Hindawi
Discrete Dynamics in Nature and Society
Volume 2019, Article ID 9290582, 15 pages
https://doi.org/10.1155/2019/9290582

Research Article

On the Solutions of a Porous Medium Equation with Exponent Variable

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Received 24 April 2019; Accepted 1 July 2019; Published 1 August 2019

Academic Editor: Genni Fragnelli

The paper studies the initial-boundary value problem of a porous medium equation with exponent variable. How to deal with nonlinear term with the exponent variable is the main dedication of this paper. The existence of the weak solution is proved by the monotone convergent method. Moreover, according to the different boundary value conditions, the stability of weak solutions is studied. In some special cases, the stability of weak solutions can be proved without any boundary value condition.

1. Introduction

Let ρ be the density, let V be the velocity, and let p be the pressure of the ideal barotropic gas through a porous medium. The motion is governed by the mass conservation law

$$\rho_t + \operatorname{div}(\rho V) = 0, \tag{1}$$

the Darcy law

$$V = -k(x)\nabla p, \tag{2}$$

and the equation of stage

$$p = P(\rho), \tag{3}$$

where $k(x)$ is a given matrix. One of the most common cases is $P(s) = \mu s^{\alpha}$ with $\mu, \alpha = \text{const}$. Then we obtain a semilinear parabolic equation on the density

$$\rho_t = \frac{\mu\alpha}{1+\alpha}\operatorname{div}\left(k(x)\nabla\rho^{1+\alpha}\right). \tag{4}$$

If we additionally assume that p may explicitly depend on x and has the form $p = \mu\rho^{\gamma(x)}$, then equation for ρ becomes

$$\rho_t = \mu\operatorname{div}\left(k(x)\rho\nabla\rho^{\gamma(x)}\right), \tag{5}$$

and can be written as

$$\rho_t = \mu\operatorname{div}\left(k(x)\gamma(x)\rho^{\gamma(x)}\nabla\rho) + \rho\log\rho k(x)\nabla\gamma\right). \tag{6}$$

If $k(x) = a(x)I$, where $a(x)$ is a function and I is the unit matrix, then (4) becomes

$$\rho_t = \frac{\mu\alpha}{1+\alpha}\operatorname{div}\left(a(x)\nabla\rho^{1+\alpha}\right) = \mu\alpha\operatorname{div}\left(a(x)\rho^{\alpha}\nabla\rho\right), \tag{7}$$

and (6) has the form

$$\rho_t = \mu\operatorname{div}\left(a(x)\gamma(x)\rho^{\gamma(x)}\nabla\rho + \rho\log\rho a(x)I\cdot\nabla\gamma\right). \tag{8}$$

In this paper, we generalized (8) to the following type:

$$u_t = \operatorname{div}\left(a(x)|u|^{m(x)}\nabla u\right) + \sum_{i=1}^{N}\frac{\partial b_i\left(u^{m(x)+1}\right)}{\partial x_i}, \quad (x,t)\in Q_T = \Omega\times(0,T), \tag{9}$$

and consider the initial-boundary value problem, where $m(x) > 0$ is a $C^1(\overline{\Omega})$ function, $b_i(s) \in C^1(\mathbb{R})$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$.

If $a(x) \equiv 1$, $b_i = 0$, $m(x) = m-1$ is a constant, (9) is equivalent to the so-called porous medium equation

$$u_t = \triangle u^m. \tag{10}$$

In this case, there exists an abundant literature; one can refer to the survey books [1–6] and the references therein.

If $a(x) \geq 0$, in one way, (9) can be regarded as a special case of reaction-diffusion equation

$$u_t = \operatorname{div}(a(u,x,t)\nabla u) + \operatorname{div}\left(\overrightarrow{b(u)}\right), \tag{11}$$

there are also many papers devoted to its well-posedness problem. The most striking part of this equation is that if there is an interior point of the set

$$\{x \in \Omega : a(\cdot,x,t) = 0\}, \tag{12}$$

then the uniqueness of weak solution can be proved only under the entropy condition; one can refer to [7–15]. Moreover, if $a(\cdot,x,t)$ is degenerate on the boundary, how to impose a suitable boundary value condition to study the well-posedness of weak solutions to (11) has attracted extensive attentions and has been widely studied for a long time. In the other word, though the initial value

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{13}$$

is always imposed, the Dirchilet boundary condition

$$u(x,t) = 0, \quad (x,t) \in S_T = \partial\Omega \times (0,T), \tag{14}$$

may not be imposed or be imposed in a weaker sense than the traditional trace. One can refer to [7–12] for the details.

In another way, the evolutionary equations with variable exponents, especially the so-called electrorheological fluids equations with the form

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + f(x,t,u,\nabla u), \tag{15}$$

have been brought to the forefront by many scholars since the beginning of this century; one can refer to [16–23] and the references therein. But we noticed that, compared with (15), the papers devoted to the equations with the type

$$u_t = \operatorname{div}\left(|u|^{\gamma(x)}\nabla u\right) + f(x,t,u,\nabla u), \tag{16}$$

seem much fewer. The existence, uniqueness, and localization properties of solutions to (16) have been studied by Antontsev-Ahmarev in [24]. The free boundary problem and the numerical study were researched in [25] by Duque et al. Different from these papers [16–20, 24, 25], we enable the diffusion $a(x)$ in (9) to be degenerate on the boundary. In detail, we suppose that

$$\begin{aligned} &a(x) > 0, \quad x \in \Omega; \\ &a(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{17}$$

$b_i(s)$ is a $C^1(\mathbb{R}$ function, and

$$\begin{gathered} 0 \leq u_0 \in L^\infty(\Omega), \\ \sqrt{a(x)}\nabla u_0^m \in L^2(\Omega). \end{gathered} \tag{18}$$

Definition 1. If a nonnegative function $u(x,t)$ satisfies

$$\begin{gathered} u \in L^\infty(Q_T), \\ \sqrt{a(x)}\,|u|^{m(x)}\,|\nabla u| \in L^\infty\left(0,T;L^2(\Omega)\right), \end{gathered} \tag{19}$$

and for any function $\varphi \in C^1(Q_T)$, $\varphi|_{t=T} = 0$, $\varphi|_{\partial\Omega} = 0$, there holds

$$\begin{aligned} &\iint_{Q_T}\left(-\frac{\partial\varphi}{\partial t}u + a(x)\,|u|^{m(x)}\,\nabla u\nabla\varphi\right)\mathrm{d}x\mathrm{d}t \\ &\quad + \sum_{i=1}^{N}\iint_{Q_T} b_i\left(u^{m(x)+1}\right)\varphi_{x_i}(x,t)\,dxdt \\ &= \int_\Omega u_0\varphi(x,0)\,dx, \end{aligned} \tag{20}$$

then we say $u(x,t)$ is weak solution of (9) with the initial value (13) in the sense

$$\lim_{t\to 0}\int_\Omega |u(x,t) - u_0(x)|\,dx = 0. \tag{21}$$

If $u(x,t)$ satisfies (14) in the sense of the trace in addition, then we say it is a weak solution of the initial-boundary value problem of (9).

Theorem 2. *If $m(x) > 0$ is a $C^1(\overline{\Omega})$ function, $b_i(s) \in C^1(\mathbb{R})$, $b_i(0) = 0$, $i = 1,2,\ldots,N$, $u_0(x) \geq 0$ satisfies (18), then (9) with initial value (13) has a nonnegative solution.*

Based on the usual Dirichlet boundary value condition, we have the following.

Theorem 3. *If $m(x) > 0$ is a $C^1(\overline{\Omega})$ function, $b_i(s) \in C^1(\mathbb{R})$, $i = 1,2,\ldots,N$,*

$$\int_\Omega a^{-1}(x)\,dx < \infty, \tag{22}$$

then the initial-boundary value problems (9), (13) and (14) have a uniqueness solution.

In some cases, we can establish the stability of the weak solutions without any boundary value condition.

Theorem 4. *If $a(x)$ satisfies (17) and (22)*

$$|\nabla a| = 0, \quad x \in \partial\Omega, \tag{23}$$

$u(x,t)$ is a solution of (9) with the initial value (13) but without the boundary value condition, $u(x,t)$ satisfies

$$\int_\Omega a(x)\left[1 + (m(x)+1)\log u\right]^2 |\nabla m|^2\,dx \leq c, \tag{24}$$

then $u(x,t)$ is the unique solution.

