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Research Article

Averaged and Integrated Estimations of Varying-Coefficient Regression Models with Dependent Observations

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In practical applications, lots of data such as sequentially collected economic data often exhibit some evident dependence. This paper studies the varying-coefficient regression models with different smoothing variables when the data form a stationary α-mixing sequence. Both the averaged and integrated estimators of coefficient functions are proposed. The asymptotic normalities of the proposed averaged and integrated estimators are also established.

1. Introduction

Regression analysis is one of the most mature and widely applied branches of statistics. Various regression models have been studied by many authors; e.g., Liang and Fan [1] studied Berry-Esseen type bounds of estimators in a semiparametric model with linear process errors; Fan, Liang, and Xu [2] considered empirical likelihood confidence regions for heteroscedastic partial linear model and so on. Recently, varying models which were proposed by Hastie and Tibshirani [3] and Chen and Tsay [4] have received more and more attention. This is mainly because varying model offer a flexible but parsimonious alternative to nonparametric model and has been used in many contexts such as it has been successfully applied to multidimensional nonparametric regression, generalized linear model, time series analysis, longitudinal and functional data analysis, and time-varying model in finance. A nice feature of varying coefficients is to allow appreciable flexibility on the structure of fitted models without suffering from the “curse of dimensionality”.

Consider the following varying-coefficient model with different smoothing variables in different coefficient functions:

$$Y=\sum_{\alpha=1}^{p}a_{\alpha}\left(X_{\alpha}\right)Z_{\alpha}+\sigma\left(\mathbf{X},\mathbf{Z}\right)\varepsilon, \tag{1}$$

where $(Y,\mathbf{X}^{\tau},\mathbf{Z}^{\tau})$ is a random vector, $\mathbf{X}=(X_1,\dots,X_p)^{\tau}\in R^p$ and $\mathbf{Z}=(Z_1,\dots,Z_p)^{\tau}\in R^p$; $\{a_{\alpha}(\cdot)\}_{\alpha=1}^{p}$ are some measurable functions from R to R; $\sigma(\cdot,\cdot)$ is a measurable function from R^{2p} to R and ε is independent of $(\mathbf{X},\mathbf{Z})$ with mean zero and variance 1.

Most of the work focused on the case where all coefficient functions share a single smoothing variable in a model, i.e. $X_1=X_2=\cdots=X_p$ in model (1). In particular, the asymptotic normalities of the estimators of these coefficient functions by local polynomial technique are obtained, such as Shen et al. [5], Wang and Lin [6], and Fan, Liang, and Zhu [7]. However, the varying-coefficient regression models on which different coefficient functions share different variables have been given less attention because the local polynomial technique is not adequate. To solve this difficulty, when $(\mathbf{X}_i,\mathbf{Z}_i,\varepsilon_i)$, which come from $(\mathbf{X},\mathbf{Z},\varepsilon)$, are independent and identically distributed (i.i.d.), Zhang and Li [8] employed the local linear technique to obtain an initial value for the estimate of each coefficient function in model (1), and, then, the integrated estimate of each coefficient function is defined by integrating its initial value on these variables which the coefficient function does not share; Zhang and Li [9] used the local linear technique to give an initial value for every estimate of the coefficient function in model (1). Then the averaged estimate of each coefficient function is defined by averaging its initial value on these variables which the coefficient function does not share; Yang [10] proposed estimators for this model under random right censoring case by using mean-preserving transformation and established their asymptotic properties. The estimation procedure is based on the profiling and the smooth backfitting techniques.

However, the above related papers need the i.i.d. assumption for the data. The independence assumptions are not always valid in applications, especially for sequentially collected economic data, which often exhibit evident dependence. In the sequel, $(\mathbf{X}_i, \mathbf{Z}_i, \varepsilon_i)$ are assumed to form a strong stationary α-mixing sequence. Fan et al. [11] studied the varying-coefficient errors-in-variables models when the data form a stationary α-mixing sequence of random variables. Recall that a sequence $\{\zeta_i,\ i \geq 1\}$ is said to be α-mixing if the α-mixing coefficient

$$\alpha(m) := \sup_{k\geq 1} \sup \left\{ |P(AB) - P(A)P(B)| : A \in \mathcal{F}_{m+k}^{\infty},\ B \in \mathcal{F}_1^k \right\} \tag{2}$$

converges to zero as $m \longrightarrow \infty$, where $\mathcal{F}_l^m = \sigma\{\zeta_l, \zeta_{l+1}, \ldots, \zeta_m\}$ denotes the σ-algebra generated by $\zeta_l, \zeta_{l+1}, \ldots, \zeta_m$ with $l \leq m$. Among various mixing conditions used in the literature, α-mixing is reasonably weak and has many practical applications. In fact, under very mild assumptions linear autoregressive and more generally bilinear time series models are α-mixing with mixing coefficients decaying exponentially; i.e., $\alpha(k) = O(\rho^k)$ for some $0 < \rho < 1$; see Doukhan [12], page 99, for more details.

The paper is organized as follows. In Section 2, we list some assumptions and present the averaged and integrated estimators of the coefficient functions, as well as the asymptotical normalities for these estimators. Proofs of the main results and the preliminary lemmas are provided in Section 3. Concluding remarks are given in Section 4.

2. Methodology and Main Results

Suppose that $\{Y_i, \mathbf{X}_i, \mathbf{Z}_i,\ i = 1, \ldots, n\}$ is a sample from model (1), $(\mathbf{X}_i, \mathbf{Z}_i, \varepsilon_i)$ are strong stationary α-mixing sequence, and that for every $\alpha = 1, \ldots, p$, $a_\alpha(\cdot)$ has a Lipschitz continuous second derivative. Then $a_\alpha(X_\alpha)$ can be approximate locally by a linear function, $a_\alpha(X_\alpha) \approx a_\alpha(x_\alpha) + a'_\alpha(x_\alpha)(X_\alpha - x_\alpha)$, at a neighborhood of x_α which is in the support of X_α. We minimize

$$\sum_{i=1}^{n} \left[Y_i - \sum_{\alpha=1}^{p} a_\alpha Z_{i\alpha} - \sum_{\alpha=1}^{p} b_\alpha (X_{i\alpha} - x_\alpha) Z_{i\alpha} \right]^2 \cdot \prod_{\alpha=1}^{p} K_{\alpha,h_\alpha}(X_{i\alpha} - x_\alpha) \tag{3}$$

with respect to $\{a_\alpha\}_{\alpha=1}^p$ and $\{b_\alpha\}_{\alpha=1}^p$, where $X_{i\alpha}$ and $Z_{i\alpha}$ are the αth entries of the ith observation $\mathbf{X}_i$ and $\mathbf{Z}_i$, respectively; $K_{\alpha,h_\alpha}(\cdot) = h_\alpha^{-1} K_\alpha(\cdot/h_\alpha)$, $K_\alpha(\cdot)$ is a bounded, nonnegative, compactly supported symmetric about zero and Lipschitz continuous density function; $h_\alpha = h_{n\alpha}$ is a sequence of positive numbers and called a bandwidth.

