

Research Article Winner's Optimal Reimbursement in Contest

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This paper explores a designer-optimal reimbursement scheme in all-pay auctions with winner's reimbursement. Assuming the reimbursement is a linear function of the cost of effort, we obtain analytical solutions for the contestants' symmetrical equilibrium effort and the contest organizer's expected revenue. We show that if the effort cost function is concave, the optimal reimbursement scheme is to return the full cost to the winner. On the contrary, if the effort cost function is convex, the optimal reimbursement scheme is not to compensate the winner. Moreover, we find that the organizer's expected revenue may increase or decrease as the number of contestants increases when the winner is fully reimbursed.

1. Introduction

It is well known that the all-pay auction (contest) is commonly used to model applications, such as political campaigns, rent seeking, R&D competitions, job promotions, etc. One of the possible solutions of eliciting higher expected revenue involves reimbursement of the contestants' efforts. Several studies contribute to the research of contestants' reimbursement in all-pay auctions.

Cohen and Sela [1] first consider the winner's reimbursement in Tullock contest with two asymmetric players. They show that the expected revenue is higher when the winner's cost of effort is reimbursed than no reimbursement. Matros and Armanios [2] extend the work of Cohen and Sela [1]. They consider Tullock contest with symmetric players and find that the winner's reimbursement causes an increase in the expected revenue, while all losing players' reimbursement leads to a decrease in the expected revenue. Later, this model is generalized by Matros [3] and Yates [4]. Matros [3] focuses on the winner's reimbursement in Tullock contest with asymmetric players. He describes all equilibria and discusses their properties. Yates [4] analyzes the unique equilibrium of a class of two-player contests, including Tullock contests, where all the other players' costs are reimbursed except for the winner. Minchuk and Sela [5] investigate the all-pay auction with

insurance, which is a form of loser reimbursement for an additional fee. More types of reimbursement in contest are studied by Kaplan et al. [6], Baye et al. [7], Chowdhury and Sheremeta [8], etc.

This paper focuses on all-pay auctions where the winner's reimbursement is a linear function of the cost of effort. Closely related to our paper, Minchuk [9] analyzes the all-pay auctions where the winner gets a full refund of his effort. He shows that if the cost of effort is concave, then the expected revenue of the contest organizer is higher when the winner is fully reimbursed opposed to receiving no reimbursement. However, the result is reversed if the cost of effort is convex. This study only considers the case if the winner is fully compensated or not compensated at all. It is unclear how much reimbursement would be optimal for the organizer to pay the winner. By introducing a linear compensation function in all-pay auctions, we demonstrate that a full reimbursement for the winner maximizes the organizer's expected revenue if the cost of effort is concave. On the contrary, no reimbursement for the winner maximizes the organizer's expected revenue if the cost of effort is convex. The linear compensation function is also employed by Matros and Armanios [2] and Baye et al. [10]. Matros and Armanios [2] analyze the Tullock contest where the winner and the losers are all compensated and the reimbursement is a linear function of effort. Baye et al. [10] characterize

symmetric litigation environments with a simple auction-theoretic framework to compare different litigation systems.

The remainder of the paper is organized as follows. Section 2 introduces our model of the all-pay auctions with winner's reimbursement. Section 3 characterizes the equilibrium effort and expected revenue and shows the effect of reimbursement and the number of participants on both of them. Section 4 concludes the paper.

2. Model

Consider the all-pay auctions where there are $n \ge 2$ risk-neutral contestants competing for a single prize. Each contestant i's value of winning v_i is independently and identically distributed on [0, 1] according to a continuously differentiable distribution function F(v) with a positive and continuous density function f(v), which is assumed to be commonly known. Let all the contestants share the cost of the effort x, y(x), satisfying $\gamma(0) = 0$ and $\gamma' > 0$ (This form of cost function in all-pay auctions is considered in many studies. For example, Moldovanu and Sela [11, 12].). Denote the reverse function of $\gamma(x)$ by $g = \gamma^{-1}$. The contestant with the highest effort wins the prize. If contestant *i* with the value v_i and effort x_i wins the prize, he also gets reimbursement which is a linear function of the cost of effort, $k\gamma(x_i)$, where $0 \le k \le 1$ is a constant; if he loses, he pays his cost of effort $\gamma(x_i)$. It is reasonable to assume that each contestant's effort equals zero if his value of winning is zero.

