13.610000	17.450480	18.766460	9.504395	2.694746
		7.0	2.984502	13.012700	6.135907	3.184979	12.656000	13.609970	17.450300	18.766500	9.504434	2.694711
		7.4	2.975387	12.991980	6.127092	3.176411	12.680580	13.618370	17.461910	18.778570	9.501705	2.687995
0.95	7.740	6.1	2.995548	13.063250	6.263475	3.197506	12.593050	13.586360	17.342490	18.742500	9.512746	2.703092
		6.8	2.984508	13.012730	6.135875	3.184984	12.655980	13.609970	17.450320	18.766500	9.504429	2.694716
		7.6	2.975396	12.991980	6.127152	3.176420	12.680580	13.618350	17.461860	18.778560	9.501717	2.688002
0.96	8.024	6.3	2.984523	13.012790	6.135794	3.184996	12.655920	13.609980	17.450380	18.766480	9.504417	2.694727
		7.5	2.984487	13.012640	6.135989	3.184966	12.656060	13.609960	17.450240	18.766520	9.504446	2.694700
		7.9	2.975394	12.991980	6.127137	3.176418	12.680580	13.618360	17.461870	18.778560	9.501714	2.688000
0.97	8.374	6.7	3.107925	13.246470	6.249412	3.300299	12.367660	13.505560	17.298960	18.604680	9.533868	2.785177
		7.4	2.984490	13.012650	6.135973	3.184969	12.656050	13.609960	17.450260	18.766510	9.504444	2.694703
		8.2	2.984465	13.012560	6.136104	3.184949	12.656140	13.609950	17.450170	18.766540	9.504463	2.694685
0.98	8.845	7.4	2.984490	13.012650	6.135974	3.184969	12.656050	13.609960	17.450260	18.766510	9.504444	2.694702
		8.3	3.075341	13.108540	6.426811	3.273873	12.499240	13.483630	17.175920	18.633460	9.561397	2.761813
		8.7	2.984450	13.012500	6.136186	3.184936	12.656200	13.609940	17.450110	18.766550	9.504475	2.694674
0.99	9.597	7.4	2.984489	13.012650	6.135975	3.184969	12.656050	13.609960	17.450250	18.766510	9.504444	2.694702
		8.0	2.984471	13.012580	6.136072	3.184954	12.656120	13.609950	17.450190	18.766530	9.504458	2.694689
		9.4	2.967558	12.953210	6.191147	3.170496	12.715800	13.608250	17.415440	18.783990	9.511675	2.682452

TABLE 4: The optimal portfolio strategies under ellipsoid and generalized budget perturbation set (%).