At last, we assume that

$$\left|b_i(s_1) - b_i(s_2)\right| \leq c_i\left|s_1 - s_2\right|, \quad i = 1,2,\ldots,N. \tag{25}$$

and probe the stability of weak solutions based on a partial boundary value condition.

Theorem 5. *Let u, v be two solutions of (9) with the initial values $u_0(x), v_0(x)$, respectively, and with a partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T). \tag{26}$$

It is supposed that, for every $i \in \{1, 2, \ldots, N\}$, either $b_i'(s) \geq 0$ or $b_i'(s) \leq 0$, $a(x)$ satisfies (17) and

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^2 dx\right)^{1/2} \leq c, \tag{27}$$

u and v satisfy

$$\begin{gathered}\int_\Omega a(x)\left[1 + (m(x)+1)\log u\right]^2 |\nabla m|^2 dx < \infty, \\ \int_\Omega a(x)\left[1 + (m(x)+1)\log v\right]^2 |\nabla m|^2 dx < \infty.\end{gathered} \tag{28}$$

Then

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \leq c\int_\Omega |u_0(x) - v_0(x)|\, dx. \tag{29}$$

Here, if $b_i'(s) \geq 0$, $1 \leq i \leq N$, then

$$\Sigma_1 = \left\{x \in \partial\Omega : \sum_{i=1}^N c_i a_{x_i} < 0\right\}. \tag{30}$$

However, if $b_i'(s) \geq 0$, $1 \geq i \leq N$, then

$$\Sigma_1 = \left\{x \in \partial\Omega : \sum_{i=1}^N c_i a_{x_i} > 0\right\}. \tag{31}$$

To show that the partial boundary value condition (26) with the expression (30) or (31) is reasonable, let us review the equation

$$\begin{aligned}&u_t - \operatorname{div}(a(x)\nabla u) - \sum_{i=1}^N b_i(x) D_i u + c(x,t)u \\ &\quad = f(x,t).\end{aligned} \tag{32}$$

According to Fichera-Oleinik theory [26–29], the boundary value condition matching up with (32) is

$$u(x,t) = 0, \quad (x,t) \in \Sigma \times [0,T), \tag{33}$$

with that

$$\Sigma = \{x \in \partial\Omega : b_i(x)n_i(x) < 0\}, \tag{34}$$

where $\overrightarrow{n} = \{n_i\}$ is the inner normal vector of Ω. Since (9) is nonlinear, Fichera-Oleinik theory is invalid; whether the partial boundary Σ_1 in (26) can be expressed similar to (34) has become an interesting problem. Theorem 5 partially answers this question. One can see that if $a(x) = d(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary, $a_{x_i} = d_{x_i} = n_i$, the expression (30) or (31) is similar to (34). In fact, instead of (9), if we consider the equation

$$u_t - \operatorname{div}\left(d(x)|u|^{m(x)}\nabla u\right) - \sum_{i=1}^N b_i(x) D_i u = 0, \tag{35}$$

by a similar method as the proof of Theorem 5, we can show that the partial boundary value condition matching up with (35) has the same expression as (34). Thus, the partial boundary value condition (26) with the expression (30) or (31) is reasonable.

At the end of the Introduction section, we would like to suggest that if $m(x) = m$ is a constant, then condition (24) in Theorem 4 and condition (28) in Theorem 5 are naturally true. Actually, when $m(x) = m$ is a constant, $a(x) = d^\alpha(x)$; (9) has been studied by the author in [29]. But, one can see that, the results (Theorems 4 and 5) are much better and clearer than the results in [29].

2. The Proof of Theorem 2

Proof of Theorem 2. We suppose that $u_0 \in C_0^\infty(\Omega)$ and $0 \leq u_0 \leq M$, and consider the following regularized problem:

$$\begin{aligned}u_{nt} &= \operatorname{div}\left(\left(a(x) + \frac{1}{n}\right)\left(|u_n|^{m(x)} + \frac{1}{n}\right)\nabla u\right) \\ &\quad + \sum_{i=1}^N \frac{\partial b_i\left(u_n^{m(x)+1}\right)}{\partial x_i}, (x,t), \quad (x,t) \in Q_T, \\ u_n(x,t) &= \frac{1}{n}, \quad (x,t) \in \partial\Omega \times (0,T), \\ u_n(x,0) &= u_{0n}(x) = u_0(x) + \frac{1}{n}, \quad x \in \Omega.\end{aligned} \tag{36}$$

According to the standard parabolic equation theory, there is a weak solution

$$\begin{gathered}u_n \in L^\infty(Q_T), \\ \left(a(x) + \frac{1}{n}\right)^{1/2}\left(|u_n|^{m(x)} + \frac{1}{n}\right)^{1/2}\nabla u \in L^2(Q_T),\end{gathered} \tag{37}$$

and

$$\frac{1}{n} \leqslant u_n(x,t) \leq \|u_0\|_{L^\infty(\Omega)} + \frac{1}{n}, \quad (x,t) \in Q_T. \tag{38}$$

Moreover, by comparison theorem, we clearly have

$$u_{n+1}(x,t) \leqslant u_n(x,t), \tag{39}$$

which yields

$$u(x,t) = \lim_{n\to\infty} u_n(x,t), \tag{40}$$

and

$$|u(x,t)| \leq M + 1. \tag{41}$$

In what follows, we are able to prove that the limit function u is a weak solution of (9) with the initial value (13).

Multiplying both sides of the first equation in (36) by $\phi = u_n^{m(x)+1} - (1/n)^{m(x)+1}$, and integrating it over Q_T, we have

$$\begin{aligned}
&\iint_{Q_T} u_{nt}\left(u_n^{m(x)+1} - \left(\frac{1}{n}\right)^{m(x)+1}\right) dxdt \\
&= \iint_{Q_T} \operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)\nabla u\right] \\
&\cdot \left(u_n^{m(x)+1} - \frac{1}{n^{m(x)+1}}\right) dxdt \\
&+ \sum_{i=1}^{N}\iint_{Q_T} \frac{\partial b_i\left(u_n^{m(x)+1}\right)}{\partial x_i}\left(u_n^{m(x)+1}\right. \\
&\left. - \left(\frac{1}{n}\right)^{m(x)+1}\right) dxdt.
\end{aligned} \tag{42}$$

Let us analyse every term in (42):

$$\begin{aligned}
&\iint_{Q_t} u_{nt}\left(u_n^{m(x)+1} - \left(\frac{1}{n}\right)^{m(x)+1}\right) dxdt \\
&= \int_\Omega \frac{1}{m(x)+2}\left[u_n^{m(x)+2}(x,t)\right. \\
&\left. - u_n^{m(x)+2}(x,0)\right] dx - \int_\Omega \left(\frac{1}{n}\right)^{m(x)+1} \\
&\cdot \left[u_n(x,t) - u_n(x,0)\right] dx.
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\iint_{Q_T} \operatorname{div}\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)\nabla u_n\right]\left(u_n^{m(x)+1}\right. \\
&\left. - \frac{1}{n^{m(x)}+1}\right) dxdt \\
&= -\iint_{Q_T}\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)\nabla u_n\right] \\
&\cdot \nabla\left(u_n^{m(x)+1} - \frac{1}{n^{m(x)}+1}\right) dxdt \\
&= -\iint_{Q_T}\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)\nabla u_n\right] \\
&\cdot \left[\log u_n u^{m(x)+1}\nabla m(x) + (m(x)+1)\, u_n^{m(x)}\nabla u_n\right. \\
&\left. - \left(\frac{1}{n}\right)^{m(x)+1}\log n \nabla m(x)\right] dxdt \\
&= -\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)(m(x)+1) \\
&\cdot u_n^{m(x)}\left|\nabla u_n\right|^2 dxdt - \iint_{Q_T}\left(a(x)+\frac{1}{n}\right) \\
&\cdot \left(|u_n|^{m(x)}+\frac{1}{n}\right)\nabla u_n \nabla m(x)\cdot\left[\log u_n u^{m(x)+1}\right. \\
&\left. - \left(\frac{1}{n}\right)^{m(x)+1}\log n\right] dxdt \\
&\le -\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)(m(x)+1) \\
&\cdot u_n^{m(x)}\left|\nabla u_n\right|^2 dxdt + \iint_{Q_T}\left(a(x)+\frac{1}{n}\right) \\
&\cdot \left(|u_n|^{m(x)}+\frac{1}{n}\right)\cdot\left\{\frac{1}{2}(m(x)+1)\,u_n^{m(x)}\left|\nabla u_n\right|^2\right. \\
&\left. + \frac{1}{2}\left[(m(x)+1)\,u_n^{m(x)}\right]^{-1}\left|\nabla m(x)\right|^2\right\} \\
&\cdot \left|\log u_n u_n^{m(x)+1} - \left(\frac{1}{n}\right)^{m(x)+1}\log n\right| dxdt \\
&= -\frac{1}{2}\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)(m(x)+1) \\
&\cdot u_n^{m(x)}\left|\nabla u_n\right|^2 dxdt + \frac{1}{2}\iint_{Q_T}\left(a(x)\right. \\
&\left. + \frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)\left[(m(x)+1)\,u_n^{m(x)}\right]^{-1} \\
&\cdot \left|\nabla m(x)\right|^2\cdot\left|\log u_n u_n^{m(x)+1}\right. \\
&\left. - \left(\frac{1}{n}\right)^{m(x)+1}\log n\right| dxdt \le -\frac{1}{2} \\
&\cdot \iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)(m(x)+1) \\
&\cdot u_n^{m(x)}\left|\nabla u_n\right|^2 dxdt + c.
\end{aligned} \tag{44}$$

