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Research Article

Bifurcations of a New Fractional-Order System with a One-Scroll Chaotic Attractor

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In this paper, a new fractional-order system which has a chaotic attractor of the one-scroll structure is presented. Firstly, the stability of the equilibrium points of the system is investigated. And based on the stability analysis, the generation conditions of the one-scroll structure for the attractor are determined. In a commensurate-order case, bifurcations with the variation of a system parameter are investigated as derivative orders decrease from 0.99. In an incommensurate-order case, bifurcations with the variation of a derivative order are analyzed as other orders decrease from 1.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Although the seeds of fractional derivatives were planted over 300 years ago, the development of fractional calculus is very slow at an early stage for the absence of geometrical interpretation and applications. In the recent several decades, it has been applied to almost every field of science, engineering, economics, secures communication, and so on [1–5].

It is well known that fractional calculus is very suitable for the description of properties of various real materials. Meanwhile, fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials [6].

Many new fractional-order systems were presented in recent years. Meanwhile, rich and complex dynamics, such as periodic solutions, chaos windows, all kinds of bifurcations, boundary, and interior crises were observed in these systems. For example, Chua system with a derivative order 2.7 makes chaos motion [7]. Chaotic dynamics of a damped van der Pol equation with fractional order is investigated in [8]. Chen studied the nonlinear dynamics and chaos in a fractional-order financial system [9]. A periodically forced complex Duffing's oscillator was proposed, and chaos for the system was studied in detail [10]. Bifurcation, chaos control, and synchronization were investigated for a fractional-order Lorenz system with the complex variables [11]. In [12], boundary and interior crises were determined in a fractional-order Duffing system by a global numerical computation method. A proposed standard for the publication of new chaotic systems was studied in [13]. The authors investigated a simple chaotic flow with a plane of equilibria in [14].

It is well known that bifurcation theory concerns the changes in qualitative or topological structures of limiting motions such as equilibriums, periodic solutions, homoclinic orbits, heteroclinic orbits, and invariant tori for nonlinear evolution equations as some relevant parameters in the equations vary. Generally, the subject can be traced back to the very earlier work of Poincare around 1892 [15]. Nowadays, it is a fundamental tool to analyze nonlinear problems which enables us to understand how and when a system organizes new states and patterns near the original "trivial" one when a control parameter crosses a critical value. For fractional-order systems, a bifurcation implies a qualitative or topological change in dynamics with a variation of a system parameter or derivative order, and bifurcation analysis becomes harder due to the nonlocal property of the operator of fractional calculus. Many references have studied the bifurcations of fractional-order systems [16–20]. However, these investigations mainly focus on the bifurcation of a fractional-order system as a system parameter or a derivative order varies. To our knowledge, few works concerns bifurcations with the variation of both a system parameter and a derivative order or with the variation of both a derivative order and other orders.

Compared with integer-order chaotic systems, fractional-order chaotic systems with more complex dynamic characteristics and more system parameters can provide higher security for secure communication [21, 22]. In [23], the authors investigated the synchronization of a three-dimensional integer-order system. The differential equations of the system with simple structure were similar to those of the Lorenz system. It is well known that the Lorenz system has a chaotic attractor with double-scroll structure. The system in [23] has a chaotic attractor with only one-scroll and very abundant dynamic behaviors. Motivated by the above, in this paper, a corresponding fractional-order system is proposed and studied. Firstly, the stability of equilibrium points of the system is investigated. In a commensurate-order case, bifurcations with the variation of a system parameter are investigated as derivative orders decrease from 0.99. In an incommensurate-order case, bifurcations with the variation of a derivative order are analyzed as the other orders decrease from 1. Period-doubling and saddle-node bifurcations can be observed from the bifurcation diagrams by numerical simulations.

The remainder of the paper is organized as follows. In Section 2, the definitions of the fractional calculus and related preliminaries are given. A new fractional-order system with one-scroll attractor is presented in Section 3. In Section 4, bifurcations in the two cases of commensurate-order and incommensurate-order are analyzed, respectively. Conclusions of the paper are drawn in Section 5.

2. Fractional Derivatives and Preliminaries

2.1. Definitions. Fractional calculus can be considered as a generalization of integration and differentiation. The operator ${}_aD_t^q$ of fractional calculus can be defined by

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q} & R(q) > 0 \\ 1 & R(q) = 0 \\ \displaystyle\int_a^t (d\tau)^q & R(q) < 0, \end{cases} \tag{1}$$

where q denotes the derivative order and $R(q)$ corresponds to the real part of q. The numbers a and t represent the limits of the operator.

In general, three definitions of fractional derivative are used frequently, namely, the Grunwald-Letnikov definition, the Riemann-Liouville, and the Caputo definitions [6, 24].

The Grunwald-Letnikov definition (GL) derivative with fractional order q can be described by

$$ {}_a^{GL}D_t^q = \lim_{h\longrightarrow 0} \frac{1}{h^q} \sum_{j=0}^{[(t-a)/h]} (-1)^q \binom{q}{j} f(t - jh), \tag{2}$$

where the symbol $[\cdot]$ represents the integer part.

The Riemann-Liouville (RL) definition is

$$ {}_a^{RL}D_t^q f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-q)} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau, \qquad n-1 < q < n, \tag{3}$$

where $\Gamma(\cdot)$ denotes the gamma function.

The Caputo (C) fractional derivative is defined as follows:

$$ {}_a^{C}D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau)\, d\tau, \qquad n-1 < q < n. \tag{4}$$

For a fractional differential equation which is defined by Caputo derivatives, the initial condition takes on the same form as those for the integer-order ones, which can be measured easily in applications. For this reason, the Caputo derivative will be adopted in the rest of the paper.

2.2. Numerical Methods. Due to the nonlocal property of the operator of fractional calculus, it is not easy to obtain the numerical solutions for a fractional differential equation. Generally speaking, two approximation methods are frequently used, namely, an improved version of Adams-Bashforth-Moulton algorithm based on the predictor-correctors scheme and the frequency domain approximation [25–28]. For the accuracy [29], we will employ the improved predictor-corrector algorithm to solve a fractional differential equation in this paper.

In order to get the approximate solution of a fractional-order chaotic system by the improved predictor-corrector algorithm, the following equation is considered:

$$ \begin{gathered} \frac{d^q x}{dt^q} = f(t, x), \quad 0 \le t \le T \\ x^k(0) = x_0^{(k)} \quad k = 0, 1, \cdots, \lceil q \rceil - 1, \end{gathered} \tag{5}$$

which is equivalent to the Volterra integral equation

$$ \begin{aligned} x(t) = {} & \sum_{k=0}^{\lceil q \rceil - 1} x_0^{(k)} \frac{t^k}{k!} \\ & + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{(q-1)} f(\tau, x(t))\, d\tau. \end{aligned} \tag{6}$$

Now, set $h = T/N$, $t_j = jh$, $(j = 0, 1, \cdots, N)$. The corrector formula for (6) can thus be discretized as follows:

$$x_h(t_{n+1}) = \sum_{k=0}^{\lceil q \rceil - 1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, x_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,n+1} f(t_j, x_h(t_j)), \quad (7)$$

where predicted values $x_h(t_{n+1})$ are determined by the following formula:

$$x_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q \rceil - 1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,n+1} f(t_j, x_h(t_j)), \quad (8)$$

and

$$\alpha_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1. \end{cases} \quad (9)$$

$$\beta_{j,n+1} = \frac{h^q}{q}\left((n-j+1)^q - (n-j)^q\right), \quad 1 \le j \le n$$

The error estimate of this approach is $\max_{j=0,1,\cdots,N} |x(t_j) - x_h(t_j)| = O(h^P)$, where $p = \min(2, 1+q)$.