Let $\tilde{a}_\alpha(\mathbf{x})$, $\alpha = 1, \ldots, p$, be the first p entries of the minimizers of (3); then, it follows from the least squares theory that

$$\tilde{a}_\alpha(\mathbf{x}) = \sum_{i=1}^{n} e_{\alpha,2p}^{\tau} \left(U^\tau W U\right)^{-1} U_i W_{ii} Y_i, \quad \alpha = 1, \ldots, p, \tag{4}$$

where $\mathbf{x} = (x_1, \ldots, x_p)^\tau$, $e_{\alpha,2p}$ is a unite vector with 1 at its αth position and U is an $n \times 2p$ matrix with $U_i^\tau = (Z_{i1}, \ldots, Z_{ip}, ((X_{i1} - x_1)/h_1)Z_{i1}, \ldots, ((X_{ip} - x_p)/h_p)Z_{ip})$ as its ith row, $W = \operatorname{diag}(W_{11}, \ldots, W_{nn})$ with $W_{ii} = \prod_{\alpha=1}^{p} K_{\alpha,h_\alpha}(X_{i\alpha} - x_\alpha)$. $\tilde{a}_\alpha(\mathbf{x})$ is called as a initial value of the estimate of the coefficient function $a_\alpha(x)$.

The averaged estimate of the coefficient function $a_\alpha(x_\alpha)$ is defined as (see [9])

$$\hat{a}_\alpha(x_\alpha) = \frac{1}{n} \sum_{j=1}^{n} \tilde{a}_\alpha\left(X_{j1}, \ldots, X_{j,\alpha-1}, x_\alpha, X_{j,\alpha+1}, \ldots, X_{jp}\right), \quad \alpha = 1, \ldots, p. \tag{5}$$

Next, we define the integrated estimate of the coefficient function. Let $Q_{-\alpha}(\mathbf{x}_{-\alpha})$ be a deterministic weight function with $\int dQ_{-\alpha}(\mathbf{x}_{-\alpha}) = 1$, for each $\alpha = 1, \ldots, p$, where $\mathbf{x}_{-\alpha} = (x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_p)^\tau$. We allow for both discrete and continuous $Q_{-\alpha}(\mathbf{x}_{-\alpha})$ and integrals should be interpreted in the Stieltjes sense. Let $q_{-\alpha}(\mathbf{x}_{-\alpha})$ be the density of $Q_{-\alpha}(\mathbf{x}_{-\alpha})$ with respect to either Lebesgue or a counting measure. Suppose that the support of $Q_{-\alpha}(\cdot)$ is contained within that of $\mathbf{x}_{-\alpha}$. Then, the integrated estimate of the coefficient function $a_\alpha(x_\alpha)$ is defined as (see [8])

$$\breve{a}_\alpha(x_\alpha) = \int \tilde{a}_\alpha(\mathbf{x})\, dQ_{-\alpha}(\mathbf{x}_{-\alpha}), \quad \alpha = 1, \ldots, p. \tag{6}$$

In order to formulate the main results, we need the following assumptions.

(A1)

(i) The joint density $p(\mathbf{x})$ of $\mathbf{X}$ and the marginal density $p_\alpha(x_\alpha)$ of X_α, $\alpha = 1, \ldots, p$, are compactly supported, bounded, Lipschitz continuous, and bounded away from zero by a constant.

(ii) The conditional probabilities satisfy $P(\mathbf{X}_i, \mathbf{X}_j \mid \mathbf{X}_k) = P(\mathbf{X}_i \mid \mathbf{X}_k) \cdot P(\mathbf{X}_j \mid \mathbf{X}_k)$ for different $1 \leq i, j, k \leq n$.

(A2) The conditional expects $E(Z_\alpha Z_{\alpha'} \sigma^2(\mathbf{X}, \mathbf{Z}) \mid \mathbf{X} = \mathbf{x})$ and $E(Z_\alpha^{\lambda_1} Z_{\alpha'}^{\lambda_2} \mid \mathbf{X} = \mathbf{x})$ are Lipschitz continuous, $\lambda_1 + \lambda_2 = 2$ or 4, $\lambda_1 \geq 0, \lambda_2 \geq 0, \alpha, \alpha' = 1, \ldots, p$.

(A3) $E(|\varepsilon| + \|\mathbf{Z}\| + \|\mathbf{X}\|)^{2+\delta} < \infty$, for some $\delta > 0$.

(A4)

(i) $\alpha(n) = O(n^{-\gamma})$, for some $\gamma > 1 + 2/\delta$.

(ii) There exist positive integers $r \coloneqq r(n)$ and $q \coloneqq q(n)$ such that for sufficiently large n, $r + q \le n$, $n\alpha(q)/r = o(1)$ and $r^{\lambda/2} = o(n^{(5\lambda+2\lambda\delta-\delta)/5(2+\delta)})$, for some $0 < \lambda < \delta$.

Remark 1. Conditions (A1)(i) and (A2) have been assumed in Zhang and Li [8, 9]. (A1)(ii) holds naturally when $\mathbf{X}_i$ are i.i.d. (A3) has been used by Fan et al. [2]. Assumption (A4)(i) is a mild condition. In the particular case of exponential decay, $\alpha(k) = O(\rho^k)$ for some $0 < \rho < 1$, we have that $\alpha(k) = O(k^{-\gamma})$ for sufficient large γ, and hence assumption (A3) is satisfied. The technical condition (A4) is very easy to be satisfied. For example, when $\delta = 1$ and $\alpha(n) = O(n^{-4})$, choosing $r = [n^{1/2}]$, $q = [n^{1/6}]$, and $\lambda = 1/2$, we have $n\alpha(q)/r = O(n^{-1/6})$ and $r^{\lambda/2}/n^{(5\lambda+2\lambda\delta-\delta)/5(2+\delta)} = O(n^{-1/24})$.