Here, we have used (41) and the fact

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right) \\
&\cdot \left[(m(x)+1)\,u_n^{m(x)}\right]^{-1}\left|\nabla m(x)\right|^2 \\
&\cdot \left|\log u_n u_n^{m(x)+1} - \left(\frac{1}{n}\right)^{m(x)+1}\log n\right| = a(x) \\
&\cdot (m(x)+1)^{-1}\left|\nabla m(x)\right|^2 u^{m(x)+1}\left|\log u\right| < \infty.
\end{aligned} \tag{45}$$

In addition, by the fact

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial b_i\left(u_n^{m(x)+1}\right)}{\partial x_i} u_n^{m(x)+1} dxdt \\
&= -\iint_{Q_T} b_i\left(u_n^{m(x)+1}\right)\frac{\partial}{\partial x_i} u_n^{m(x)+1} dxdt
\end{aligned}$$

$$= -\iint_{Q_T}\left[\frac{\partial}{\partial x_i}\int_{(1/n)^{m(x)+1}}^{u_n^{m(x)+1}} b_i(s)\,ds + b_i\left(n^{-m(x)-1}\right)\cdot\left(\frac{1}{n}\right)^{m(x)+1}\log n m_{x_i}(x)\right]dxdt$$
$$= -\iint_{Q_T} b_i\left(n^{-m(x)-1}\right)\cdot\left(\frac{1}{n}\right)^{m(x)+1}\log n m_{x_i}(x)\,dxdt, \tag{46}$$

using the assumption that $b_i(0) = 0$, we have

$$\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i\left(u_n^{m(x+1)}\right)}{\partial x_i}\left(u_n^{m(x)+1}-\left(\frac{1}{n}\right)^{m(x)+1}\right)dxdt = -\sum_{i=1}^{N}\iint_{Q_T} b_i\left(n^{-m(x)-1}\right)\left(\frac{1}{n}\right)^{m(x)+1}\cdot\log n m_{x_i}(x)\,dxdt - \sum_{i=1}^{N}\iint_{Q_T} b_i\left(u_n^{m(x)+1}\right)\cdot\left(\frac{1}{n}\right)^{m(x)+1}\log n m_{x_i}(x)\,dxdt, \tag{47}$$

accordingly

$$\lim_{n\longrightarrow\infty}\left|\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i\left(u_n^{m(x)+1}\right)}{\partial x_i}\left(u_n^{m(x)+1}-\left(\frac{1}{n}\right)^{m(x)+1}\right)dxdt\right| = 0, \tag{48}$$

and so

$$\left|\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i\left(u_n^{m(x)+1}\right)}{\partial x_i}\left(u_n^{m(x)+1}-\left(\frac{1}{n}\right)^{m(x)+1}\right)dxdt\right| \le c. \tag{49}$$

Then by (41), (42), (43), (44), and (49), we have

$$\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}|\nabla u_n|^2\,dxdt \le c. \tag{50}$$

There is a $\overrightarrow{\zeta} \in L^2(Q_T)$ and

$$\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{1/2}\nabla u_n \rightharpoonup \overrightarrow{\zeta}, \tag{51}$$

weakly in $L^2(Q_T)$. We now prove that

$$\overrightarrow{\zeta} = a(x)^{1/2}|u|^{m(x)}\nabla u. \tag{52}$$

For any $\forall\psi \in C_0^\infty(\Omega)$, we have

$$\iint_{Q_T}\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{1/2}\nabla u_n\psi dxdt = \iint_{Q_T}\nabla\left\{\left[\left(a(x)+\frac{1}{n}\right)\cdot\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{1/2}u_n\right\}\psi dxdt - \frac{1}{2}\cdot\iint_{Q_T}\psi u_n\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{-1/2}\cdot\nabla\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]dxdt, \tag{53}$$

$$\lim_{n\longrightarrow\infty}\iint_{Q_T}\nabla\left\{\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{1/2}\cdot u_n\right\}\psi dxdt = -\lim_{n\longrightarrow\infty}\iint_{Q_T}\left\{\left[\left(a(x)+\frac{1}{n}\right)\cdot\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right]^{1/2}u_n\right\}\nabla\psi dxdt = -\iint_{Q_T}a(x)u^{m(x)+1}\nabla\psi dxdt. \tag{54}$$

Denoting that $A_n = [(a(x)+1/n)(|u_n|^{m(x)}+1/n)u_n^{m(x)}]$, then

$$\nabla A_n = \nabla\left[\left(a(x)+\frac{1}{n}\right)\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right] = \nabla a\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)} + \left(a(x)+\frac{1}{n}\right)\nabla\left[\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)}\right] = \nabla a\left(|u_n|^{m(x)}+\frac{1}{n}\right)\cdot u_n^{m(x)} + \left(a(x)+\frac{1}{n}\right)\left(2u_n^{m(x)}+\frac{1}{n}\right)\left(u^{m(x)}\cdot\log u_n\nabla m + u^{m(x)-1}m(x)\nabla u\right) = \nabla a\left(|u_n|^{m(x)}+\frac{1}{n}\right)u_n^{m(x)} + \left(a(x)+\frac{1}{n}\right)\left(2u_n^{m(x)}+\frac{1}{n}\right)u^{m(x)}$$

$$\cdot \log u_n \nabla m + \left(a(x) + \frac{1}{n}\right)\left(2u_n^{m(x)} + \frac{1}{n}\right) \cdot u^{m(x)-1} m(x) \nabla u = I_1 + I_2 + I_3. \quad (55)$$

$$-\frac{1}{2}\iint_{Q_T} \psi u_n \left[\left(a(x) + \frac{1}{n}\right)\left(|u_n|^{m(x)} + \frac{1}{n}\right) u_n^{m(x)}\right]^{-1/2} \cdot \nabla \left[\left(a(x) + \frac{1}{n}\right)\left(|u_n|^{m(x)} + \frac{1}{n}\right) u_n^{m(x)}\right] dxdt = -\frac{1}{2}\iint_{Q_T} \psi u_n A_n^{-1/2} \nabla A_n dxdt = -\frac{1}{2} \cdot \iint_{Q_T} \psi u_n A_n^{-1/2}\left(I_1 + I_2 + I_3\right) dxdt \quad (56)$$

$$-\lim_{n\longrightarrow\infty} \frac{1}{2}\iint_{Q_T} \psi u_n A_n^{-1/2} I_3 dxdt = \lim_{n\longrightarrow\infty} \frac{1}{2} \cdot \iint_{Q_T} \psi u_n A_n^{1/2} \nabla u m(x) \frac{2u_n^{m(x)} + 1/n}{u_n^{m(x)} + 1/n} = -\iint_{Q_T} m(x) \psi \vec{\zeta} dxdt. \quad (57)$$