$1 - \epsilon$	γ	Ω	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.91	6.940	5.0	2.984245	13.011700	6.137194	3.184769	12.656970	13.609830	17.449420	18.766740	9.504627	2.694327
		6.0	2.984154	13.011350	6.137643	3.184694	12.657310	13.609780	17.449110	18.766820	9.504694	2.694462
		6.8	2.984081	13.011070	6.137998	3.184634	12.657580	13.609740	17.448870	18.766890	9.504748	2.694410
0.92	7.107	5.1	2.984236	13.011670	6.137242	3.184761	12.657000	13.609820	17.449380	18.766740	9.504634	2.694521
		6.1	2.984145	13.011310	6.137690	3.184686	12.657340	13.609770	17.449080	18.766830	9.504701	2.694455
		7.0	2.984063	13.011000	6.138089	3.184619	12.657650	13.609730	17.448810	18.766900	9.504761	2.694397
0.93	7.293	5.2	2.984227	13.011630	6.137290	3.184754	12.657040	13.609820	17.449350	18.766750	9.504641	2.694514
		6.3	2.984127	13.011240	6.137782	3.184672	12.657410	13.609760	17.449020	18.766840	9.504715	2.694442
		7.1	2.984054	13.010960	6.138136	3.184612	12.657680	13.609720	17.448770	18.766910	9.504768	2.694390
0.94	7.501	5.4	2.984209	13.011560	6.137383	3.184739	12.657100	13.609800	17.449290	18.766770	9.504655	2.694501
		6.3	2.984127	13.011240	6.137785	3.184672	12.657410	13.609760	17.449010	18.766840	9.504715	2.694442
		7.3	2.984036	13.010890	6.138227	3.184598	12.657750	13.609710	17.448710	18.766930	9.504782	2.694378
0.95	7.740	5.5	2.984200	13.011520	6.137432	3.184732	12.657140	13.609800	17.449250	18.766780	9.504662	2.694495
		6.5	2.984109	13.011170	6.137877	3.184657	12.657480	13.609750	17.448950	18.766860	9.504729	2.694429
		7.5	2.984019	13.010820	6.138319	3.184583	12.657820	13.609700	17.448650	18.766940	9.504796	2.694365
0.96	8.024	5.7	2.984182	13.011450	6.137526	3.184717	12.657210	13.609790	17.449190	18.766790	9.504676	2.694482
		6.8	2.984082	13.011060	6.138015	3.184635	12.657580	13.609740	17.448850	18.766880	9.504750	2.694410
		7.6	2.984010	13.010790	6.138367	3.184576	12.657850	13.609700	17.448610	18.766950	9.504803	2.694358
0.97	8.374	6.1	2.984145	13.011310	6.137710	3.184687	12.657340	13.609770	17.449060	18.766830	9.504704	2.694456
		6.9	2.984073	13.011030	6.138064	3.184628	12.657620	13.609730	17.448820	18.766890	9.504757	2.694404
		7.9	2.983983	13.010680	6.138504	3.184554	12.657950	13.609680	17.448520	18.766970	9.504823	2.694339
0.98	8.845	6.4	2.984112	13.011180	6.137884	3.184660	12.657470	13.609750	17.448940	18.766860	9.504730	2.694432
		7.3	2.984037	13.010890	6.138248	3.184599	12.657750	13.609710	17.448690	18.766920	9.504784	2.694378
		8.3	2.979340	12.993050	6.159839	3.180717	12.675100	13.607610	17.434070	18.771330	9.507953	2.691004
0.99	9.597	6.8	2.984075	13.011040	6.138075	3.184630	12.657610	13.609730	17.448810	18.766890	9.504758	2.694406
		7.9	2.983975	13.010650	6.138563	3.184548	12.657980	13.609670	17.448480	18.766980	9.504832	2.694334
		8.9	2.979341	12.993050	6.159864	3.180718	12.675100	13.607610	17.434050	18.771320	9.507958	2.691005

FIGURE 8: The optimal values generated by nominal distribution model (31) under risk parameter.

FIGURE 9: The comparison of optimal values of nominal model and distributionally robust model in *Case II*.

distributionally robust model, the investor should allocate his/her money to ten stocks. The results show that the optimal portfolio strategies of the two methods are completely different, and the distributionally robust optimization method can better resist the distributional uncertainty.

5. Conclusions

In this paper, we studied portfolio optimization problem under distribution uncertainty from a new perspective. The obtained major results include the following three aspects.

First, to model the distribution of model parameters varying in a prescribed ambiguity set, we developed an ambiguous VaR optimization framework for distributionally robust linear optimization problem. The objective of the proposed model is to optimize the worst case VaR over an ambiguity set. As a result, the obtained robust solution provides immunization against the distribution uncertainty.

Second, we addressed the tractability of the ambiguous probabilistic constraints under two new combinations of perturbation sets. The first ambiguity set is the intersection of ball with budget, while the second one is the intersection of ellipsoid with generalized budget. Depending on the choice of the two new ambiguity sets, the resulting framework of the

safe approximation can be reduced to a system of efficiently computable convex constraints.

Finally, to show the advantage of the proposed approach, we applied our approach to a practical case about the Chinese stock market. By comparison study, we concluded that the optimal solution to the proposed robust optimization approach provides immunization against distribution uncertainty, and even small perturbation of nominal probability distribution cannot be ignored in practical portfolio optimization problems.

Appendix

Proofs of Main Results

In this appendix, we only present the proofs of Theorems 1 and 3, and Theorems 2 and 4 can be proved similarly.

We first prove Theorem 1.