Theorem 2. *Suppose that (A1)–(A4) hold and the bandwidths are such that $n\prod_{\alpha=1}^{p} h_\alpha \longrightarrow \infty$, and for each $\alpha = 1, \dots, p$, $h_\alpha = C_\alpha n^{-1/5}$, and $nh_{\alpha'}^5 \longrightarrow 0$, $\alpha' \ne \alpha$, $\alpha' = 1, \dots, p$, C_α is a constant; then*

$$\sqrt{nh_\alpha}\left\{\hat{a}_\alpha(x_\alpha) - a_\alpha(x_\alpha) - \frac{1}{2}\mu_2(K_\alpha)\, a_\alpha''(x)\, h_\alpha^2\right\} \xrightarrow{d} N\left(0, v_0(K_\alpha)\, v_\alpha^2(x_\alpha)\right), \tag{7}$$

where $\mu_\lambda(K_\alpha) = \int u^\lambda K_\alpha(u)du$, $v_\lambda(K_\alpha) = \int u^\lambda K_\alpha^2(u)du$, $v_\alpha^2(x_\alpha) = \int (p_{-\alpha}^2(\mathbf{x}_{-\alpha})/p(\mathbf{x}))E[\{\sum_{\alpha'=1}^{p} S^{\alpha\alpha'}(\mathbf{x})Z_{\alpha'}\}^2\sigma^2(\mathbf{X},\mathbf{Z}) \mid \mathbf{X} = \mathbf{x}]d\mathbf{x}_{-\alpha}$, $S(\mathbf{x}) = E(\mathbf{Z}\mathbf{Z}^\tau \mid \mathbf{X} = \mathbf{x}) = \{S_{\lambda_1,\lambda_2}(\mathbf{x})\}_{1\le\lambda_1,\lambda_2\le p}$, and its inverse $S^{-1}(\mathbf{x}) = \{S^{\lambda_1,\lambda_2}(\mathbf{x})\}_{1\le\lambda_1,\lambda_2\le p}$, $p_{-\alpha}(\mathbf{x}_{-\alpha})$ is the marginal density of $\mathbf{X}_{-\alpha} = (X_1, \dots, X_{\alpha-1}, X_{\alpha+1}, \dots, X_p)^\tau$. In particular, when $\sigma(\mathbf{X},\mathbf{Z})$ is a constant σ, $v_\alpha^2(x_\alpha) = \sigma^2\int\{p_{-\alpha}^2(\mathbf{x}_{-\alpha})/p(\mathbf{x})\}\cdot S^{\alpha\alpha}(\mathbf{x})d\mathbf{x}_{-\alpha}$.

Theorem 3. *Under the conditions of Theorem 2, we have*

$$\sqrt{nh_\alpha}\left\{\breve{a}_\alpha(x) - a_\alpha(x_\alpha) - \frac{1}{2}\mu_2(K_\alpha)\, a_\alpha''(x)\, h_\alpha^2\right\} \xrightarrow{d} N\left(0, v_0(K_\alpha)\, \breve{v}_\alpha^2(x_\alpha)\right), \tag{8}$$

where $\breve{v}_\alpha^2(x_\alpha) = \int(q_{-\alpha}^2(\mathbf{x}_{-\alpha})/p(x))$ $E[\{\sum_{\alpha'=1}^{p} S^{\alpha\alpha'}(\mathbf{x})\mathbf{Z}_{\alpha'}\}^2\sigma^2(\mathbf{X},\mathbf{Z}) \mid \mathbf{X} = \mathbf{x}]d\mathbf{x}_{-\alpha}$. In particular, when $\sigma(\mathbf{X},\mathbf{Z})$ is a constant σ, $\breve{v}_\alpha^2(x_\alpha) = \sigma^2\int\{q_{-\alpha}^2(\mathbf{x}_{-\alpha})/p(\mathbf{x})\}S^{\alpha\alpha}(\mathbf{x})d\mathbf{x}_{-\alpha}$.

3. Proof of the Main Results

In this section, we list some preliminary Lemmas. Let $\{\mathscr{Z}_i, i \ge 1\}$ be a stationary α-mixing sequence of random variables with mixing coefficients $\{\alpha(m)\}$. In the sequel, let C denote positive constant whose value may vary at each occurrence. In case of no loss of generalization and expositional purpose, we shall work with the special case $p = 2$.

Lemma 4 (see [13]). *Let $V_1, \dots, V_n$ be α-mixing random variables measurable with respect to the σ-algebra $\mathscr{F}_{i_1}^{j_1}, \dots, \mathscr{F}_{i_m}^{j_m}$, respectively, with $1_1 \le i_1 < j_1 < \cdots < j_m \le n$, $i_{l+1} - j_l \ge w \ge 1$ and $|V_j| \le 1$ for $l, j = 1, 2, \dots, m$. Then*

$$\left|E\left(\prod_{j=1}^{m} V_j\right) - \prod_{j=1}^{m} EV_j\right| \le 16(m-1)\alpha_w, \tag{9}$$

where $\mathscr{F}_a^b = \sigma\{V_i, a < i \le b\}$ denotes σ-field generated by $V_{a+1}, V_{a+2}, \dots, V_b$, α_n is the mixing coefficient.

Lemma 5 (see [14], Corollary A.2, p. 278). *Suppose that X and Y are random variables such that $E|X|^s < \infty$, $E|Y|^t < \infty$, where $s, t > 1$, $s^{-1} + t^{-1} < 1$. Then*

$$|EXY - EXEY| \le 8\|X\|_s\|Y\|_t \cdot \left\{\sup_{A\in\sigma(X), B\in\sigma(Y)} |P(AB) - P(A)P(B)|\right\}^{1-s^{-1}-t^{-1}}. \tag{10}$$

Lemma 6 (see [15], Theorem 4.1). *Let $2 < s < t \le \infty$, $2 < k \le t$ and $E\mathscr{Z}_n = 0$. Assume that $\alpha(n) \le Cn^{-\gamma}$ for some $C > 0$ and $\gamma > 0$. If $\gamma > k/(k-2)$ and $\gamma \ge (s-1)t/(t-s)$. then, for any $\epsilon > 0$, there exists $Q = Q(\epsilon, s, t, k, \gamma, C) < \infty$ such that*

$$E\left|\sum_{i=1}^{n}\mathscr{Z}_i\right|^s \le Q\left(n^{s/2}\max_{1\le i\le n}\|\mathscr{Z}_i\|_k^s + n^{1+\epsilon}\max_{1\le i\le n}\|\mathscr{Z}_i\|_t^s\right). \tag{11}$$

Lemma 7 (see [16], Lemma 2.3). *Assume $\alpha(k) \le C_1k^{-r}$, for some $r > 1$, $C_1 > 0$. Let $\sup_{1\le i,j\le n, i\ne j}|\operatorname{cov}(\mathscr{Z}_i, \mathscr{Z}_j)| \coloneqq R^*(n) < \infty$ be satisfied. Moreover, let $R_m(n) < \infty$ for some m, $2r/(r-1) < m \le \infty$, where $R_m(n) = \sup_{1\le i\le n}(E|\mathscr{Z}_i|^m)^{1/m}$, for $1 \le m < \infty$, and $R_\infty(n) = \sup_{1\le i\le n}\operatorname{ess\,sup}_{w\in\Omega}|\mathscr{Z}_i|$. Then*

$$\operatorname{var}\left(\sum_{i=1}^{n}\mathscr{Z}_i\right) \le n\left\{C_2(r,m)\left(R_m(n)\right)^{2m/(r(m-2))} \cdot \left(R^*(n)\right)^{1-m/(r(m-2))} + R_2^2(n)\right\} \tag{12}$$

holds with $C_2(r,m) \coloneqq ((20r - 40r/m)/(r - 1 - 2r/m))C_1^{1/r}$.