2.3. The Stability of a Fractional-Order System. For fractional-order systems, the stability analysis of equilibrium points is complex and difficult due to the nonlocal property of fractional calculus. Here, the definitions of commensurate-order and incommensurate-order fractional-order systems will be given firstly.

Definition 1. For a fractional-order system, which can be described by $d^{\mathbf{q}}\mathbf{x}/dt^{\mathbf{q}} = f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ is the state vector, $\mathbf{q} = (q_1, q_2, \cdots, q_n)^{\mathrm{T}}$ is the fractional derivative orders vector, and $q_i > 0$. The fractional-order system is commensurate-order when all the derivative orders satisfy $q_1 = q_2 = \cdots = q_n$; otherwise it is an incommensurate-order system [30].

In order to investigate the stability of equilibrium points for fractional-order systems, the following lemma is used frequently.

Lemma 2. *For a commensurate fractional-order system, the equilibrium points of the system are asymptotically stable if all the eigenvalues at the equilibrium* E^* *satisfy the following condition:*

$$|\arg(eig(\mathbf{J}))| = |\arg(\lambda_j)| > \frac{\pi}{2}q, \quad j = 1, 2, \cdots, n, \quad (10)$$

where $\mathbf{J}$ *is the Jacobian matrix of the system evaluated at the equilibria* E^* *[31].*

3. A New Fractional-Order System

In this section, a new fractional-order system which consist of three differential equations is proposed and can be denoted as follows:

$$\begin{aligned} D^{q_1}x &= ax - y \\ D^{q_2}y &= x - z - by \\ D^{q_3}z &= ax + 4z(y-2) \end{aligned} \quad (11)$$

where x, y, and z are state variables of the system, a and b the system parameters, and q_1, q_2, and q_3 derivative orders.

When the derivative orders are selected as $q_1 = q_2 = q_3 = q = 0.99$, the system parameters $a = 0.3$ and $b = 0.02$, and initial conditions $(x_0, y_0, z_0) = (-0.5, -1, 1)$, system (11) is chaotic. In Figures 1(a) and 1(b), a chaotic attractor with a one-scroll on three-dimensional space and projected onto $x - y$ plane are depicted. In this case, the corresponding Lyapunov exponents are $\lambda_1 = 0.117$, $\lambda_2 = 0$, and $\lambda_3 = -10.49$. When the derivative order $q = 0.985$ and $q = 0.965$, the corresponding attractors are displayed in Figures 1(c)–1(f).

The equilibrium points of system (11) can be calculated by solving the equations $D^{q_1}x = 0$, $D^{q_2}y = 0$, and $D^{q_3}z = 0$. The system contains two equilibriums, i.e.,

$$\begin{aligned} &\mathrm{E}_1\,(0,0,0) \\ &\mathrm{E}_2\left(\frac{2}{a} - \frac{1}{4(1-ab)}, a\left(\frac{2}{a} - \frac{1}{4(1-ab)}\right), (1-ab) \cdot \left(\frac{2}{a} - \frac{1}{4(1-ab)}\right)\right). \end{aligned} \quad (12)$$

The equilibrium E_2 exists when the system parameters satisfy the condition $1 - ab \neq 0$ or $1 - ab \neq a$.

The Jacobin matrix for system (11) evaluated at the equilibrium point (x^*, y^*, z^*) is given by

$$J(x^*, y^*, z^*) = \begin{pmatrix} a & -1 & 0 \\ 1 & -b & -1 \\ a & 4z^* & 4(y^*-2) \end{pmatrix}. \quad (13)$$

Based on the matrix, the characteristic equation at the equilibriums E_1 is

$$\lambda^3 + (8+b-a)\lambda^2 + [8(b-a) + 1 - ab]\lambda + 8(1-ab) - a = 0. \quad (14)$$

By the Routh-Hurwitz test, the equilibrium point E_1 has three roots with the negative real parts if and only if $8 + b - a > 0$ and $0 < 8(1-ab) - a < (8+b-a)[8(b-a)+1-ab]$. In our case, when the values of the system parameters are taken as $a = 0.3$, $b = 0.02$, the corresponding eigenvalues for the equilibrium point E_1 are $\lambda_1 = -8.0047$ and $\lambda_{2,3} = -0.1576 \pm 1.0089i$. According to Lemma 2, the equilibrium point E_1 is locally stable.