If there is a symmetric monotonically increasing equilibrium effort function $\chi_k(v_i) : [0, 1] \rightarrow [0, 1]$, then $\chi_k(0) = 0$ and the expected function of contestant *i* with the value v_i and the effort $\chi_k(v_i)$ is

$$U(v_i) = (v_i - (1 - k)\gamma(\chi_k(v_i)))G(v_i) - \gamma(\chi_k(v_i))(1 - G(v_i)),$$

for $i = 1, 2, ..., n,$ (1)

where $G(v_i) \equiv P\{\chi_k(v_i) > \chi_k(v_j), j \neq i\} = P\{v_i < v_j, j \neq i\} = F^{n-1}(v_i)$ is the probability that contestant *i* with the value v_i has the highest effort among all *n* contestants.

3. Main Results

3.1. *The Effect of Reimbursement.* The following proposition first gives expressions of the equilibrium effort of each contestant and the expected revenue of contest organizer.

Proposition 1. In an all-pay auction with winner's reimbursement,

(i) the equilibrium effort of contestant i is

$$\chi_k(v_i) = g\left(\frac{1}{1 - kG(v_i)} \int_0^{v_i} sG'(s) ds\right), \text{ for } i = 1, 2, ..., n.$$
(2)

(ii) the contest organizer's expected revenue is given by

$$R_{k} = n \int_{0}^{1} \left(g \left(\frac{1}{1 - kG(v)} \int_{0}^{v} sG'(s) ds \right) -G(v) g \left(\frac{k}{1 - kG(v)} \int_{0}^{v} sG'(s) ds \right) \right) f(v) dv.$$
(3)

Proof. (i) Suppose that the symmetrical equilibrium effort function $\chi_k(v)$ exists and strictly increases with v. With all other contestants using $\chi_k(v)$, as per (1) the expected function of contestant *i* with the value v_i and the effort $\chi_k(t)$ is given by

$$U(t) = (v_i - (1 - k))\gamma(\chi_k(t))G(t) - \gamma(\chi_k(t))(1 - G(t)), \quad (4)$$

where $G(t) = F^{n-1}(t)$ is the probability that contestant *i* with the value *t* wins. In equilibrium, the derivative of U(t) at $t = v_i$ equals zero, i.e.,

$$G'(v_i)(v_i - (1 - k)\gamma(\chi_k(v_i))) - (1 - k)G(v_i)(\gamma(\chi_k(v_i)))' -(\gamma(\chi_k(v_i)))'(1 - G(v_i)) + \gamma(\chi_k(v_i))G'(v_i) = 0.$$
(5)

Rearranging yields

$$(1 - kG(v_i))(\gamma(\chi_k(v_i)))' - kG'(v_i)\gamma(\chi_k(v_i)) = v_iG'(v_i).$$
(6)

Solving the first order nonhomogeneous differential equation and noting that $\gamma(\chi_k(0)) = 0$, we have

$$\gamma(\chi_k(v_i)) = \frac{1}{1 - kG(v_i)} \int_0^{v_i} sG'(s) ds.$$
(7)

Noting that $g = \gamma^{-1}$ and rearranging yields (2).