Lemma 8. *If assumptions (A1)–(A4) hold and $h_1 = Cn^{-1/5}$, $nh_2^5 \longrightarrow 0$, then*

$$S_n(\mathbf{x}) = \frac{1}{n}(U^\tau WU) = p(\mathbf{x})\, S_0(\mathbf{x})\left(1 + o_p(1)\right)$$
$$\text{and } S_n^{-1}(\mathbf{x}) = p^{-1}(\mathbf{x})\, S_0^{-1}(\mathbf{x})\left(1 + o_p(1)\right), \tag{13}$$

in the sense that each element converges, where

$$S_0(\mathbf{x}) = \begin{pmatrix} E\left(Z_1^2 \mid \mathbf{X}=\mathbf{x}\right) & E\left(Z_1 Z_2 \mid \mathbf{X}=\mathbf{x}\right) & 0 & 0 \\ E\left(Z_1 Z_2 \mid \mathbf{X}=\mathbf{x}\right) & E\left(Z_2^2 \mid \mathbf{X}=\mathbf{x}\right) & 0 & 0 \\ 0 & 0 & \mu_2(K_1) E\left(Z_1^2 \mid \mathbf{X}=\mathbf{x}\right) & 0 \\ 0 & 0 & 0 & \mu_2(K_2) E\left(Z_2^2 \mid \mathbf{X}=\mathbf{x}\right) \end{pmatrix}. \quad (14)$$

Proof. Let

$$S_{\lambda_1,\lambda_2,r_1,r_2}(\mathbf{x}) = \frac{1}{n}\sum_{l=1}^{n} Z_{l\lambda_1} Z_{l\lambda_2} \left(\frac{X_{l\lambda_1}-x_{\lambda_1}}{h_{\lambda_1}}\right)^{r_1} \cdot \left(\frac{X_{l\lambda_2}-x_{\lambda_2}}{h_{\lambda_2}}\right)^{r_2} K_{1,h_1}\left(X_{l\lambda_1}-x_{\lambda_1}\right) \cdot K_{2,h_2}\left(X_{l\lambda_2}-x_{\lambda_2}\right); \quad (15)$$

then $S_n(\mathbf{x})$ can be presented

$$S_n(\mathbf{x}) = \begin{pmatrix} S_{1,1,0,0}(\mathbf{x}) & S_{1,2,0,0}(\mathbf{x}) & S_{1,1,1,0}(\mathbf{x}) & S_{1,2,0,1}(\mathbf{x}) \\ S_{1,2,0,0}(\mathbf{x}) & S_{2,2,0,0}(\mathbf{x}) & S_{1,2,1,0}(\mathbf{x}) & S_{2,2,0,1}(\mathbf{x}) \\ S_{1,1,1,0}(\mathbf{x}) & S_{1,2,1,0}(\mathbf{x}) & S_{1,1,2,0}(\mathbf{x}) & S_{1,2,1,1}(\mathbf{x}) \\ S_{1,2,0,1}(\mathbf{x}) & S_{2,2,0,1}(\mathbf{x}) & S_{1,2,1,1}(\mathbf{x}) & S_{2,2,0,2}(\mathbf{x}) \end{pmatrix}. \quad (16)$$

Note that $(\mathbf{X}_i, \mathbf{Z}_i)$ is a strong stationary sequence of α-mixing random variables; then we have

$$E\left(S_{\lambda_1,\lambda_2,r_1,r_2}(\mathbf{x})\right) = E\left[E\left(Z_{\lambda_1} Z_{\lambda_2} \mid \mathbf{X}=\mathbf{x}\right)\left(\frac{X_{\lambda_1}-x_{\lambda_1}}{h_{\lambda_1}}\right)^{r_1}\left(\frac{X_{\lambda_2}-x_{\lambda_2}}{h_{\lambda_2}}\right)^{r_2} K_{1,h_1}\left(X_{\lambda_1}-x_{\lambda_1}\right) K_{2,h_2}\left(X_{\lambda_2}-x_{\lambda_2}\right)\right]$$
$$= \begin{cases} p(\mathbf{x}) E\left(Z_{\lambda_1} Z_{\lambda_2} \mid \mathbf{X}=\mathbf{x}\right)\{1+o(1)\} & r_1 = r_2 = 0 \\ 0 & r_1 = 1 \text{ or } r_2 = 1 \\ p(\mathbf{x}) E\left(Z_{\lambda_1} Z_{\lambda_2} \mid \mathbf{X}=\mathbf{x}\right)\mu_{r_1}\left(K_{\lambda_1}\right)\mu_{r_2}\left(K_{\lambda_2}\right)\{1+o(1)\} & r_1 = 0,\ r_2 = 2 \text{ or } r_1 = 2,\ r_2 = 0. \end{cases} \quad (17)$$

Set $M_l = Z_{l\lambda_1} Z_{l\lambda_2}((X_{l\lambda_1} - x_{\lambda_1})/h_{\lambda_1})^{r_1}((X_{l\lambda_2} - x_{\lambda_2})/h_{\lambda_2})^{r_2} K_1((X_{l\lambda_1} - x_{\lambda_1})/h_{\lambda_1}) K_2((X_{l\lambda_2} - x_{\lambda_2})/h_{\lambda_2})$. It is easy to see

$$\sup_{1\le l\le n}\left[E\left|M_l\right|^m\right]^{1/m} = O\left(h_{\lambda_1} h_{\lambda_2}\right), \quad \text{for } 1 \le m < \infty, \quad (18)$$
$$\text{and} \sup_{1\le i,j\le n, i\ne j}\left|\operatorname{cov}\left(M_i, M_j\right)\right| = O\left(h_{\lambda_1}^2 h_{\lambda_2}^2\right),$$