$$-\lim_{n\longrightarrow\infty} \frac{1}{2}\iint_{Q_T} \psi u_n A_n^{-1/2} I_2 dxdt = -\lim_{n\longrightarrow\infty} \frac{1}{2} \cdot \iint_{Q_T} \left[\left(a(x) + \frac{1}{n}\right)\left(|u_n|^{m(x)} + \frac{1}{n}\right) u_n^{m(x)}\right]^{-1/2} \cdot \psi u_n \left(a(x) + \frac{1}{n}\right)\left(2u_n^{m(x)} + \frac{1}{n}\right) u^{m(x)} \cdot \log u_n \nabla m dxdt = -\iint_{Q_T} \psi a(x)^{1/2} u^{m(x)} \cdot \log u \nabla m dxdt. \quad (58)$$

$$-\lim_{n\longrightarrow\infty} \frac{1}{2}\iint_{Q_T} \psi u_n A_n^{-1/2} I_1 dxdt = -\lim_{n\longrightarrow\infty} \frac{1}{2} \cdot \iint_{Q_T} \left[\left(a(x) + \frac{1}{n}\right)\left(|u_n|^{m(x)} + \frac{1}{n}\right) u_n^{m(x)}\right]^{-1/2} \cdot \psi u_n \nabla a \left(|u_n|^{m(x)} + \frac{1}{n}\right) u_n^{m(x)} dxdt = -\frac{1}{2} \cdot \iint_{Q_T} \psi a(x)^{-1/2} u^{m(x)+1} \nabla a dxdt. \quad (59)$$

Let $n \longrightarrow \infty$ in (53). We obtain that

$$\begin{aligned}\iint_{Q_T} (m(x) + 1) \zeta \psi dxdt &= -\iint_{Q_T} \psi a(x)^{1/2} u^{m(x)} \log u \nabla m \psi dxdt - \frac{1}{2}\iint_{Q_T} \psi a(x)^{-1/2} u^{m(x)+1} \nabla a \psi dxdt \\ &= -\iint_{Q_T} u \nabla \left(a(x)^{1/2} u^{m(x)}\right) \psi dxdt + \iint_{Q_T} a(x)^{1/2} m(x) u^{m(x)} \nabla u \psi dxdt,\end{aligned} \quad (60)$$

which implies

$$\iint_{Q_T} \zeta \psi dxdt = \iint_{Q_T} a(x)^{1/2} u^{m(x)} \nabla u \psi dxdt, \quad (61)$$

Since $b_i \in C^1(\mathbb{R})$, by (40), we have

$$\lim_{n\longrightarrow\infty} b_i\left(u_n^{m(x)+1}\right) = b_i\left(u^{m(x)+1}\right). \quad (62)$$

Letting $n \longrightarrow \infty$ in (42), by (51), (52), and (62), we know $u(x,t)$ satisfies (20). At the same time, the initial value (13) can be proved in a similar way as that when $m(x) = m - 1$ is a constant; one can refer to [5] for the details. Thus, u is a solution of (9) with the initial value (13). If u_0 only satisfies (18), by considering the problem of (36) with the initial value $u_{0\varepsilon}$ which is the mollified function of u_0, then we can get the conclusion by a process of limitation. Certainly, the solution $u(x,t)$ generally is not continuous at $t = 0$, but satisfies (19) and (20). Theorem 2 is proved. □

3. The Stability Based on the Dirichlet Boundary Value Condition

Lemma 6. *If $\int_\Omega a(x)^{-1} dx \leqslant c$, then $\int_\Omega |\nabla u^{m(x)+1}| dx \leqslant c$.*

Proof. Since $\|\sqrt{a(x)} u^{m(x)} |\nabla u|\|_{L^2(Q_T)} \leq c$,

$$\iint_{Q_T} a(x) u^{2m(x)} |\nabla u|^2 dxdt \leq c, \quad (63)$$

which yields

$$\begin{aligned}&\iint_{Q_T} a(x) \left|\nabla u^{m(x)+1}\right|^2 dxdt \\ &\leq 2\iint_{Q_T} a(x) \left|u^{m(x)+1} \log u \nabla m\right|^2 dxdt + 2\iint_{Q_T} a(x) \left|(m(x) + 1) u^{m(x)} \nabla u\right|^2 dxdt \leq c.\end{aligned} \quad (64)$$

Then

$$\begin{aligned}\int_\Omega \left|\nabla u^{m(x)+1}\right| dx &= \int_{\{x\in\Omega: a^{1/2}|\nabla u^{m(x)+1}|\leqslant 1\}} \left|\nabla u^{m(x)+1}\right| dx + \int_{\{x\in\Omega: a^{1/2}|\nabla u^{m(x)+1}|>1\}} \left|\nabla u^{m(x)+1}\right| dx \\ &\leqslant \int_\Omega a(x)^{-1/2} dx + \int_\Omega a(x) \left|\nabla u^{m(x)+1}\right|^2 dx \leqslant c.\end{aligned} \quad (65)$$

Thus $u^{m(x)+1}$ can be defined by the trace on the boundary in the traditional way. By the definition of the trace, we also know that u can be defined by the trace on the boundary in the traditional way. The lemma is proved. □

For every fixed $t \in [0,T)$, we define the Banach space $V_t(\Omega)$ by

$$V_t(\Omega) = \{u(x,t) : u(x,t) \in L^2(\Omega) \bigcap W_0^{1,1}(\Omega),\ |\nabla u(x,t)|^2 \in L^1(\Omega)\}, \quad \|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{2,\Omega}, \tag{66}$$

and denote by $V_t'(\Omega)$ its dual space. In addition, we denote the Banach space $\mathbf{W}(Q_T)$ by

$$\mathbf{W}(Q_T) = \{u : [0,T] \longrightarrow V_t(\Omega) \mid u \in L^2(Q_T), |\nabla u|^2 \in L^1(Q_T),\ u = 0 \text{ on } \Gamma = \partial\Omega\}, \quad \|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{2,Q_T} + \|u\|_{2,Q_T}, \tag{67}$$

and denote by $\mathbf{W}'(Q_T)$ its dual space. According to [18], we know that

$$w \in \mathbf{W}'(Q_T) \Longleftrightarrow \begin{cases} w = w_0 + \sum_{i=1}^{N} D_i w_i, & w_0 \in L^2(Q_T),\ w_i \in L^2(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T), & \ll w, \phi \gg = \iint_{Q_T} \left(w_0 \phi + \sum_i^N w_i D_i \phi \right) dx\,dt. \end{cases} \tag{68}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\{\ll v, \phi \gg \mid \phi \in \mathbf{W}(\mathbf{Q_T}),\ \|\phi\|_{\mathbf{W}(Q_T)} \le 1\}. \tag{69}$$

Lemma 7. *If $u(x,t)$ is a weak solution of (9) with the initial value (13), then $u_t \in \mathbf{W}'(Q_T)$.*

Proof. For any $v \in \mathbf{W}(Q_T)$ and $\|v\|_{W(Q_T)} = 1$, there holds

$$\langle u_t, v \rangle = -\iint_{Q_T} a(x) u^{m(x)} \nabla u \nabla v\,dx\,dt - \sum_{i=1}^{N} \iint_{Q_T} v_{x_i} b_i\left(u^{m(x)+1}\right) dx\,dt. \tag{70}$$

By Young's inequality, it follows from (70) that

$$|\langle u_{\varepsilon t}, v \rangle| \le c \left[\iint_{Q_T} a(x) |u|^{2m(x)} |\nabla u|^2\,dx\,dt + \iint_{Q_T} \left(|v|^2 + |\nabla v|^2 \right) dx\,dt + 1 \right] \le c, \tag{71}$$

we have

$$\|u_t\|_{\mathbf{W}'(Q_T)} \le c. \tag{72}$$

□

Lemma 8. *Suppose that $u \in \mathbf{W}(Q_T)$ and $u_t \in \mathbf{W}'(Q_T)$. For any continuous function $h(s)$, let $H(s) = \int_0^s h(s)ds$. For a.e. $t_1, t_2 \in (0,T)$, there holds*

$$\int_{t_1}^{t_2} \int_\Omega h(u) u_t\,dx\,dt = \left[\int_\Omega \left(H(u)(x,t_2) - H(u)(x,t_1) \right) dx \right]. \tag{73}$$

This lemma can be found in [18].