Now, Eq. (2) is merely a necessary condition for $t = v_i$ to maximize U(t). Next, we claim that it is also a sufficient condition. Using (4) we have

$$U'(t) = (v_i + k\gamma(\chi_k(t)))G'(t) - (1 - kG(t))(\gamma(\chi_k(t)))'.$$
 (8)

As per (7), we get

$$\gamma(\chi_k(t)) = \frac{1}{1 - kG(t)} \int_0^t sG'(s) ds.$$
(9)

$$\left(\gamma(\chi_k(t))\right)' = \frac{kG'(t)}{\left(1 - kG(t)\right)^2} \int_0^t sG'(s)ds + \frac{tG'(t)}{1 - kG(t)}.$$
 (10)

Substituting (9) and (10) into (8) and simplifying, we obtain

$$U'(t) = (v_i - t)G'(t),$$
(11)

which implies that U'(t) > 0 if $t < v_i$ and U'(t) < 0 if $t > v_i$. Thus, U(t) is maximized at $t = v_i$.

Finally, we need to prove that $\chi_k(v_i)$ is strictly increasing. In fact, since $G(v_i) = F^{n-1}(v_i) \in [0, 1]$ is strictly increasing and $0 \le k \le 1, 1/(1 - k G(v_i))$ is increasing. Clearly, $\int_0^{v_i} sG'(s) ds$ is strictly increasing. Thus, the right hand side of (7) is strictly increasing in v_i . Hence, $\gamma(\chi_k(v_{i1})) < \gamma(\chi_k(v_{i2}))$ if $v_{i1} < v_{i2}$. Since γ is strictly increasing, we have $\chi_k(v_{i1}) < \chi_k(v_{i2})$ if $v_{i1} < v_{i2}$.

(ii) The contest organizer's expected revenue is

$$R_{k} = n \int_{0}^{1} \chi_{k}(v) f(v) dv - \int_{0}^{1} g(k\gamma(\chi_{k}(v))) dF^{n}(v), \quad (12)$$

where the first part is the expected revenue from all efforts of the contestants, while the second part is the expected reimbursement cost. Substituting (2) into the above equation and rearranging yields (3).

The equilibrium effort and the expected revenue in the standard all-pay auction without reimbursement (see, for example Minchuk and Sela, [5]) are, respectively, given by

$$\chi_{all}(v_i) = g\left(\int_0^{v_i} sG'(s)ds\right),\tag{13}$$

$$R_{all} = n \int_0^1 g \bigg(v G(v) - \int_0^v G(s) ds \bigg) f(v) dv, \qquad (14)$$

or equivalently,

$$R_{all} = n \int_0^1 g\left(\int_0^v sG'(s)ds\right) f(v)dv.$$
(15)

It is easy to see that when k = 0, $\chi_k(v_i)$ and R_k degenerate to $\chi_{all}(v_i)$ and R_{all} , respectively. Namely, $\chi_0(v_i) = \chi_{all}(v_i)$, and $R_0 = R_{all}$.

The equilibrium effort and the expected revenue in the all-pay auction with winner's full reimbursement are (Minchuk [9] considers the all-pay auction with winner's reimbursement, in which the winner receives a full refund of his effort in addition to the contest prize), respectively, given by

$$\chi_{full}(v_i) = g\left(\frac{1}{1 - G(v_i)} \int_0^{v_i} sG'(s) ds\right),\tag{16}$$

$$R_{full} = n \int_{0}^{1} g \left(\frac{vG(v)}{1 - G(v)} - \frac{\int_{0}^{v} G(s) ds}{1 - G(v)} \right) f(v) dv - \int_{0}^{1} g \left(\frac{vG(v)}{1 - G(v)} - \frac{\int_{0}^{v} G(s) ds}{1 - G(v)} \right) dF^{n}(v),$$
(17)

or equivalently,

$$R_{full} = n \int_{0}^{1} (1 - G(v)) g\left(\frac{1}{1 - G(v)} \int_{0}^{v} sG'(s) ds\right) f(v) dv.$$
(18)

It is clear that when k = 1, $\chi_k(v_i)$, and R_k become $\chi_{full}(v_i)$ and R_{full} , namely, $\chi_1(v_i) = \chi_{full}(v_i)$, and $R_1 = R_{full}$.

The following proposition explores the effect of reimbursement on the equilibrium effort of each participant.