Proof. We already know that (11)-(12) represent the RC of (4), with the perturbation set being (6). Now let us prove that if ζ_l , $l = 1, \dots, L$, satisfy hypothesis (15) and x, z, ω are feasible for (11)-(12), then x is feasible for (4) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

TABLE 5: The optimal portfolio strategies produced by nominal distribution model (31) (%).

$1 - \epsilon$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0.91	-	-	-	-	-	-	-	-	23.805630	76.194370
0.92	-	-	-	-	-	-	-	-	25.430380	74.569620
0.93	-	-	-	-	-	-	-	0.341380	26.757220	72.901400
0.94	-	-	-	-	-	-	-	3.227114	27.332420	69.440470
0.95	-	-	-	-	-	-	-	5.835254	27.852130	66.312620
0.96	-	-	-	-	-	-	-	8.179620	28.319590	63.500790
0.97	-	-	-	-	-	-	-	10.621300	28.806190	60.572500
0.98	-	-	-	-	-	-	-	13.112000	29.302940	57.585060
0.99	-	-	-	-	-	-	-	16.123170	29.902920	53.973910

We pass from (4) to the following inequality

$$-\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x. \quad (\text{A.1})$$

Thus, the following equivalent representation holds:

$$\begin{aligned} & \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x \right\} \\ & \geq 1 - \epsilon \iff \\ & \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t > d^0 + [c^0]^T x \right\} \\ & \leq \epsilon. \end{aligned} \quad (\text{A.2})$$

Indeed, when $\zeta_l, l = 1, \dots, L$, are mutually independent and $\|\zeta\|_\infty \leq 1$, we have the following.

$$\begin{aligned} x \text{ is infeasible for (4)} & \implies -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \\ & > d^0 + [c^0]^T x \\ & \implies -\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l + t \\ & > d^0 + [c^0]^T x \quad [by (12)] \\ & \implies -\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l + t \\ & > \Omega \sqrt{\sum_{l=1}^L z_l^2} + \gamma \max_l |\omega_l| + t \\ & \quad [by (11)] \end{aligned} \quad (\text{A.3})$$

Let (x, z, ω) be feasible for (11)-(12). Then we have

$$\begin{aligned} \|\omega\|_2^2 &= \sum_{l=1}^L \omega_l^2 \leq \sum_{l=1}^L |\omega_l| \|\omega\|_\infty \leq \|\omega\|_\infty \sum_{l=1}^L |\omega_l| \\ &\leq \|\omega\|_\infty \sqrt{L} \|\omega\|_2. \end{aligned} \quad (\text{A.5})$$

Thus,

$$\|\omega\|_2 \leq \sqrt{L} \|\omega\|_\infty. \quad (\text{A.6})$$

As a result, we have

$$-\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l > \Omega \sqrt{\sum_{l=1}^L z_l^2} + \frac{\gamma}{\sqrt{L}} \sqrt{\sum_{l=1}^L \omega_l^2}. \quad (\text{A.7})$$

Setting $s = \min\{\Omega, \gamma/\sqrt{L}\}$, we have

$$-\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \left(\sqrt{\sum_{l=1}^L z_l^2} + \sqrt{\sum_{l=1}^L \omega_l^2} \right). \quad (\text{A.8})$$

According to the triangle inequality

$$\sqrt{\sum_{l=1}^L z_l^2} + \sqrt{\sum_{l=1}^L \omega_l^2} > \sqrt{\sum_{l=1}^L (z_l + \omega_l)^2}, \quad (\text{A.9})$$

we have

$$-\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \sqrt{\sum_{l=1}^L (z_l + \omega_l)^2}. \quad (\text{A.10})$$

On the basis of the conclusion of proposition 2.3.1 in [13], we have the following result:

$$\begin{aligned} & \Pr_{\zeta \sim P} \{x \text{ is infeasible for (4)}\} \\ & \leq \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \sqrt{\sum_{l=1}^L (z_l + \omega_l)^2} \right\} \\ & \leq \exp \left\{ -\frac{s^2}{2} \right\}, \end{aligned} \quad (\text{A.11})$$

with $s = \min\{\Omega, \gamma/\sqrt{L}\}$, $s > 0$, and thus

$$\begin{aligned} & \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x \right\} \\ & \geq 1 - \exp \left\{ -\frac{s^2}{2} \right\}. \end{aligned} \quad (\text{A.12})$$

The proof of theorem is complete. $\square$

Next we prove Theorem 3.