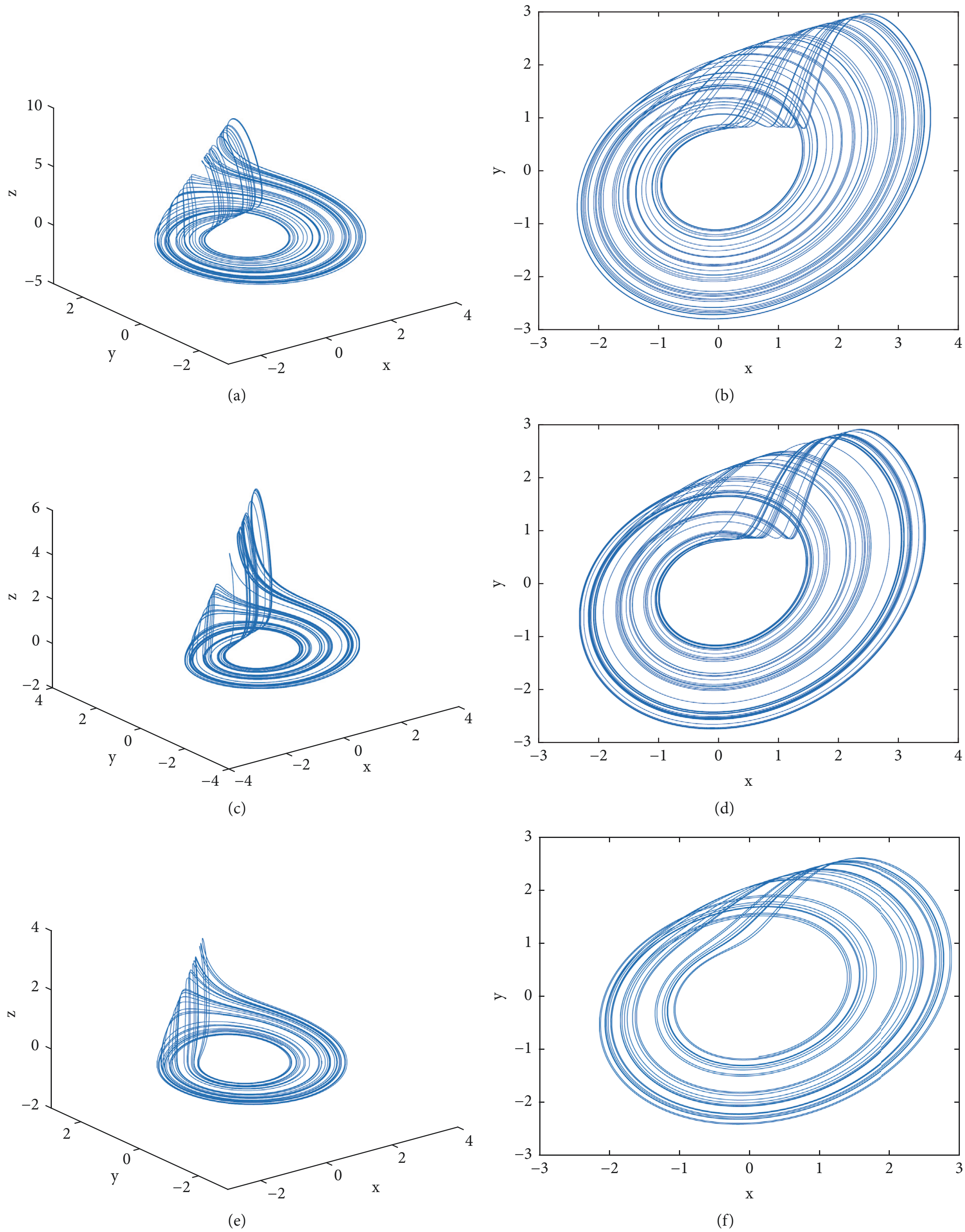


Figure 1: The chaotic attractor of system (11). (a) 3D plot of the attractor when $q = 0.99$; (b) the attractor projected onto $x - y$ plane when $q = 0.99$; (c) 3D plot of the attractor when $q = 0.985$; (d) the attractor projected onto $x - y$ plane when $q = 0.985$; (e) 3D plot of the attractor when $q = 0.965$; (f) the attractor projected onto $x - y$ plane when $q = 0.965$.

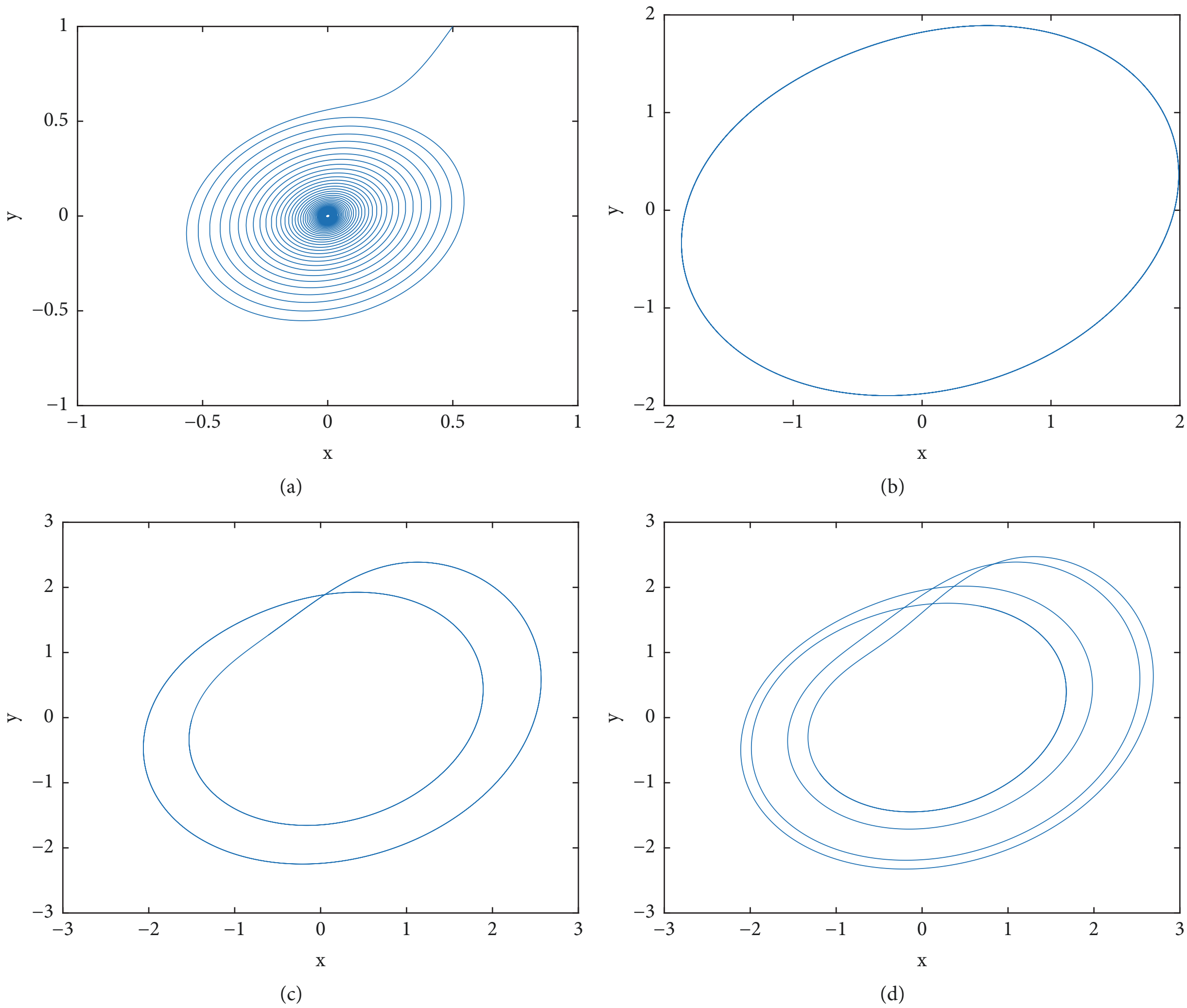


Figure 2: The attractors for different values of the derivative order q. (a) $q = 0.90$; (b) $q = 0.92$; (c) $q = 0.95$; (d) $q = 0.955$.

The characteristic equation at equilibrium E_2 is

$$\begin{aligned}&\lambda^3 + [(b-a) - 4(y-2)]\lambda^2 \\ &\quad - [4(y-2)(b-a) + ab - (4z+1)]\lambda \\ &\quad + 4(y-2)(ab-1) - a(1+4z) = 0.\end{aligned} \tag{15}$$

By the Routh-Hurwitz test, the equilibrium point E_2 has three roots with the negative real parts if and only if $(b-a) - 4(y-2) > 0$ and

$$\begin{aligned}0 &< 4(y-2)(ab-1) - a(1+4z) \\ &< [(b-a) - 4(y-2)] \\ &\quad \cdot [4(y-2)(b-a) + ab - (4z+1)].\end{aligned} \tag{16}$$

The corresponding eigenvalues for the equilibrium point E_2 are $\lambda_1 = -7.4038$ and $\lambda_{2,3} = 0.1029 \pm 0.9653i$ when the system parameters take the same values as before. Then based on Lemma 2, we can get that the fixed point E_2 is unstable.