Proposition 2. *In an all-pay auction with winner's reimbursement, the equilibrium effort of each contestant increases monotonically in parameter k.*

Proof. Since $G(v_i) \ge 0, 1/(1 - kG(v_i))$ increases in parameter k. Noting $\int_0^{v_i} sG'(s) ds \ge 0$ and g' > 0, hence $\chi_k(v_i)$ increases in parameter k.

Corollary 1. *In an all-pay auction with winner's reimbursement, we have*

$$\chi_{all}(v_i) = \chi_0(v_i) \le \chi_k(v_i) \le \chi_1(v_i) = \chi_{full}(v_i).$$
(19)

Proposition 2 and Corollary 1 imply that the winner's reimbursement causes an increase in the equilibrium effort. The higher the reimbursement the winner receives, the more effort each contestant exerts. In particular, full reimbursement enables each contestant to exert their best effort, namely, k = 1. This finding is consistent with the result that winner's reimbursement parameter increases an equilibrium effort in Tullock contest, see Matros and Armanios [2]. Intuitively, if the winner gets reimbursed, this increases the actual prize, and as a result, this increases the competition in the contest and therefore all the contestants exert higher effort.

The following proposition shows the effect of reimbursement on the organizer's expected revenue.

Proposition 3. In an all-pay auction with winner's reimbursement,

(i) if the effort cost function $\gamma(\cdot)$ is concave, then the organizer's expected revenue increase monotonically in parameter k; (ii) if the effort cost function $\gamma(\cdot)$ is convex, then the organizer's expected revenue decrease monotonically in parameter k.

Proof. (i) If the effort cost function $\gamma(\cdot)$ is concave, it follows from $\gamma' > 0$ and $g = \gamma^{-1}$ that $g(\cdot)$ is convex, which implies that g' > 0 and $g'' \ge 0$. Let

$$h(v,k) = g\left(\frac{1}{1-kG(v)}\int_{0}^{v} sG'(s)ds\right)$$
(20)
- $G(v)g\left(\frac{k}{1-kG(v)}\int_{0}^{v} sG'(s)ds\right).$

Differentiating both sides with respect to *k* yields

$$\frac{d}{dk}h(v,k) = \frac{G(v)}{(1-kG(v))^2}g'\Big(\frac{1}{1-kG(v)}\int_0^v sG'(s)ds\Big)\int_0^v sG'(s)ds - \frac{G(v)}{(1-kG(v))^2}g'\Big(\frac{k}{1-kG(v)}\int_0^v sG'(s)ds\Big)\int_0^v sG'(s)ds = \frac{G(v)}{(1-kG(v))^2}\int_0^v sG'(s)ds\Big(g'\Big(\frac{1}{1-kG(v)}\int_0^v sG'(s)ds\Big) - g'\Big(\frac{k}{1-kG(v)}\int_0^v sG'(s)ds\Big)\Big) \ge 0,$$
(21)

since $G(v)/(1 - kG(v))^2 \ge 0$, $\int_0^v sG'(s)ds \ge 0$, $g'' \ge 0$ and $1/(1 - kG(v)) \ge k/(1 - kG(v))$. Noting that $R_k = n \int_0^1 h(v, k) dv$, we have $dR_k/dk \ge 0$. The proof is completed.

(ii) If the effort cost function $\gamma(\cdot)$ is convex, noting that $\gamma' > 0$ and $g = \gamma^{-1}$, then $g(\cdot)$ is concave, which implies that g' > 0 and $g'' \le 0$. The remaining proof is similar to that of part (i).