Which, combing with Lemma 7, (A4) and $h_1 = Cn^{-1/5}$, $nh_2^5 \longrightarrow 0$, yield that, for $m > 2$,

$$\operatorname{var}\left(S_{\lambda_1,\lambda_2,r_1,r_2}(\mathbf{x})\right) \le C\frac{n}{n^2 h_1^2 h_2^2}\left\{\left(h_{\lambda_1} h_{\lambda_2}\right)^{2m/r(m-2)} \cdot \left(h_{\lambda_1}^2 h_{\lambda_2}^2\right)^{1-m/r(m-2)} + \left(h_{\lambda_1} h_{\lambda_2}\right)^2\right\} = O\left(\frac{h_{\lambda_1}^2 h_{\lambda_2}^2}{nh_1^2 h_2^2}\right) = o(1). \quad (19)$$

Conjoining (17) and (19), we get the result of this lemma. □

Proof of Theorem 2. Note that $\tilde{a}_1(\mathbf{x})$ and $\tilde{a}_2(\mathbf{x})$ can be represented that

$$\tilde{a}_1(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} p^{-1}(\mathbf{x})\left(S^{11}(\mathbf{x}) Z_{i1} + S^{12}(\mathbf{x}) Z_{i2}\right) \cdot K_{1,h_1}\left(X_{i1}-x_1\right) K_{2,h_2}\left(X_{i2}-x_2\right) Y_i\left\{1+o_p(1)\right\},$$
$$\tilde{a}_2(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} p^{-1}(\mathbf{x})\left(S^{12}(\mathbf{x}) Z_{i1} + S^{22}(\mathbf{x}) Z_{i2}\right) \cdot K_{1,h_1}\left(X_{i1}-x_1\right) K_{2,h_2}\left(X_{i2}-x_2\right) Y_i\left\{1+o_p(1)\right\}. \quad (20)$$

Since

$$\hat{a}_1(x_1) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{p\left(x_1, X_{j2}\right)}\left(S^{11}\left(x_1, X_{j2}\right) Z_{i1} + S^{12}\left(x_1, X_{j2}\right) Z_{i2}\right) K_{1,h_1}\left(X_{i1}-x_1\right) K_{2,h_2}\left(X_{i2} - X_{j2}\right) \times Y_i\left\{1+o_p(1)\right\},$$
$$a_1(x_1) = \frac{1}{n}\sum_{j=1}^{n} a_1(x_1) = \frac{1}{n}\sum_{j=1}^{n} e_{1,4}^{\tau}\left(U^{\tau} W U\right)^{-1}$$

$$\cdot U^{\tau} W U \left(a_1(x_1), a_2(X_{j2}), h_1 a_1'(x_1), h_2 a_2'(X_{j2})\right)^{\tau}$$
$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{1,h_1}(X_{i1} - x_1) K_{2,h_2}(X_{i2}$$
$$- X_{j2}) \times w_i(x_1, X_{j2})\left\{1 + o_p(1)\right\}, \quad (21)$$

where $w_i(x_1, X_{j2}) = a_1(x_1)Z_{i1} + a_2(X_{j2})Z_{i2} + a_1'(x_1)(X_{i1} - x_1)Z_{i1} + a_2'(X_{j2})(X_{i2} - x_2)Z_{i2}$. Then we can obtain that

$$\widehat{a}_1(x_1) - a_1(x_1) = (A_{11} + A_{12} + A_2)\left\{1 + o_p(1)\right\}, \quad (22)$$

where

$$A_{11} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{1,h_1}(X_{i1} - x_1) K_{2,h_2}(X_{i2}$$
$$- X_{j2}) \times \left(a_1(X_{i1}) Z_{i1} - a_1(x_1) Z_{i1}\right.$$
$$\left. - a_1'(x_1)(X_{i1} - x_1) Z_{i1}\right),$$
$$A_{12} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{1,h_1}(X_{i1} - x_1) K_{2,h_2}(X_{i2}$$
$$- X_{j2}) \times \left(a_2(X_{i2}) Z_{i2} - a_2(X_{j2}) Z_{i2}\right.$$
$$\left. - a_2'(X_{j2})(X_{i2} - x_2) Z_{i2}\right),$$

$$A_2 = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{1,h_1}(X_{i1} - x_1) K_{2,h_2}(X_{i2}$$
$$- X_{j2}) \times \sigma(\mathbf{X}_i, \mathbf{Z}_i)\,\varepsilon_i. \quad (23)$$

Next, we establish that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j2})$$
$$= \frac{p_2(X_{i2})}{p(x_1, X_{i2})}\left(S^{11}(x_1, X_{i2}) Z_{i1}\right.$$
$$\left. + S^{12}(x_1, X_{i2}) Z_{i2}\right)\left\{1 + o_p(1)\right\}. \quad (24)$$

In order to prove (24), we need only to show

$$E\left\{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.\right.$$
$$\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j2}) - E\left[\frac{1}{n}\right.$$
$$\cdot \sum_{j=1}^{n}\frac{1}{p(x_1, X_{j2})}$$
$$\cdot \left(S^{11}(x_1, X_{j2}) Z_{i1} + S^{12}(x_1, X_{j2}) Z_{i2}\right)$$
$$\left.\left. \cdot K_{2,h_2}(X_{i2} - X_{j2}) \mid X_{i2}, Z_{i1}, Z_{i2}\right]\right\}^2 \longrightarrow 0. \quad (25)$$

In fact, according to (A1)(ii), (A3), and Lemma 5, we have

$$E\left\{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1} + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j2}) - E\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p(x_1, X_{j2})}\left(S^{11}(x_1, X_{j2}) Z_{i1}\right.\right.\right.$$
$$\left.\left.\left. + S^{12}(x_1, X_{j2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j2}) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\right\}^2$$
$$= \frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} E\left\{\left[\frac{1}{p(x_1, X_{j_1 2})}\left(S^{11}(x_1, X_{j_1 2}) Z_{i1} + S^{12}(x_1, X_{j_1 2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j_1 2})\right.\right.$$
$$\left. - E\left(\frac{1}{p(x_1, X_{j_1 2})}\left(S^{11}(x_1, X_{j_1 2}) Z_{i1} + S^{12}(x_1, X_{j_1 2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j_1 2}) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\right]$$
$$\cdot \left[\frac{1}{p(x_1, X_{j_2 2})}\left(S^{11}(x_1, X_{j_2 2}) Z_{i1} + S^{12}(x_1, X_{j_2 2}) Z_{i2}\right) K_{2,h_2}(X_{i2} - X_{j_2 2})\right.$$