In a nonlinear dynamical system, a saddle point is an equilibrium point on which the equivalent linearized model has at least one eigenvalue in the stable region and one eigenvalue in the unstable region. In the same system, a saddle point is called saddle point of index 1 if one of the eigenvalues is unstable and other eigenvalues are stable. Also, a saddle point of index 2 is a saddle point with one stable eigenvalue and two unstable eigenvalues. In chaotic systems, it is proved that scrolls are generated only around the saddle points of index 2. Moreover, saddle points of index 1 are responsible only for connecting scrolls [32, 33].

From the above analysis we can see that the equilibrium point E_2 is a saddle point of index 2. In chaotic systems, it is proved that the scrolls of a chaotic attractor are generated only around the saddle points of index 2. Moreover, the saddle points of index 1 are responsible for connecting the scrolls. In the fractional-order system (11), the equilibrium E_1 is not a saddle point of index 1. The necessary condition for the existence of double-scroll attractor in system (11) cannot be satisfied. Therefore, the chaotic attractor of system (11) is one scroll.

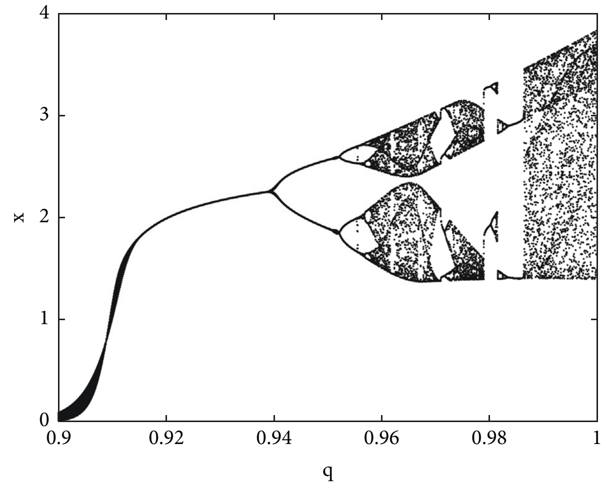


Figure 3: The bifurcation diagram of system (11) with the variation of the derivative order q.

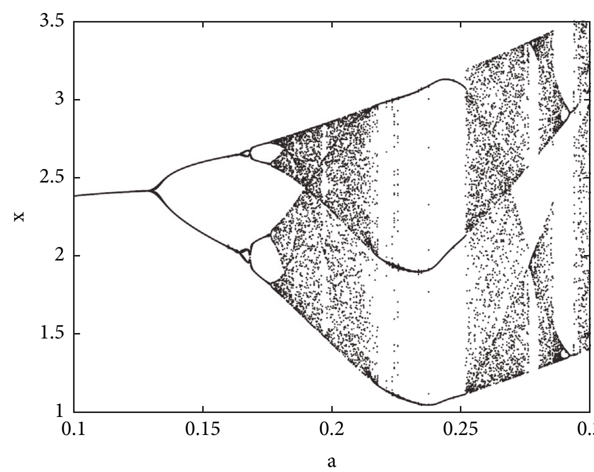


Figure 4: The bifurcation diagram of system (11) with the variation of the system parameter a.

Compared with an integer-order system, the derivative order is an important parameter for a fractional-order system. For system (11), the system parameters and the initial conditions are fixed. The phase diagrams with different values of the derivative order q are employed to demonstrate the behavior of system (11), as shown in Figure 2. From which it can be seen that system (11) converges to a fixed point as $q = 0.90$, and it is period-1 for $q = 0.92$, period-2 for $q = 0.95$, and period-4 for $q = 0.955$.

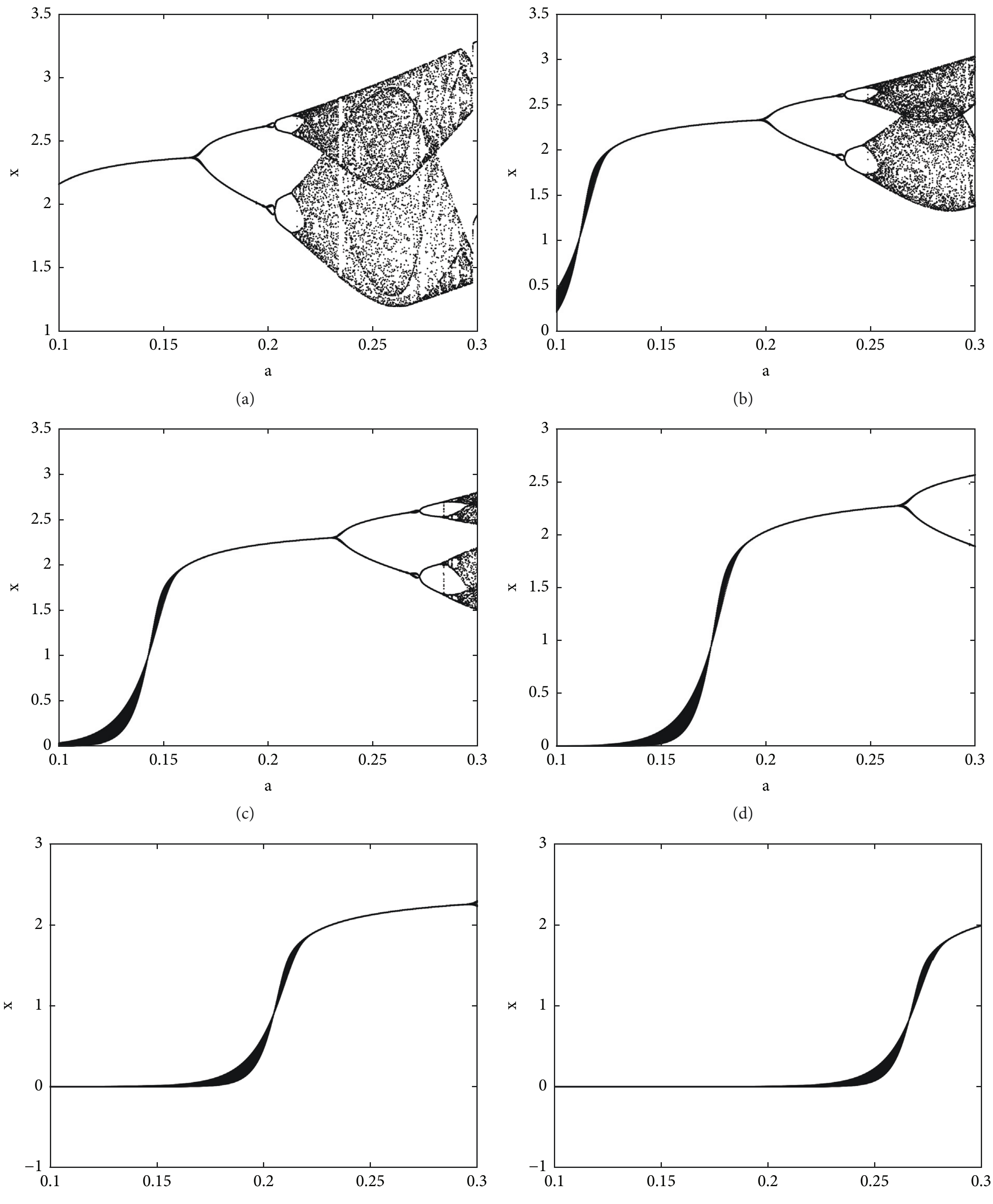
4. Bifurcations of the New System

Bifurcations play a vital role in dynamics research for fractional-order systems. Therefore, in this section, bifurcation analysis is conducted to study the rich dynamical behavior of the fractional-order system (11) in the two cases of commensurate-order and incommensurate-order, respectively.