Proof. We already know that (20)-(21) represent the RC of (4), with the perturbation set being (16). Now let us prove that if the independent random variables $\zeta_l, l = 1, \dots, L$, satisfy hypotheses P.1-2 and x, z, ω are feasible for (20)-(21), then x is feasible for (4) with probability of at least $1 - \exp\{-s^2/2\}$, $s = \min\{\Omega, \gamma/\sqrt{L}\}$.

We can pass from (4) to the following inequality

$$-\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x. \quad (\text{A.13})$$

Thus, the following equivalent representation holds:

$$\begin{aligned} \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x \right\} \\ \geq 1 - \epsilon \iff \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \right. \\ \left. > d^0 + [c^0]^T x \right\} \leq \epsilon. \end{aligned} \quad (\text{A.14})$$

Without loss of generality, when $|\zeta_l| \leq 1$ and $\zeta_l, l = 1, \dots, L$ are mutually independent, we have

$$\begin{aligned} x \text{ is infeasible for (4)} &\implies -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l \\ &> d^0 + [c^0]^T x \\ &\implies -\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l + t > d^0 + [c^0]^T x \\ &\quad [by (21)] \quad (\text{A.15}) \\ &\implies -\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l + t \\ &> \Omega \sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \gamma \max_l |\sigma_l \omega_l| + t \quad [by (20)]. \end{aligned}$$

Let (x, z, ω) be feasible for (20)-(21). Then we have

$$\begin{aligned} \|\omega\|_2^2 = \sum_{l=1}^L \omega_l^2 &\leq \sum_{l=1}^L |\omega_l| \|\omega\|_\infty \leq \|\omega\|_\infty \sum_{l=1}^L |\omega_l| \\ &\leq \|\omega\|_\infty \sqrt{L} \|\omega\|_2. \end{aligned} \quad (\text{A.16})$$

Thus, we have

$$\|\omega\|_2 \leq \sqrt{L} \|\omega\|_\infty. \quad (\text{A.17})$$

As a result, we have

$$-\sum_{l=1}^L z_l \zeta_l - \sum_{l=1}^L \omega_l \zeta_l > \Omega \sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \frac{\gamma}{\sqrt{L}} \sqrt{\sum_{l=1}^L \sigma_l^2 \omega_l^2}. \quad (\text{A.18})$$

Setting $s = \min\{\Omega, \gamma/\sqrt{L}\}$, we have

$$-\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \left(\sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \sqrt{\sum_{l=1}^L \sigma_l^2 \omega_l^2} \right). \quad (\text{A.19})$$

According to the triangle inequality

$$\sqrt{\sum_{l=1}^L \sigma_l^2 z_l^2} + \sqrt{\sum_{l=1}^L \sigma_l^2 \omega_l^2} > \sqrt{\sum_{l=1}^L \sigma_l^2 (z_l + \omega_l)^2}, \quad (\text{A.20})$$

we have

$$-\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \sqrt{\sum_{l=1}^L \sigma_l^2 (z_l + \omega_l)^2}. \quad (\text{A.21})$$

On the basis of the conclusion of proposition 2.4.2 in [13], we obtain the following result:

$$\begin{aligned} \Pr_{\zeta \sim P} \{x \text{ is infeasible for (4)}\} \\ \leq \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L (z_l + \omega_l) \zeta_l > s \sqrt{\sum_{l=1}^L \sigma_l^2 (z_l + \omega_l)^2} \right\} \\ \leq \exp \left\{ -\frac{s^2}{2} \right\}, \end{aligned} \quad (\text{A.22})$$

with $s = \min\{\Omega, \gamma/\sqrt{L}\}$, $s > 0$ thus

$$\begin{aligned} \Pr_{\zeta \sim P} \left\{ -\sum_{l=1}^L \left([c^l]^T x + d^l \right) \zeta_l + t \leq d^0 + [c^0]^T x \right\} \\ \geq 1 - \exp \left\{ -\frac{s^2}{2} \right\}. \end{aligned} \quad (\text{A.23})$$

The proof of the theorem is complete. $\square$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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