$$
\begin{aligned}
&- E\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\Bigg]\Bigg\} \\
&= \frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} E\Bigg\{E\Bigg\{\Bigg[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right) \\
&- E\left(\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\Bigg] \\
&\cdot \Bigg[\frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right) \\
&- E\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\Bigg]\Bigg\} \mid X_{i2}, Z_{i1}, Z_{i2}\Bigg\} \\
&= \frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} E\Bigg\{E\Bigg[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right) \\
&\cdot \frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right)\Bigg] \\
&- E\left[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right] \\
&\cdot E\left[\frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right) \mid X_{i2}, Z_{i1}, Z_{i2}\right]\Bigg\} \\
&= \frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} E\Bigg\{E\Bigg[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right) \\
&\cdot \frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right)\Bigg] \cdot P\left(X_{j_1 2}X_{j_2 2} \mid X_{i2}, Z_{i1}, Z_{i2}\right) \\
&- E\left[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right)\right] \cdot E\Bigg[\frac{1}{p\left(x_1, X_{j_2 2}\right)}\Big(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} \\
&+ S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\Big)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right)\Bigg] \cdot P\left(X_{j_1 2} \mid X_{i2}, Z_{i1}, Z_{i2}\right) \cdot P\left(X_{j_2 2} \mid X_{i2}, Z_{i1}, Z_{i2}\right)\Bigg\} \\
&= \frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} E\left[P\left(X_{j_1 2}X_{j_2 2} \mid X_{i2}, Z_{i1}, Z_{i2}\right)\right] \cdot \Bigg\{E\Bigg[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\Big(X_{i2} \\
&- X_{j_1 2}\Big) \cdot \frac{1}{p\left(x_1, X_{j_2 2}\right)}\left(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right)\Bigg] \\
&- E\left[\frac{1}{p\left(x_1, X_{j_1 2}\right)}\left(S^{11}\left(x_1, X_{j_1 2}\right)Z_{i1} + S^{12}\left(x_1, X_{j_1 2}\right)Z_{i2}\right)K_{2,h_2}\left(X_{i2} - X_{j_1 2}\right)\right] \cdot E\Bigg[\frac{1}{p\left(x_1, X_{j_2 2}\right)}\Big(S^{11}\left(x_1, X_{j_2 2}\right)Z_{i1} \\
&+ S^{12}\left(x_1, X_{j_2 2}\right)Z_{i2}\Big)K_{2,h_2}\left(X_{i2} - X_{j_2 2}\right)\Bigg]\Bigg\} \le Cn^{-2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\alpha^{\delta/(2+\delta)}\left(j_2 - j_1\right) \le C\frac{1}{n^2}\sum_{j_1=1}^{n}\sum_{k=1}^{\infty}\alpha^{\delta/(2+\delta)}(k) \le Cn^{-1} \longrightarrow 0.
\end{aligned}
\tag{26}
$$

Thus, (24) holds.

Then similar to the proof of Theorem in Zhang and Li [9], we have

$$A_{11} = \frac{1}{2}\mu_2(K_1)a_1''(x_1)h_1^2\{1+o_p(1)\}$$
$$\text{and } A_{12} = O_p(h_2^2). \tag{27}$$

Hence, in order to prove Theorem 2, we need only to show that

$$\sqrt{nh_1}A_2 \xrightarrow{d} N(0, v_0(K_1)v_1^2(x_1)). \tag{28}$$

Denote $D_i = \sqrt{h_1}(p_2(X_{i2})/p(x_1,X_{i2}))(S^{11}(x_1,X_{i2})Z_{i1} + S^{12}(x_1,X_{i2})Z_{i2})K_{1,h_1}(X_{i1}-x_1)\sigma(\mathbf{X}_i,\mathbf{Z}_i)\varepsilon_i$. Then A_2 can be represented as

$$A_2 = \frac{1}{n\sqrt{h_1}}\sum_{i=1}^{n}D_i\{1+o_p(1)\}. \tag{29}$$

In order to prove (28), we need only to show

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}D_i \xrightarrow{d} N(0, v_0(K_1)v_1^2(x_1)). \tag{30}$$

Let $\eta_{mn}, \eta'_{mn}, \eta'_{wn}$ be defined as follows:

$$\eta_{mn} = \sum_{i=k_m}^{k_m+r-1}D_i,$$
$$\eta'_{mn} = \sum_{j=l_m}^{l_m+q-1}D_j, \tag{31}$$
$$\eta''_{wn} = \sum_{k=w(r+q)+1}^{n}D_k,$$

where $k_m=(m-1)(r+q)+1$, $l_m=(m-1)(r+q)+r+1$, $m=1,\ldots,w$. Then

$$n^{-1/2}\sum_{i=1}^{n}D_i = n^{-1/2}\left\{\sum_{m=1}^{w}\eta_{mn} + \sum_{m=1}^{w}\eta'_{mn} + \eta''_{wn}\right\}$$
$$:= n^{-1/2}\{S'_n + S''_n + S'''_n\}. \tag{32}$$

Hence, in order to prove (30), it suffices to show that

$$\frac{1}{n}E(S''_n)^2 \longrightarrow 0,$$
$$\frac{1}{n}E(S'''_n)^2 \longrightarrow 0, \tag{33}$$

$$\operatorname{var}(n^{-1/2}S'_n) \longrightarrow v_0(K_1)v_1^2(x_1), \tag{34}$$

$$\left|E\exp\left(it\sum_{m=1}^{w}\eta_{mn}\right) - \prod_{m=1}^{w}E\exp(it\eta_{mn})\right| \longrightarrow 0, \tag{35}$$

$$\frac{1}{n}\sum_{m=1}^{w}E\eta_{mn}^2 I(|\eta_{mn}| > \epsilon\sqrt{n}) \longrightarrow 0, \quad \forall\epsilon > 0. \tag{36}$$

Relation (33) implies that S''_n and S'''_n are asymptotically negligible, (35) shows that the summands η_{mn} in S'_n are asymptotically independent, and (34) and (36) are the standard Lindeberg-Feller conditions for asymptotic normality of S'_n under independence.