4.1. The Commensurate-Order Case. Firstly, with the system parameters fixed and the derivative order varying on the closed interval $q \in [0.9, 1]$, bifurcation diagram for system (11) is depicted in Figure 3. Clearly, the evolution of the period-doubling scenario and saddle-node bifurcation can be observed from this figure. When the order satisfies $q \leq 0.958$, the route of leading to chaos for system (11) is period-doubling bifurcation. When $q = 0.978$, the chaotic solutions disappear suddenly, and two new period-1 solutions appear, which means that the saddle-node bifurcation occurs as the order q varies.

Secondly, bifurcations with the variation of the system parameter a are studied for $q = 0.99$ and $b = 0.02$. In Figure 4, it can be seen that a series of period-doubling bifurcations occurs as the system parameter a decreases from 0.3 to 0.1. Meanwhile, the period-3 and chaos windows can also be observed from the bifurcation diagram.

In order to get a better understanding of the dynamics of system (11), bifurcations with the variation of the parameter a for the different values of derivative order q are given. With the parameter a varying, the corresponding bifurcation diagrams with several specified values of the derivative order like $q = 0.98$, $q = 0.97$, $q = 0.96$, $q = 0.95$, $q = 0.94$, and $q = 0.93$ are plotted in Figure 5 for system (11). Compared these figures, it can be seen that the bifurcation structure



(a)

(b)

(c)

(d)

(e)

(f)

Figure 5: The bifurcation diagrams of system (11) when the parameter a varied with the different values of the order q: (a) $q = 0.98$; (b) $q = 0.97$; (c) $q = 0.96$; (d) $q = 0.95$; (e) $q = 0.94$; (f) $q = 0.93$.

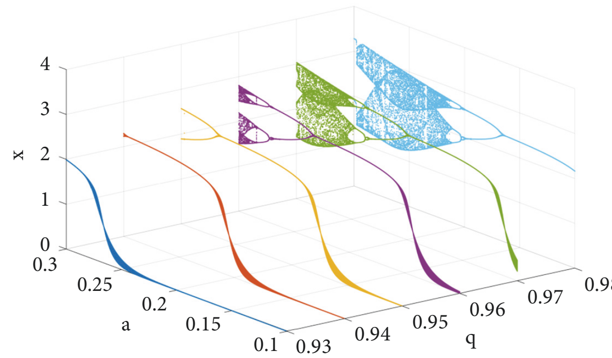


Figure 6: The bifurcation diagram of system (11) in three-dimensional space with the variation of both the parameter a and the order q.

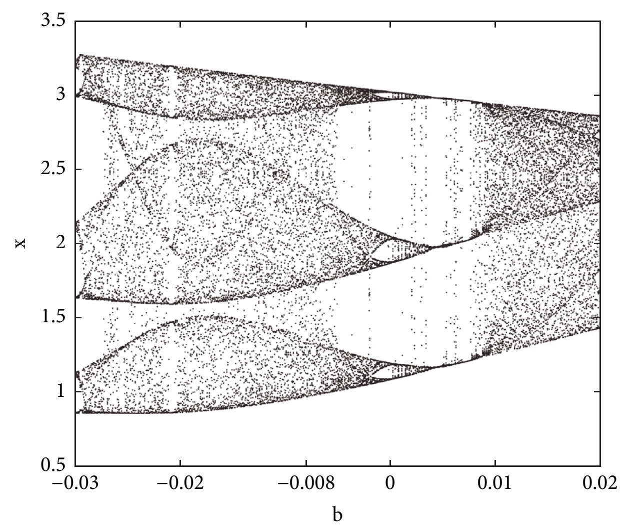


Figure 7: The bifurcation diagram of system (11) with the variation of the parameter b.

of system (11) changes qualitatively with the variation of the parameter a and the order q. The area of the chaotic motion decreases with the augment of the periodic motion as the derivative order decreases. From Figures 5(a)–5(c), it is clear that the route leading to chaos is period-doubling bifurcation. Meanwhile, a typical period-doubling bifurcation, of which numerical solutions change from period-1 to period-2, can be seen clearly from Figures 5(d) and 5(e). The system becomes totally periodic as the order q decreases to 0.93; see Figure 5(f). In order to further discuss the bifurcations with the variation of both the system parameter a and derivative order q, a bifurcation diagram in the three-dimensional space is depicted in Figure 6. It can be seen that dynamics of system (11) becomes simple as the derivative order decreases from 0.98 to 0.93.

Thirdly, the bifurcation of system (11) with the variation of the parameter b is studied for $q = 0.99$ and $a = 0.3$. In Figure 7, there is a long time of the chaotic window for $b \leq -0.008$. Meanwhile, the route leading to chaos is the period-doubling bifurcation for system (11).

Bifurcations with the variation of the parameter b are studied for different values of the derivative order q. When the derivative order q decreases from 0.98 to 0.93, the corresponding bifurcation diagrams are plotted for the fractional-order system (11) when the parameter $b \in [-0.03, 0.02]$; see Figure 8, from which, it is clear that structure of dynamics of system (11) evolves as the order q varies. The interval of the chaotic motion increases and that of the periodic motion decreases. Meanwhile, system (11) is periodic completely when the order $q = 0.94$, and period-4 and period-2 motions can be obtained from Figure 8(e). When the order $q = 0.93$, only period-2 and period-1 motions exist; see Figure 8(f). In order to further discuss the bifurcations with the variation of both the system parameter b and derivative order q, a bifurcation diagram in the three-dimensional space is plotted in Figure 9. It can be seen that dynamics of system (11) becomes complex as the derivative order increases from 0.93 to 0.98.

4.2. The Incommensurate-Order Case. In order to learn more about the characteristics of the new fractional-order system (11), the dynamics of system (11) with the variation of the different derivative orders will be investigated in this subsection. In the following work, the system parameters are taken as $a = 0.3$ and $b = 0.02$.