Theorem 9. *Let $\int_\Omega a^{-1}(x)dx < \infty$ and let u and v be two solutions of (9) with the initial values $u_0(x), v_0(x)$, respectively, and with the same homogeneous boundary value conditions*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{74}$$

Then

$$\int_\Omega |u(x,t) - v(x,t)| \le c \int_\Omega |u_0(x) - v_0(x)|\,dx. \tag{75}$$

Proof. For any given positive integer n, let $g_n(s) = \int_0^s h_n(\tau)d\tau$, $h_n(s) = 2n(1 - n|s|)_+$. Then $h_n(s) \in C(\mathbb{R})$, and we have

$$h_n(s) \ge 0, \quad |s h_n(s)| \le 1, \quad |g_n(s)| \le 1, \tag{76}$$

and

$$\lim_{n\to\infty} g_n(s) = \operatorname{sgn} s, \quad \lim_{n\to\infty} s g_n'(s) = 0. \tag{77}$$

By the definition of weak solutions, we have

$$\iint_{Q_T} u_t \varphi(x,t)\,dx\,dt + \iint_{Q_T} \frac{a(x)}{m(x)+1} \nabla u^{m(x)+1} \nabla\varphi\,dx\,dt - \iint_{Q_T} \frac{a(x)}{m(x)+1} u^{m(x)+1} \log u \nabla m \nabla\varphi\,dx\,dt + \sum_{i=1}^{N} \iint_{Q_T} b_i\left(u^{m(x)+1}\right) \varphi_{x_i}\,dx\,dt = 0. \tag{78}$$

Since $\int_{\Omega} a(x)^{-1}dx \leqslant c$, $u = v = 0$ on the boundary, we can choose $g_n(u^{m(x)+1} - v^{m(x)+1})$ as the test function. Then

$$\begin{aligned}
&\int_{Q_T} g_n\left(u^{m(x)+1} - v^{m(x)+1}\right)\frac{\partial (u-v)}{\partial t}dxdt \\
&+ \iint_{Q_T} \frac{a(x)}{m(x)+1}\left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2 \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)dxdt \\
&- \iint_{Q_T} \frac{a(x)}{m(x)+1}\left(u^{m(x)+1}\log u - v^{m(x)+1}\log u\right) \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)\nabla m \nabla\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt \\
&+ \sum_{i=1}^{N}\iint_{Q_T}\left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right] \\
&\cdot\left(u^{m(x)+1} - v^{m(x)+1}\right)_{x_i} \\
&\cdot g_n'\left(\left(u^{m(x)+1} - v^{m(x)+1}\right)\right)dxdt = 0.
\end{aligned} \tag{79}$$

$$\begin{aligned}
&\iint_{Q_T} \frac{a(x)}{m(x)+1}\left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2 g_n'\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt \geq 0.
\end{aligned} \tag{80}$$

and

$$\begin{aligned}
&- \iint_{Q_T} \frac{a(x)}{m(x)+1}\left(u^{m(x)+1}\log u - v^{m(x)+1}\log u\right) \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)\nabla m \nabla\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt \geq -\frac{1}{2} \\
&\cdot \iint_{Q_T} \frac{a(x)}{m(x)+1}\left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2 g_n'\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt - \frac{1}{2} \\
&\cdot \iint_{Q_T} \frac{a(x)}{m(x)+1}\left[\left(u^{m(x)+1}\log u - v^{m(x)+1}\log u\right)\right. \\
&\left.\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)\right]^2 dxdt.
\end{aligned} \tag{81}$$

For simplism, in what follows, we denote that

$$\begin{aligned}
D_n &= \left\{\Omega : \left|u^{m(x)+1} - v^{m(x)+1}\right| < \frac{1}{n}\right\}, \\
D_0 &= \{x \in \Omega : |u - v| = 0\}
\end{aligned} \tag{82}$$

and, clearly,

$$\lim_{n\longrightarrow\infty} D_n = D_0. \tag{83}$$

We have

$$\begin{aligned}
&\left|\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right]\left(u^{m(x)+1} - v^{m(x)+1}\right)_{x_i}\right. \\
&\left.\cdot g_n'\left(\left(u^{m(x)+1} - v^{m(x)+1}\right)\right)dx\right| \\
&= \left|\int_{D_n}\left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right]\cdot g_n'\left(u^{m(x)+1}\right.\right. \\
&\left.\left.- v^{m(x)+1}\right)\left(u^{m(x)+1} - v^{m(x)+1}\right)_{x_i}dx\right| \\
&\leqslant c\int_{D_n}\left|\frac{b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)}{u^{m(x)+1} - v^{m(x)+1}}\right|\left|\left(u^{m(x)+1}\right.\right. \\
&\left.\left.- v^{m(x)+1}\right)_{x_i}\right|dx = c\int_{D_n}\left|a^{-1/2}\right. \\
&\left.\cdot\frac{b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)}{u^{m(x)+1} - v^{m(x)+1}}\right|\left|a(x)\left(u^{m(x)+1}\right.\right. \\
&\left.\left.- v^{m(x)+1}\right)_{x_i}\right|dx \leqslant c\left[\int_{D_n}\left(a^{-1/2}\right.\right. \\
&\left.\left.\cdot\frac{b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)}{u^{m(x)+1} - v^{m(x)+1}}\right)^2 dx\right]^{1/2} \\
&\cdot\left[\int_{D_n} a(x)\left|\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right)\right|^2 dx\right]^{1/2}.
\end{aligned} \tag{84}$$

Since $\int_{\Omega} a^{-1}(x)dx < \infty$,

$$\begin{aligned}
&\int_{D_n}\left(a^{-1/2}\frac{b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)}{u^{m(x)+1} - v^{m(x)+1}}\right)^2 dx \\
&\leqslant c\int_{D_n} a(x)^{-1}dx \leqslant c.
\end{aligned} \tag{85}$$

If $D_0 = \{x \in \Omega : |u^{m(x)+1} - v^{m(x)+1}| = 0\}$ is with 0 measure, we have

$$\lim_{n\longrightarrow\infty}\int_{D_n} a(x)^{-1}dx = \int_{D_0} a(x)^{-1}dx = 0. \tag{86}$$

If the set $D_0 = \{x \in \Omega : |u^{m(x)+1} - v^{m(x)+1}| = 0\}$ has a positive measure, then,

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{D_n} a(x)\left|\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right)\right|^2 dx \\
&= \int_{D_0} a(x)\left|\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right)\right|^2 dx = 0.
\end{aligned} \tag{87}$$

Therefore, in both cases,

$$\lim_{n\to\infty}\int_\Omega \left(b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right) \cdot g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)\left(u^{m(x)+1}-v^{m(x)+1}\right)_{x_i}dx = 0. \tag{88}$$

At last,

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega g_n\left(u^{m(x)+1}-v^{m(x)+1}\right)\frac{\partial(u-v)}{\partial t}dx &= \int_\Omega \operatorname{sgn}\left(u^{m(x)+1}-v^{m(x)+1}\right)\frac{\partial(u-v)}{\partial t}dx \\ &= \int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t} = \frac{d}{dt}\|u-v\|_1.\end{aligned} \tag{89}$$

Let $n\longrightarrow\infty$ in (79). By (80), (81), (88), and (89), we have

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \leqslant \int_\Omega \left|u_0(x)-v_0(x)\right|dx = 0. \tag{90}$$

□

Corollary 10. *Theorem 3 is true.*

4. The Stability without the Boundary Value Condition

In this section, we will prove Theorem 4.