We first establish (33). Obviously

$$E(S''_n)^2 = \sum_{m=1}^{w}\sum_{i=l_m}^{l_m+q-1}ED_i^2$$
$$+ 2\sum_{m=1}^{w}\sum_{l_m\le i<j\le l_m+q-1}\operatorname{cov}(D_i,D_j) \tag{37}$$
$$+ 2\sum_{1\le i<j\le w}\operatorname{cov}(\eta'_{in},\eta'_{jn}) := J_{1n}+J_{2n}+J_{3n}.$$

Note that

$$ED_i^2 = h_1E\left[\frac{p_2^2(X_2)}{p^2(x_1,X_2)}\left(S^{11}(x_1,X_2)Z_1\right.\right.$$
$$\left.\left.+ S^{12}(x_1,X_2)Z_2\right)^2K_{1,h_1}^2(X_1-x_1)\sigma^2(\mathbf{X},\mathbf{Z})\right]$$
$$= h_1E\left[\frac{p_2^2(X_2)}{p^2(x_1,X_2)}E\left\{\left(S^{11}(x_1,X_2)Z_1\right.\right.\right.$$
$$\left.\left.+ S^{12}(x_1,X_2)Z_2\right)^2\sigma^2(\mathbf{X},\mathbf{Z}) \mid \mathbf{X}=\mathbf{x}\right\}K_{1,h_1}^2(X_1 \tag{38}$$
$$\left.-x_1)\right]$$
$$= v_0(K_1)\int\frac{p_2^2(X_2)}{p(x_1,x_2)}E\left\{\left(S^{11}(x_1,x_2)Z_1\right.\right.$$
$$\left.\left.+ S^{12}(x_1,x_2)Z_2\right)^2\sigma^2(\mathbf{X},\mathbf{Z}) \mid \mathbf{X}=\mathbf{x}\right\}dx_2\{1$$
$$+ o(1)\} = v_0(K_1)v_1^2(x_1)\{1+o(1)\},$$

which yields that $J_{1n}=O(wq/n)=o(1)$ from (A4). Since

$$|J_{2n}| \le \frac{2}{n}\sum_{1\le i<j\le n}|\operatorname{cov}(D_i,D_j)|$$
$$\text{and } |J_{3n}| \le \frac{2}{n}\sum_{1\le i<j\le n}|\operatorname{cov}(D_i,D_j)|, \tag{39}$$

to prove $|J_{2n}|=o(1)$ and $|J_{3n}|=o(1)$, it suffices to show that

$$\frac{1}{n}\sum_{1\le i<j\le n}|\operatorname{cov}(D_i,D_j)| \longrightarrow 0. \tag{40}$$

Next, let c_n (specified below) be a sequence of integers such that $c_n \longrightarrow \infty$ and $c_nh_1 \longrightarrow 0$. Put

$$S_1 = \{(i,j) \mid i,j\in\{1,2,\ldots,n\},\ 1\le j-i\le c_n\}$$
$$S_2 \tag{41}$$
$$= \{(i,j) \mid i,j\in\{1,2,\ldots,n\},\ c_n+1\le j-i\le n-1\}.$$

We write

$$\frac{1}{n}\sum_{1\le i<j\le n}\left|\operatorname{cov}\left(D_i,D_j\right)\right| = \frac{1}{n}\sum_{S_1}\left|\operatorname{cov}\left(D_i,D_j\right)\right| + \frac{1}{n}\sum_{S_2}\left|\operatorname{cov}\left(D_i,D_j\right)\right|. \quad (42)$$

It is easy to see that

$$\left|\operatorname{cov}\left(D_i,D_j\right)\right| \le E\left|D_iD_j\right| + E\left|D_i\right|E\left|D_j\right| = O\left(h_1\right). \quad (43)$$

Hence,

$$\frac{1}{n}\sum_{S_1}\left|\operatorname{cov}\left(D_i,D_j\right)\right| = O\left(c_nh_1\right) \longrightarrow 0. \quad (44)$$

On the other hand, it follows from Lemma 5 that

$$\left|\operatorname{cov}\left(D_i,D_j\right)\right| \le C\left[\alpha\left(j-i\right)\right]^{1-2/(2+\delta)}\left(E\left|D_i\right|^{2+\delta}\right)^{2/(2+\delta)} \quad (45)$$

and

$$E\left|D_i\right|^{2+\delta} = h_1^{1+\delta/2}E\left|\frac{p_2^{2+\delta}\left(X_2\right)}{p^{2+\delta}\left(x_1,X_2\right)}\left(S^{11}\left(x_1,X_2\right)Z_1 + S^{12}\left(x_1,X_2\right)Z_2\right)^{2+\delta}K_{1,h_1}^{2+\delta}\left(X_1-x_1\right)\sigma^{2+\delta}\left(\mathbf{X},\mathbf{Z}\right)\right| = O\left(h_1^{-\delta/2}\right). \quad (46)$$

Then we have

$$\begin{aligned}\frac{1}{n}\sum_{S_2}\left|\operatorname{cov}\left(D_i,D_j\right)\right| &\le \frac{C}{n}\sum_{j=1}^{n}\sum_{j-i=c_n+1}^{n-1}\left[\alpha\left(j-i\right)\right]^{1-2/(2+\delta)}h_1^{-\delta/(2+\delta)} \\ &\le Ch_1^{-\delta/(2+\delta)}\sum_{l=c_n}^{\infty}\alpha\left(l\right)^{\delta/(2+\delta)} \\ &\le Cc_n^{-\delta_1}h_1^{-\delta/(2+\delta)}\sum_{l=c_n}^{\infty}l^{\delta_1}\alpha\left(l\right)^{\delta/(2+\delta)}.\end{aligned} \quad (47)$$

Therefore, by choosing $c_n = h_1^{-\delta/((2+\delta)\delta_1)}$ and in view of (A4), we obtain that

$$\frac{1}{n}\sum_{S_1}\left|\operatorname{cov}\left(D_i,D_j\right)\right| = O\left(c_nh_1\right) \longrightarrow 0. \quad (48)$$

Thus, (40) is verified from (42), (44), (46), and (48).

As to $n^{-1}E(S_n''')^2$, by (A4), (38), (40), and Lemma 6, we have

$$\begin{aligned}\frac{1}{n}E\left(S_n'''\right)^2 &\le \frac{1}{n}\sum_{i=w(r+q)+1}^{n}E\left(D_i\right)^2 + \frac{2}{n}\sum_{w(r+q)+1\le i<j\le n}\left|\operatorname{cov}\left(D_i,D_j\right)\right| \\ &\le C\frac{n-w\left(p+q\right)}{n} + \frac{2}{n}\sum_{1\le i<j\le n}\left|\operatorname{cov}\left(D_i,D_j\right)\right| \longrightarrow 0.\end{aligned} \quad (49)$$

Next we establish (34). Obviously

$$\operatorname{var}\left(n^{-1/2}S_n'\right) = \frac{1}{n}\sum_{m=1}^{w}\operatorname{var}\left(\eta_{mn}\right) + \frac{2}{n}\sum_{1\le i<j\le w}\operatorname{cov}\left(\eta_{in},\eta_{jn}\right). \quad (50)$$