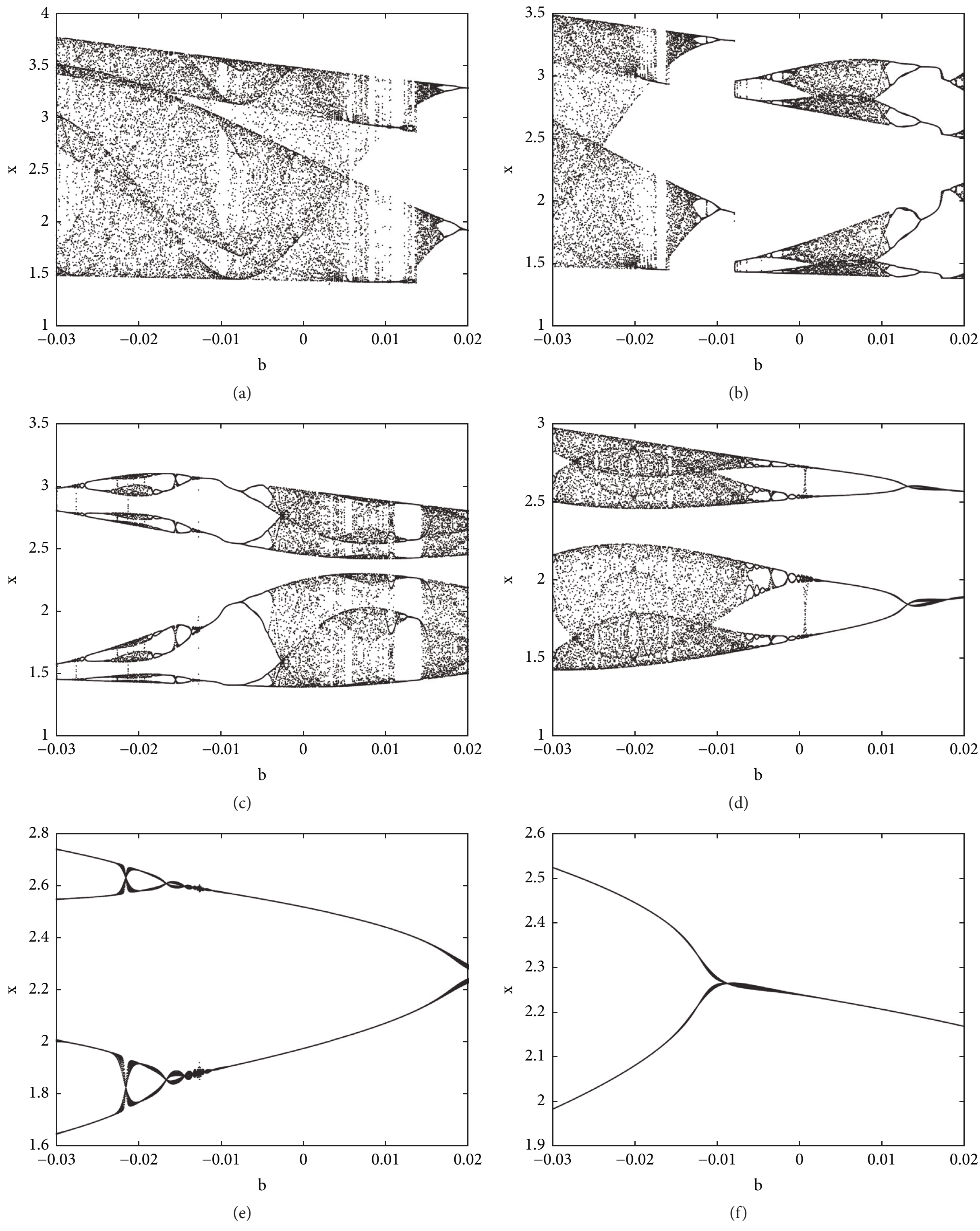


Figure 8: The bifurcation diagrams of system (11) when the parameter b varied with the different values of the order q: (a) $q = 0.98$; (b) $q = 0.97$; (c) $q = 0.96$; (d) $q = 0.95$; (e) $q = 0.94$; (f) $q = 0.93$.

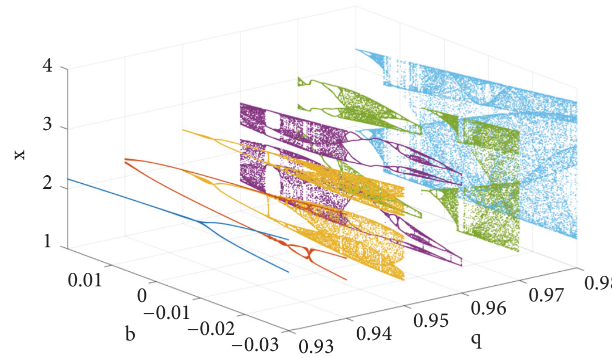


FIGURE 9: The bifurcation diagram of system (11) in three-dimensional space with the variation of both the parameter b and the order q.

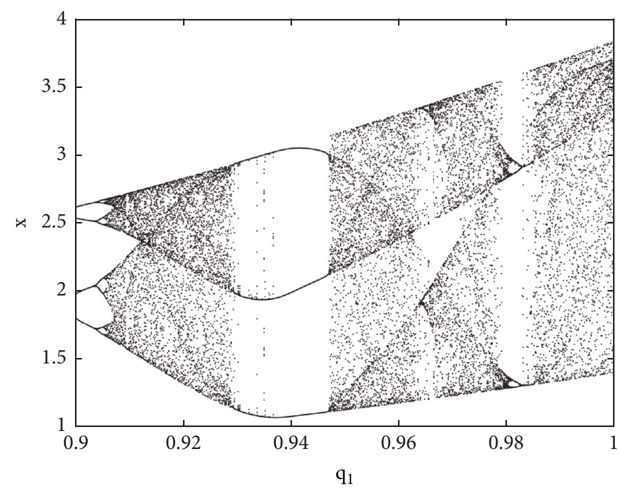


FIGURE 10: The bifurcation diagram of system (11) with the variation of the order q_1.

Firstly, the bifurcation diagram with the derivative order $q_1 \in [0.9, 1]$ as the others two derivative orders q_2, q_3 are both fixed and $q_2 = q_3 = q = 1$ is plotted. A series of period-doubling bifurcations can be seen in Figure 10. Based on this, bifurcation diagrams versus the order q_1 when the others two derivative orders decrease from 0.98 to 0.93 are given in Figure 11. For Figures 11(a)–11(c), it can be seen that the two branches of the bifurcation gradually decouple, and the area of the chaos gradually decreases with that of the period increases. For Figures 11(d)–11(f), it is clear that the fractional-order system (11) is periodic completely. A bifurcation diagram with the variation of both the order q_1 and derivative order q in the three-dimensional space is plotted in Figure 12. It is clear that dynamics of system (11) becomes simple as the derivative order decreases from 0.98 to 0.93.