Theorem 11. *Let u, v be two nonnegative solutions of (9) with the initial values $u_0(x), v_0(x)$, respectively. If $a(x)$ satisfies (17) and*

$$\int_\Omega a(x)^{-1}|\nabla a|^2\,dx \le c, \quad |\nabla a| = 0,\ x\in\partial\Omega, \tag{91}$$

u and v satisfy

$$\begin{aligned}&\int_\Omega a(x)\left[1+(m(x)+1)\log u\right]^2|\nabla m|^2\,dx\le c,\\ &\int_\Omega a(x)\left[1+(m(x)+1)\log v\right]^2|\nabla m|^2\,dx\le c,\end{aligned} \tag{92}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \le c\int_\Omega \left|u_0(x)-v_0(x)\right|dx. \tag{93}$$

Proof. For all $0\le\varphi\in C_0^1(Q_T)$, by the definition of weak solutions, for all $0\le\varphi\in C_0^1(Q_T)$, we have

$$\begin{aligned}&\iint_{Q_T} u_t\varphi(x,t)\,dxdt + \iint_{Q_T}\frac{a(x)}{m(x)+1}\nabla u^{m(x)+1}\nabla\varphi dxdt \\ &\quad - \iint_{Q_T}\frac{a(x)}{m(x)+1}u^{m(x)+1}\log u\nabla m\nabla\varphi dxdt + \sum_{i=1}^N\iint_{Q_T} b_i\left(u^{m(x)+1}\right)\varphi_{x_i}dxdt = 0.\end{aligned} \tag{94}$$

Let $\chi_{\tau,s}(t)$ be the characteristic function of $[\tau,s]\subset(0,T)$. By a process of limit, we can choose

$$\chi_{\tau,s}(t)\,g_n\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right) \tag{95}$$

as the test function; then

$$\begin{aligned}&\int_\tau^s\int_\Omega g_n\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)\frac{\partial(u-v)}{\partial t}dxdt + \int_\tau^s\int_\Omega\frac{a(x)^{r+1}}{m(x)+1}\left(\nabla u^{m(x)+1}-\nabla v^{m(x)+1}\right)\cdot\nabla\left(u^{m(x)+1}-v^{m(x)+1}\right)\\
&\cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dxdt + r\int_\tau^s\int_\Omega\frac{a(x)^r}{m(x)+1}\left(\nabla u^{m(x)+1}-\nabla v^{(m(x)+1)}\right)\cdot\nabla a\left(u^{m(x)+1}-v^{m(x)+1}\right)\\
&\cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dxdt - \int_\tau^s\int_\Omega\frac{a(x)^{r+1}}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m\cdot\nabla\left(u^{m(x)+1}-v^{m(x)+1}\right)\\
&\cdot g_n'\left(a\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dxdt - r\int_\tau^s\int_\Omega\frac{a(x)^r}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m\cdot\nabla a\left(u^{m(x)+1}-v^{m(x)+1}\right)\\
&\cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dxdt + \sum_{i=1}^N\int_\tau^s\int_\Omega\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\\
&\cdot\left[ra^{r-1}a_{x_i}(x)\left(u^{m(x)+1}-v^{m(x)+1}\right)\cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)+a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)_{x_i}g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)\right]dxdt\\
&= 0.\end{aligned} \tag{96}$$

Let us analyse every term in (96). Firstly, we have

$$\int_\Omega \frac{a(x)^{r+1}}{m(x)+1}\left(\nabla u^{m(x)+1}-\nabla v^{m(x)+1}\right) \cdot \nabla\left(u^{m(x)+1}-v^{m(x)+1}\right) \cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dx \geq 0. \tag{97}$$

Secondly, since $\int_\Omega a(x)^{-1}|\nabla a|^2 dx \leq c$,

$$\begin{aligned}&\left|\int_\Omega \frac{a(x)^r}{m(x)+1}\left(\nabla u^{m(x)+1}-\nabla v^{(m(x)+1)}\right)\cdot\nabla a\left(u^{m(x)+1}-v^{m(x)+1}\right)g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dx\right| \\ &\leq \int_\Omega \frac{a(x)^r}{m(x)+1}\left(\left|\nabla u^{m(x)+1}\right|+\left|\nabla v^{(m(x)+1)}\right|\right)|\nabla a| \cdot \left|u^{m(x)+1}-v^{m(x)+1}\right| \cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dx \leq c\left(\int_\Omega a(x)\left(\left|\nabla u^{m(x)+1}\right|^2 + \left|\nabla v^{m(x)+1}\right|^2\right)dx\right)^{1/2} \\ &\cdot\left(\int_\Omega a(x)\left[\frac{|\nabla a|}{a}\cdot a^r\left(u^{m(x)+1}-v^{m(x)+1}\right) \cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)\right]^2dx\right)^{1/2} \longrightarrow 0,\end{aligned} \tag{98}$$

as $\lambda \longrightarrow 0$.

By that

$$\begin{aligned}&\lim_{n\longrightarrow 0}\left|\int_{D_n} a(x)\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^2 g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^2|\nabla m|^2dx\right| \\ &\leq \lim_{n\longrightarrow 0}\int_{D_n} a(x)\frac{\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^2}{\left|u^{m(x)+1}-v^{m(x)+1}\right|^2}|\nabla m|^2dx \\ &\leq \int_{D_0} a(x)\left[1+(m(x)+1)\log\zeta\right]^2|\nabla m|^2dx < \infty.\end{aligned} \tag{99}$$

where $\zeta \in (u,v)$ in the mean value, we have

$$\begin{aligned}&\left|\int_{D_n} a(x)\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^2 g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^2|\nabla m|^2dx\right| \\ &\leq \int_{D_n} a(x)\frac{\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^2}{\left|u^{m(x)+1}-v^{m(x)+1}\right|^2}|\nabla m|^2\cdot\left|u^{m(x)+1}-v^{m(x)+1}\right|^2 g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^2dx \\ &< \infty.\end{aligned} \tag{100}$$

Using the Lebesgue dominated convergence theorem, we have

$$\lim_{n\longrightarrow\infty}\int_{D_n} a(x) \cdot\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^2 \cdot g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^2|\nabla m|^2dx = 0, \tag{101}$$

and so

$$\left|\int_\Omega \frac{a(x)}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m \cdot a^r\nabla\left(u^{m(x)+1}-v^{m(x)+1}\right) \cdot g_n'\left(a^r\left(u^{m(x)+1}-v^{m(x)+1}\right)\right)dxdt\right|$$

$$\begin{aligned}
&\leq \left(\int_{D_n} a(x) \left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2 dx\right)^{1/2} \\
&\cdot c\left(\int_{D_n} a(x)\right. \\
&\cdot \left[u^{m(x)+1}\log u^{m(x)+1} - v^{m(x)+1}\log v^{m(x)+1}\right]^2 \\
&\left.\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)^2 |\nabla m|^2 dx\right)^{1/2} \longrightarrow 0,
\end{aligned} \tag{102}$$

as $\lambda \longrightarrow 0$.

Thirdly, since $|\nabla a|_{x\in\partial\Omega} = 0$,

$$\begin{aligned}
&\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\left|\int_\Omega \left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\right. \\
&\left.\cdot g_n\left(u^{m(x)+1} - v^{m(x)+1}\right) dx\right| \\
&\leq \lim_{\lambda\longrightarrow 0}\int_\Omega \left|b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right|\left|\phi_{\lambda x_i}(x)\right| dx \\
&\leq c\lim_{\lambda\longrightarrow 0}\frac{1}{\lambda}\int_{\Omega\setminus\Omega_\lambda}|\nabla a|\, dx = \int_{\partial\Omega}|\nabla a|\, d\Sigma = 0.
\end{aligned} \tag{103}$$

Moreover, as in the proof of Theorem 3, we can show that

$$\begin{aligned}
&\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right] \\
&\cdot \phi_\lambda\left(u^{m(x)+1} - v^{m(x)+1}\right)_{x_i} \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right) dx dt = 0,
\end{aligned} \tag{104}$$

and, clearly,

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_\tau^s\int_\Omega g_n\left(a^r\left(u^{m(x)+1} - v^{m(x)+1}\right)\right) \\
&\cdot \phi_\lambda\frac{\partial(u-v)}{\partial t} dx = \int_\Omega |u(x,s) - v(x,s)|\, dx \\
&- \int_\Omega |u(x,\tau) - v(x,\tau)|\, dx.
\end{aligned} \tag{105}$$

At last, let $n \longrightarrow \infty$ in (96). Then

$$\begin{aligned}
&\int_\Omega |u(x,s) - v(x,s)|\, dx \\
&\quad \leq c\int_\Omega |u(x,\tau) - v(x,\tau)|\, dx.
\end{aligned} \tag{106}$$

By the arbitraries of τ, we have

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \leqslant \int_\Omega \left|u_0(x) - v_0(x)\right| dx. \tag{107}$$