From (A4), (38), and Lemma 6, we have

$$\frac{1}{n}\sum_{1\le i<j\le w}\left|\operatorname{cov}\left(\eta_{in},\eta_{jn}\right)\right| \le \frac{1}{n}\sum_{1\le i<j\le n}\left|\operatorname{cov}\left(D_i,D_j\right)\right| \longrightarrow 0. \quad (51)$$

On the other hand,

$$\begin{aligned}\sum_{m=1}^{w}\operatorname{var}\left(\eta_{mn}\right) = \sum_{i=1}^{n}\operatorname{var}D_i &+ 2\sum_{m=1}^{w}\sum_{k_m\le i<j\le k_m+r-1}\operatorname{cov}\left(D_i,D_j\right) \\ &- \sum_{m=1}^{w}\sum_{i=l_m}^{l_m+q-1}\operatorname{var}D_i - \sum_{i=w(r+q)+1}^{n}\operatorname{var}D_i.\end{aligned} \quad (52)$$

It is easy to get that

$$\begin{aligned}&\frac{1}{n}\sum_{m=1}^{w}\sum_{k_m\le i<j\le k_m+r-1}\left|\operatorname{cov}\left(D_i,D_j\right)\right| \le \frac{1}{n}\sum_{1\le i<j\le n}\left|\operatorname{cov}\left(D_i,D_j\right)\right| \longrightarrow 0, \\ &\frac{1}{n}\sum_{m=1}^{w}\sum_{i=l_m}^{l_m+q-1}\operatorname{var}D_i \le Cn^{-1}wq \le \frac{Cq}{p} \longrightarrow 0, \\ &\frac{1}{n}\sum_{i=w(p+q)+1}^{n}\operatorname{var}Z_i \le Cn^{-1}\left(n-w\left(p+q\right)\right) \le \frac{Cp}{n} \longrightarrow 0,\end{aligned} \quad (53)$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\operatorname{var} Z_i \longrightarrow \nu_0(K_1)\,\nu_1^2(x_1). \tag{54}$$

Thus, (34) is verified.

As to (35), according to Lemma 4, we have

$$\begin{aligned}&\left|E\exp\left(it\sum_{m=1}^{w}\eta_{mn}\right)-\prod_{m=1}^{w}E\exp\left(it\eta_{mn}\right)\right|\\&\quad\le 16w\alpha(q+1)\le\frac{cn\alpha(q)}{r},\end{aligned} \tag{55}$$

which tends to zero by (A4).

As to (36), by (A4), (46), and Lemma 6, for any $\epsilon_1 > 0$ and $0 < 2\epsilon_1 \le \lambda$, we have

$$\begin{aligned}&\frac{1}{n}\sum_{m=1}^{w}E\eta_{mn}^2 I\left(\left|\eta_{mn}\right|>\epsilon\right)\le Cn^{-1-\lambda/2}\sum_{m=1}^{w}E\left|\sum_{i=k_m}^{k_m+r-1}D_i\right|^{2+\lambda}\\&\quad\le Cn^{-1-\lambda/2}\sum_{m=1}^{w}\left\{r^{1+\lambda/2}\max_{1\le i\le n}\left(E\left|D_i\right|^{2+\delta}\right)^{(2+\lambda)/(2+\lambda)}\right.\\&\quad\left.+n^{1+\epsilon_1}\max_{1\le i\le n}\left(E\left|D_i\right|^{2+\delta}\right)^{(2+\lambda)/(2+\lambda)}\right\}\\&\quad\le Cn^{-\lambda/2}r^{\lambda/2}h_1^{-\delta(2+\lambda)/2(2+\delta)}\\&\quad\le\ Cr^{\lambda/2}n^{-(5\lambda+2\lambda\delta-\delta)/5(2+\delta)}\longrightarrow 0\,.\end{aligned} \tag{56}$$

Hence, (36) is shown. This completes the proof. □

Proof of Theorem 3. Note that

$$\begin{aligned}\breve{a}_1(x_1)&=\frac{1}{n}\sum_{i=1}^{n}\int p^{-1}(x_1,x_2)\\&\quad\cdot\left(S^{11}(x_1,x_2)Z_{i1}+S^{12}(x_1,x_2)Z_{i2}\right)\\&\quad\times K_{1,h_1}(X_{i1}-x_1)K_{2,h_2}(X_{i2}-x_2)\,dx_2Y_i\left\{1\right.\\&\quad\left.+o_p(1)\right\},\\a_1(x_1)&=\int a_1(x_1)q_2(x_2)\,dx_2=\int e_{1,4}^{\tau}\left(U^{\tau}WU\right)^{-1}\\&\quad\cdot U^{\tau}WU\left(a_1(x_1),a_2(x_2),h_1a_1(x_1),h_2a_2(x_2)\right)^{\tau}\\&\quad\cdot q_2(x_2)\,dx_2=\frac{1}{n}\sum_{i=1}^{n}\int p^{-1}(x_1,x_2)\\&\quad\cdot\left(S^{11}(x_1,x_2)Z_{i1}+S^{12}(x_1,x_2)Z_{i2}\right)\\&\quad\times K_{1,h_1}(X_{i1}-x_1)K_{2,h_2}(X_{i2}-x_2)\,w_i(x_1,x_2)\\&\quad\cdot q_2(x_2)\,dx_2\left\{1+o_p(1)\right\},\end{aligned} \tag{57}$$

where $w_i(x_1,x_2) = a_1(x_1)Z_{i1} + a_2(x_2)Z_{i2} + a_1'(x_1)(X_{i1} - x_1)Z_{i1} + a_2'(x_2)(X_{i2} - x_2)Z_{i2}$. Then following the method of the proof of Theorem 2.1 and Theorem 2.1 in Zhang and Li [8], we can obtain this Theorem. □

4. Conclusion Remarks

In regression analysis, α-mixing has received a lot of attentions such as Shen and Xie [17], Fan et al. [11], Chesneau et al. [18], and Guo and Liu [19]. This is because α-mixing is reasonably weak among various mixing conditions and has many practical applications. We, in this paper, focus on the varying-coefficient regression models with different smoothing variables and extend the results of Zhang and Li [8] and Zhang and Li [9] from i.i.d. assumptions to α-mixing setting. We employed the local linear technique to obtain an initial value for the estimate of each coefficient function in model (1), and, then, both averaged and integrated estimates of each coefficient function are defined by averaging and integrating its initial value on these variables which the coefficient function does not share. Under certain regularity conditions, the asymptotic properties of the proposed two estimators are established. There are some other interesting issues to be further studied; for example, the data come from survival data, such as censored data, missing data, or left truncated data. These extensions greatly expand the scope of the applicability of our methods.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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