Secondly, the bifurcation diagram versus the order $q_2 \in [0.9, 1]$ when the others two derivative orders q_1, q_3 are both fixed and $q_1 = q_3 = q = 1$ is plotted in Figure 13. Meanwhile, a series of bifurcation diagrams versus the order q_2 when the others two derivative orders decrease from 0.98 to 0.85 are given in Figure 14. By comparing Figure 14(a) and Figure 14(b), it is clear that the dynamics of system (11) becomes simple as the order q decreases. For Figures 14(b) and 14(c), it can be observed that the area of chaos increases and that of period decreases. From Figures 14(c) and 14(d), it can be seen that the two branches of the bifurcation gradually couple, and the area of the chaos gradually decreases with that of the period increases. For the rest of the figures in Figure 11, it can be seen that system (11) is totally periodic when the order $q = 0.90$. Meanwhile, the typical period-doubling bifurcation can be seen in Figure 14(g). When the order $q = 0.85$, only period-1 motion for system (11) exists. A bifurcation diagram with the variation of both the order q_2 and derivative order q in the three-dimensional space is plotted in Figure 15. It is clear that dynamics of system (11) becomes simple as the derivative order decreases from 0.98 to 0.85.

The bifurcation diagrams with the variation of the order q_3 when the other two orders q_1, q_2 decrease from 0.98 to 0.93 will not be given in here for the similarity.

5. Conclusions

In this paper, a new fractional-order system is presented. Firstly, the stability of the equilibriums is analyzed. Based on the stability analysis, the reason of the generation for the one scroll of the attractor is analyzed. Phase diagrams for the

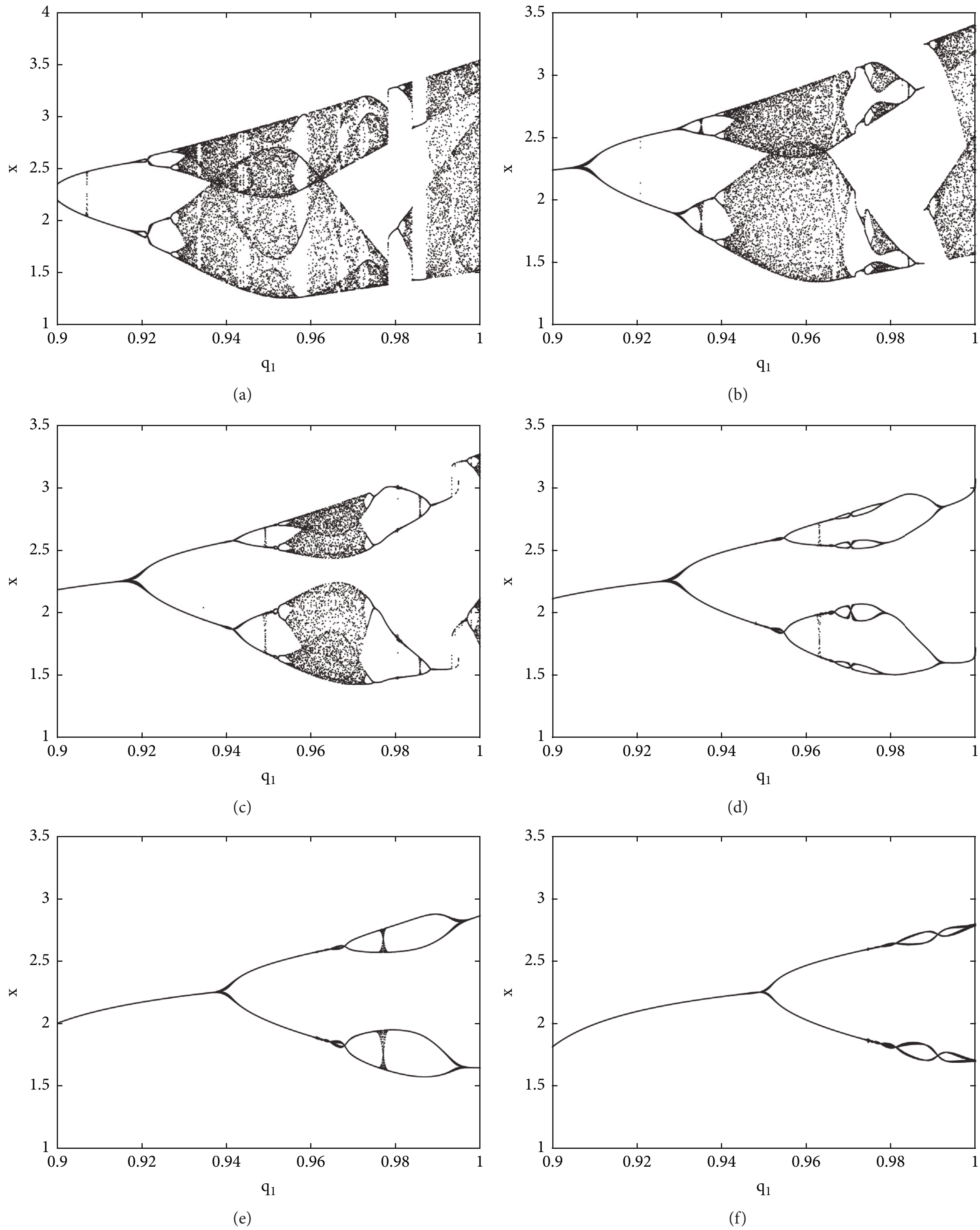


FIGURE 11: The bifurcation diagram of system (11) when the order q_1 varied with the different values of the order q: (a) $q = 0.98$; (b) $q = 0.97$; (c) $q = 0.96$; (d) $q = 0.95$; (e) $q = 0.94$; (f) $q = 0.93$.

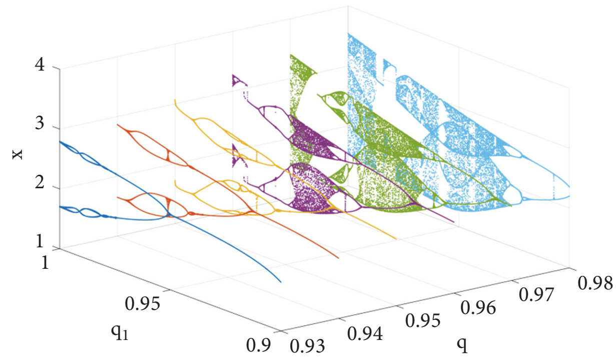


FIGURE 12: The bifurcation diagram of system (11) in three-dimensional space with the variation of both the orders q_1 and q.

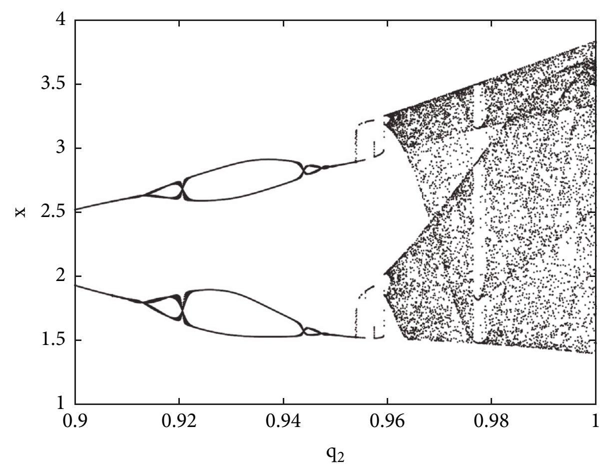


FIGURE 13: The bifurcation diagram of system (11) with the variation of the derivative order q_2.

different values of the derivative order are obtained to show the rich dynamics of the new fractional-order system.