□

5. The Stability Based on the Partial Boundary Value Condition

In this section, we will prove Theorem 5. We assume that

$$\left|b_i(s_1) - b_i(s_2)\right| \leq c_i\left|s_1 - s_2\right|, \quad i = 1, 2, \ldots, N. \tag{108}$$

we denote that $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$ as before and denote that

$$\Omega_{\lambda i1} = \left\{x \in \Omega\setminus\Omega_\lambda : a_{x_i} < 0\right\}, \quad \Sigma_{1i} = \lim_{\lambda\longrightarrow 0}\Omega_{\lambda i1}, \tag{109}$$

$$\Omega_{\lambda i2} = \left\{x \in \Omega\setminus\Omega_\lambda : a_{x_i} \geq 0\right\}, \quad \Sigma_{2i} = \lim_{\lambda\longrightarrow 0}\Omega_{\lambda i2}. \tag{110}$$

Theorem 12. *Let u, v be two solutions of (9) with the initial values $u_0(x), v_0(x)$, respectively, and with a partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T). \tag{111}$$

Here, for any given $i \in \{1, 2, \ldots, N\}$, if $b_i'(s) \geq 0$,

$$\Sigma_1 = \left\{x \in \partial\Omega : x \in \Sigma_{1i}\right\}; \tag{112}$$

if $b_i'(s) \geq 0$,

$$\Sigma_1 = \left\{x \in \partial\Omega : x \in \Sigma_{2i}\right\}. \tag{113}$$

It is supposed that $a(x)$ satisfies (17) and

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^2 dx\right)^{1/2} \leq c, \tag{114}$$

u and v satisfy

$$\begin{aligned}
&\int_\Omega a(x)\left[1 + (m(x)+1)\log u\right]^2 |\nabla m|^2 dx < \infty, \\
&\int_\Omega a(x)\left[1 + (m(x)+1)\log v\right]^2 |\nabla m|^2 dx < \infty.
\end{aligned} \tag{115}$$

Then

$$\begin{aligned}
&\int_\Omega |u(x,t) - v(x,t)|\, dx \leq c\int_\Omega \left|u_0(x) - v_0(x)\right| dx, \\
&\qquad\qquad \text{a.e.} \in [0,T).
\end{aligned} \tag{116}$$

Proof. For all $0 \leq \varphi \in C_0^1(Q_T)$, by the definition of weak solutions, for all $0 \leq \varphi \in C_0^1(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} u_t\varphi(x,t)\, dx dt \\
&\quad + \iint_{Q_T}\frac{a(x)}{m(x)+1}\nabla u^{m(x)+1}\nabla\varphi dx dt \\
&\quad - \iint_{Q_T}\frac{a(x)}{m(x)+1}u^{m(x)+1}\log u\nabla m\nabla\varphi dx dt \\
&\quad + \sum_{i=1}^N\iint_{Q_T} b_i\left(u^{m(x)+1}\right)\varphi_{x_i} dx dt = 0.
\end{aligned} \tag{117}$$

For a small positive constant $\lambda > 0$, let

$$\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}, \tag{118}$$

and

$$\phi_\lambda(x) = \begin{cases} 1, & \text{if } x \in \Omega_\lambda, \\ \dfrac{a(x)}{\lambda}, & x \in \Omega \setminus \Omega_\lambda. \end{cases} \tag{119}$$

Let $\chi_{\tau,s}(t)$ be the characteristic function of $[\tau, s] \subset (0, T)$. By a process of limit, we can choose

$$\chi_{\tau,s}(t)\,\phi_\lambda(x)\,g_n\left(u^{m(x)+1} - v^{m(x)+1}\right) \tag{120}$$

as the test function; then

$$\begin{aligned}
&\int_\tau^s \int_\Omega \phi_\lambda(x)\,g_n\left(u^{m(x)+1} - v^{m(x)+1}\right)\frac{\partial(u-v)}{\partial t}dxdt + \int_\tau^s\int_\Omega \frac{a(x)}{m(x)+1}\left(\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right)\cdot\phi_\lambda(x)\,\nabla\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)dxdt + \int_\tau^s\int_\Omega\frac{a(x)}{m(x)+1}\left(\nabla u^{m(x)+1} - \nabla v^{(m(x)+1)}\right)\cdot\nabla\phi_\lambda(x)\,g_n\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt - \int_\tau^s\int_\Omega\frac{a(x)}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m\cdot\phi_\lambda(x)\,\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right) \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)dxdt - \int_\tau^s\int_\Omega\frac{a(x)}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m\cdot\nabla\phi_\lambda(x)\,g_n\left(u^{m(x)+1}\right. \\
&\left.- v^{m(x)+1}\right)dxdt + \sum_{i=1}^N\int_\tau^s\int_\Omega\left[b_i\left(u^{m(x)+1}\right) - b_i\left(v^{m(x)+1}\right)\right] \\
&\cdot\left[\phi_{\lambda x_i}(x)\,g_n\left(u^{m(x)+1} - v^{m(x)+1}\right) + \phi_\lambda\left(u^{m(x)+1} - v^{m(x)+1}\right)_{x_i} g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)\right]dxdt = 0.
\end{aligned} \tag{121}$$

Let us analyse every term in (121):

$$\begin{aligned}
&\int_\Omega\frac{a(x)}{m(x)+1}\left(\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right) \\
&\cdot\phi_\lambda\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right) \\
&\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)dx \ge 0.
\end{aligned} \tag{122}$$

By (114), $(1/\lambda)(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^2dx)^{1/2} \le c$,

$$\begin{aligned}
&\left|\int_\Omega\frac{a(x)}{m(x)+1}\left(\nabla u^{m(x)+1} - \nabla v^{(m(x)+1)}\right)\right. \\
&\left.\cdot\nabla\phi_\lambda g_n\left(u^{m(x)+1} - v^{m(x)+1}\right)dx\right| \\
&\le \int_{\Omega\setminus\Omega_\lambda}\left|\frac{a(x)}{m(x)+1}\left(\nabla u^{m(x)+1} - \nabla v^{(m(x)+1)}\right)\right. \\
&\left.\cdot\nabla\phi_\lambda g_n(u^{m(x)+1} - v^{m(x)+1}\right|dx \\
&\le \int_{\Omega\setminus\Omega_\lambda}\frac{a(x)}{m(x)+1}\left(\left|\nabla u^{m(x)+1}\right| + \left|\nabla v^{(m(x)+1)})\right|\right) \\
&\cdot\left|\nabla\phi_\lambda\right|dx \le \frac{1}{\lambda}\int_{\Omega\setminus\Omega_\lambda}\frac{a(x)}{m(x)+1}\left(\left|\nabla u^{m(x)+1}\right|\right. \\
&\left.+\left|\nabla v^{(m(x)+1)})\right|\right)|\nabla a|\,dx \le c\left(\int_{\Omega\setminus\Omega_\lambda}a(x)\right. \\
&\left.\cdot\left(\left|\nabla u^{m(x)+1}\right|^2 + \left|\nabla v^{m(x)+1}\right|^2\right)dx\right)^{1/2} \\
&\cdot\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda}a(x)\,|\nabla a|^2\,dx\right)^{1/2} \longrightarrow 0,
\end{aligned} \tag{123}$$

as $\lambda \longrightarrow 0$.

$$\begin{aligned}
&\left|\int_\Omega\frac{a(x)}{m(x)+1}\left[u^{m(x)+1}\log u - v^{m(x)+1}\log v\right]\nabla m\right. \\
&\cdot\phi_\lambda(x)\,\nabla\left(u^{m(x)+1} - v^{m(x)+1}\right) \\
&\left.\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)dxdt\right| \le \left(\int_{D_n}a(x)\right. \\
&\left.\cdot\left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2dx\right)^{1/2}\cdot c\left(\int_{D_n}a(x)\right. \\
&\cdot\left[u^{m(x)+1}\log u^{m(x)+1} - v^{m(x)+1}\log v^{m(x)+1}\right]^2 \\
&\left.\cdot g_n'\left(u^{m(x)+1} - v^{m(x)+1}\right)^2|\nabla m|^2\,dx\right)^{1/2} \longrightarrow 0,
\end{aligned} \tag{124}$$

as $\lambda \longrightarrow 0$.