Corollary 2. In an all-pay auction with winner's reimbursement,

(*i*) *if the effort cost function* $\gamma(\cdot)$ *is concave, then*

$$R_{all} = R_0 \le R_k \le R_1 = R_{full}.$$
(22)

(ii) if the effort cost function $\gamma(\cdot)$ is convex, then

TABLE 1							
The number of participants	2	3	4	5	6	7	8
Organizer's expected revenue	0.0432	0.0663	0.0767	0.0812	0.0829	0.0830	0.0823

$$R_{full} = R_1 \le R_k \le R_0 = R_{all}.$$
 (23)

Proposition 3 and Corollary 2 imply that the winner's reimbursement leads to a higher expected revenue if the effort cost function $\gamma(\cdot)$ is concave. The more reimbursement the winner receives, the higher the expected revenue the contest organizer gains. In particular, full reimbursement maximizes the expected revenue of the organizer, which implies that a designer-optimal reimbursement scheme is to return the full cost of the winner, namely, k = 1. However, the result is reversed if the effort cost function is convex. In this case, the winner's reimbursement leads to a lower expected revenue and then no reimbursement to maximize the expected revenue of the organizer. Therefore, a designer-optimal reimbursement scheme is not to return the cost of the winner, namely, k = 0. These results are consistent with those in contest with insurance, but without an insurance fee, see Minchuk and Sela [5].

3.2. The Effect of the Number of Participants. This section we consider the two designer-optimal reimbursement schemes: the winner is fully reimbursed (i.e., k = 1) and the winner receives no reimbursement (i.e., k = 0). It is not clear whether the equilibrium effort and expected revenue increases or decreases as the number of contestants increases when k = 0, and if the equilibrium effort increases or decreases as the number of contestants increases or decreases as the number of contestants increases or decreases as the number of contestants increases when k = 1. However, the organizer's expected revenue may increase or decrease as the number of contestants increases when k = 1 (see the following example).

Example 1. Suppose k = 1, $\gamma(x) = x^{2/3}$ and F(v) is uniform on [0, 1]. Then, F(v) = v, f(v) = 1, $G(v) = F^{n-1}(v) = v^{n-1}$, $G'(v) = (n-1)v^{n-2}$, and $g(x) = \gamma^{-1}(x) = x^{3/2}$. As per (2) we have

$$\chi_1(\upsilon) = \left(\frac{n-1}{n}\frac{\upsilon^n}{1-\upsilon^{n-1}}\right)^{3/2}.$$
 (24)

Substituting $\chi_1(v)$ into (3) and rearranging yields

$$R_{1} = n \int_{0}^{1} \left(\frac{n-1}{n} \frac{v^{n}}{1-v^{n-1}} \right)^{3/2} (1-v^{n-1}) dv.$$
 (25)

The number of participants and the organizer's expected revenue are listed in Table 1 (The contest organizer's expected revenue is obtained by using the MATLAB software).

As Table 1 shows, the organizer's expected revenue increases as the number of participants increases from two to seven. However, the expected revenue decreases as the number of participants increases from seven to eight. Fu et al. [13] also find that the expected overall bid of the contest is not monotone in the number of shortlisted potential bidders under properly set mechanisms. Specifically, they find the functional relationship between the two variables to be single-peaked, which is consistent with the situation shown in Example 1. Remarkably, this result is inconsistent with the standard result in the contest literature stating that the expected revenue increases as the number of players increases.

4. Conclusion

In this study, we focus on the all-pay auction where the winner, in addition to the contest prize, is also reimbursed and the reimbursement is a linear function of the cost of effort. We find that no matter the effort cost function is concave or convex, the more reimbursement the winner receives, the more effort each contestant exerts. In particular, full reimbursement enables each contestant to exert their best effort. If the effort cost function is concave, the winner's reimbursement leads to a higher expected revenue than a regular allpay auction. The more reimbursement the winner receives, the more expected revenue the contest organizer gains. Thus, full reimbursement is the most profitable for the contest organizer. However, if the cost function is convex, the winner's reimbursement causes a decrease in the expected revenue. The more reimbursement the winner receives, the less expected revenue the contest organizer gains. In this case, having no reimbursement is the most profitable for the contest organizer. In addition, we numerically show that the organizer's expected revenue may increase or decrease as the number of participants increases when the winner is fully reimbursed. Questions that will remain for future research are, the effect of the number of contestants on the equilibrium effort, the expected revenue when there is no reimbursement for the winner, and the equilibrium effort when the winner is fully reimbursed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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