Bifurcations of the new fractional-order system in commensurate-order and incommensurate-order cases are studied in detail. In the commensurate-order case, when the derivative order decreases from 0.99, bifurcations with the variation of a system parameter are investigated. In the incommensurate-order case, bifurcations with the variation of a derivative order when the other orders decreases from 1 are analyzed. Period-doubling and saddle-node bifurcations can be observed.

What is more, it can be concluded that the dynamics of the new fractional-order system becomes periodic or simple when the derivative order approaches to 0 and chaotic or complex when the order approaches to 1 from a global perspective. These results obtained in this paper can be referenced for the bifurcation control of fractional-order systems. The generalization of the conclusion for other fractional-order systems will be our future work.

Data Availability

The data for the bifurcation diagrams used to support the findings of this study are included within the supplementary information file(s) (available here). In the manuscript, there are 31 bifurcation diagrams. For the large of the data, we will supply the partial bifurcation data calculated by the software Matlab.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

(1) adataq093: the bifurcation data of Figure 5(f). (2) adataq094: the bifurcation data of Figure 5(e). (3) adataq095: the bifurcation data of Figure 5(d). (4) adataq096: the bifurcation data of Figure 5(c). (5) adataq097: the bifurcation data of Figure 5(b). (6) adataq098: the bifurcation data of Figure 5(a). (7) q1dataq093: the bifurcation data of Figure 11(f). (8) q1dataq094: the bifurcation data of Figure 11(e). (9) q1dataq095: the bifurcation data of

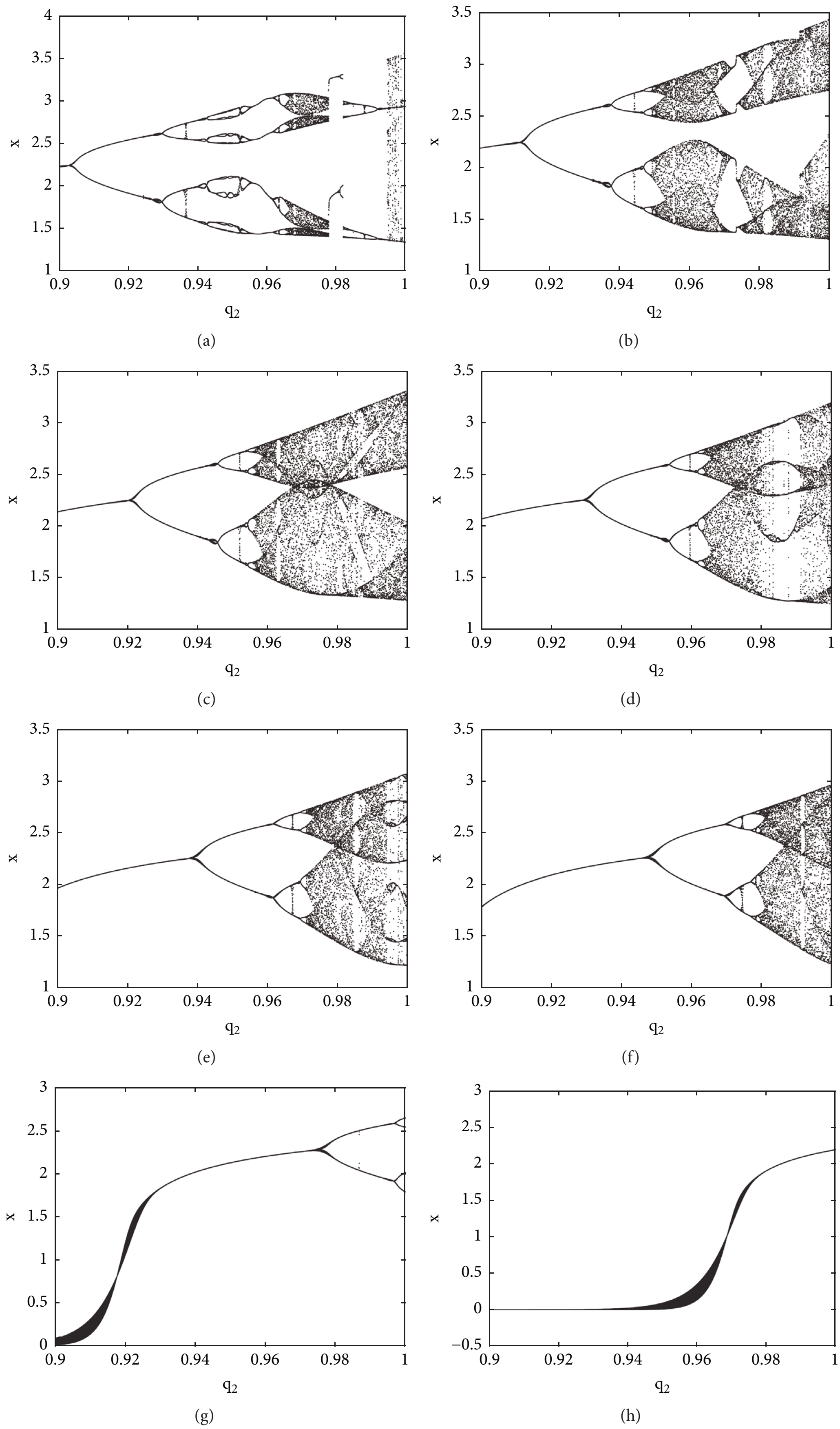

Figure 14: The bifurcation diagram of system (11) when the order q_2 varied with the different values of the order q: (a) $q = 0.98$; (b) $q = 0.97$; (c) $q = 0.96$; (d) $q = 0.95$; (e) $q = 0.94$; (f) $q = 0.93$; (g) $q = 0.90$; (h) $q = 0.85$.

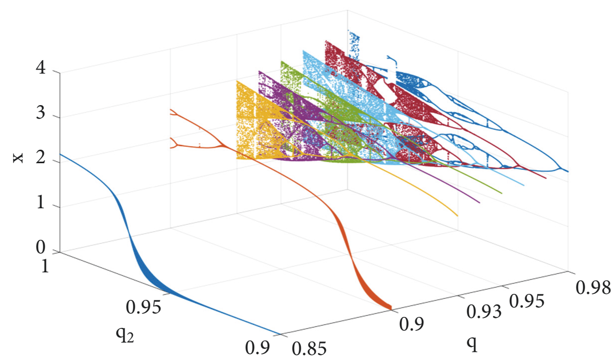

Figure 15: The bifurcation diagram of system (11) in three-dimensional space with the variation of both the orders q_2 and q.

Figure 11(d). (10) q1dataq096: the bifurcation data of Figure 11(c). (11) q1dataq097: the bifurcation data of Figure 11(b). (12) q1dataq098: the bifurcation data of Figure 11(a). *(Supplementary Materials)*

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