This is due to the fact that if $D_0 = \{x \in \Omega : |u-v| = 0\}$ is with 0 measure,

$$\begin{aligned}
&\lim_{n\longrightarrow 0}\int_{D_n}a(x)\left|\nabla u^{m(x)+1} - \nabla v^{m(x)+1}\right|^2dx \\
&\le \int_{D_0}a(x)\left(\left|\nabla u^{m(x)+1}\right|^2 + \left|\nabla v^{m(x)+1}\right|^2\right)dx = 0,
\end{aligned} \tag{125}$$

and by (115)

$$\int_{\Omega} a(x)\left[1+(m(x)+1)\log u\right]^{2}|\nabla m|^{2}\,dx<\infty, \tag{126}$$

we have

$$\begin{aligned}&\lim_{n\longrightarrow 0}\left|\int_{D_n} a(x)\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^{2} g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^{2}|\nabla m|^{2}\,dx\right|\\&\le\lim_{n\longrightarrow 0}\int_{D_n} a(x)\frac{\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^{2}}{\left|u^{m(x)+1}-v^{m(x)+1}\right|^{2}}|\nabla m|^{2}\,dx\\&\le\int_{D_0} a(x)\left[1+(m(x)+1)\log\zeta\right]^{2}|\nabla m|^{2}\,dx<\infty.\end{aligned} \tag{127}$$

where $\zeta\in(u,v)$ in the mean value. If $D_0=\{x\in\Omega:|u-v|=0\}$ has a positive measure,

$$\lim_{n\longrightarrow 0}\int_{D_n} a(x)\left|\nabla u^{m(x)+1}-\nabla v^{m(x)+1}\right|^{2}dx$$

$$=\int_{\Omega} a(x)\left(\left|\nabla u^{m(x)+1}\right|^{2}+\left|\nabla v^{m(x)+1}\right|^{2}\right)dx<\infty, \tag{128}$$

and by (115)

$$\begin{aligned}&\lim_{n\longrightarrow 0}\left|\int_{\{\Omega:|u^{m(x)+1}-v^{m(x)+1}|<1/n\}} a(x)\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^{2} g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)^{2}|\nabla m|^{2}\,dx\right|\\&\le\lim_{n\longrightarrow 0}\int_{\{\Omega:|u-v|=0\}} a(x)\frac{\left[u^{m(x)+1}\log u^{m(x)+1}-v^{m(x)+1}\log v^{m(x)+1}\right]^{2}}{\left|u^{m(x)+1}-v^{m(x)+1}\right|^{2}}|\nabla m|^{2}\,dx=0.\end{aligned} \tag{129}$$

Moreover, recall that

$$\begin{aligned}\Omega_{\lambda i1}&=\left\{x\in\Omega\setminus\Omega_\lambda: a_{x_i}<0\right\},\quad \Sigma_{1i}=\lim_{\lambda\longrightarrow 0}\Omega_{\lambda i1},\\ \Omega_{\lambda i2}&=\left\{x\in\Omega\setminus\Omega_\lambda: a_{x_i}\ge 0\right\},\quad \Sigma_{2i}=\lim_{\lambda\longrightarrow 0}\Omega_{\lambda i2}.\end{aligned} \tag{130}$$

If $b_i'(s)\ge 0$, by the partial boundary value condition (111)

$$\begin{aligned}&-\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot g_n\left(u^{m(x)+1}-v^{m(x)+1}\right)dx\\&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot\operatorname{sign}\left(u^{m(x)+1}-v^{m(x)+1}\right)dx\\&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot\operatorname{sign}\left(b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right)dx\\&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega\setminus\Omega_\lambda}\left|b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right|\frac{a_{x_i}}{\lambda}dx\end{aligned}$$

$$\begin{aligned}&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega\setminus\Omega_\lambda}\left|u^{m(x)+1}-v^{m(x)+1}\right|b_i'(\xi)\frac{a_{x_i}}{\lambda}dx\\&\le\lim_{\lambda\longrightarrow 0}\frac{1}{\lambda}\int_{\Omega_{\lambda i1}}\left(-c_i a_{x_i}\right)\left|u^{m(x)+1}-v^{m(x)+1}\right|dx\\&=\int_{\Sigma_{1i}}\left(-c_i a_{x_i}\right)\left|u^{m(x)+1}-v^{m(x)+1}\right|d\Sigma=0.\end{aligned} \tag{131}$$

If $b_i'(s)<0$, by the partial boundary value condition (111)

$$\begin{aligned}&-\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot g_n\left(u^{m(x)+1}-v^{m(x)+1}\right)dx\\&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot\operatorname{sign}\left(u^{m(x)+1}-v^{m(x)+1}\right)dx\\&=-\lim_{\lambda\longrightarrow 0}\int_{\Omega}\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right]\phi_{\lambda x_i}(x)\\&\cdot\operatorname{sign}\left(b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right)dx\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{\lambda\longrightarrow 0}\int_{\Omega\setminus\Omega_\lambda}\left|b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right|\frac{a_{x_i}}{\lambda}dx \\
&= -\lim_{\lambda\longrightarrow 0}\int_{\Omega\setminus\Omega_\lambda}\left|u^{m(x)+1}-v^{m(x)+1}\right|b_i'(\xi)\frac{a_{x_i}}{\lambda}dx \\
&\leq \lim_{\lambda\longrightarrow 0}\frac{1}{\lambda}\int_{\Omega_{\lambda i2}}\left(-c_i a_{x_i}\right)\left|u^{m(x)+1}-v^{m(x)+1}\right|dx \\
&= \int_{\Sigma_{2i}}\left(-c_i a_{x_i}\right)\left|u^{m(x)+1}-v^{m(x)+1}\right|d\Sigma = 0.
\end{aligned}
\tag{132}
$$

As in the proof of Theorem 11 we can show that

$$
\begin{aligned}
&\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_\tau^s\int_\Omega\left[b_i\left(u^{m(x)+1}\right)-b_i\left(v^{m(x)+1}\right)\right] \\
&\cdot\phi_\lambda\left(u^{m(x)+1}-v^{m(x)+1}\right)_{x_i} \\
&\cdot g_n'\left(u^{m(x)+1}-v^{m(x)+1}\right)dxdt = 0.
\end{aligned}
\tag{133}
$$

Clearly,

$$
\begin{aligned}
&\lim_{n\longrightarrow\infty}\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_\Omega g_n\left(u^{m(x)+1}-v^{m(x)+1}\right) \\
&\cdot\phi_\lambda\frac{\partial(u-v)}{\partial t}dx = \int_\Omega|u(x,s)-v(x,s)|dx \\
&-\int_\Omega|u(x,\tau)-v(x,\tau)|dx.
\end{aligned}
\tag{134}
$$

Now, after letting $\lambda\longrightarrow 0$, let $n\longrightarrow\infty$ in (121). Then

$$
\begin{aligned}
&\int_\Omega|u(x,s)-v(x,s)|dx \\
&\quad\leq c\int_\Omega|u(x,\tau)-v(x,\tau)|dx.
\end{aligned}
\tag{135}
$$

By the arbitraries of τ, we have

$$
\int_\Omega|u(x,t)-v(x,t)|dx\leqslant\int_\Omega\left|u_0(x)-v_0(x)\right|dx. \tag{136}
$$

□

Proof of Theorem 5. Since we suppose that, for every $i\in\{1,2,\ldots,N\}$, either $b_i'(s)\geq 0$ or $b_i'(s)\leq 0$, by checking the process of the proof of (131) or (132), we can easily obtain the conclusion of Theorem 5. □

6. Conclusion

The evolutionary equations with variable exponents, especially the so-called electrorheological fluids equations with the form (15), have been brought to the forefront by many scholars since the beginning of this century. There are more or less beyond one's imagination; there are only a few references devoted to the porous medium with variable exponents as (16). So, this paper fills the gaps in the related fields. Moreover, the equation considered in this paper is more general than (16). The most important characteristic lies in that there is a degenerate diffusion coefficient $a(x)$ in the equation. This characteristic may make the usual Dirichlet boundary value condition overdetermined and so a partial boundary value condition is expected. The conclusions in this paper answer the problem partially. In addition, since the equation is with variable exponents, there are many technique difficulties to be overcome. This makes our paper contain many cumbersome calculations, but it is necessary.

Data Availability

There is not any data in this paper.

Conflicts of Interest

The author declares that he has no competing interests.

Acknowledgments

The paper is supported by the Natural Science Foundation of Fujian Province (2019J01858) and the Open Research Fund Program from Fujian Engineering and Research Center of Rural Sewage Treatment and Water Safety, China.

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