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Research Article

On Third-Order Linear Recurrent Functions

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A function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Tribonacci function with period p if $\psi(x+3p) = \psi(x+2p) + \psi(x+p) + \psi(x)$, for all $x \in \mathbb{R}$. In this paper, we present some properties on the Tribonacci functions with period p. We show that if ψ is a Tribonacci function with period p, then $\lim_{x\longrightarrow\infty}(\psi(x+p)/\psi(x)) = \beta$, where β is the root of the equation $x^3 - x^2 - x - 1 = 0$ such that $1 < \beta < 2$.

1. Introduction

The most popular numbers studied in many different forms for centuries are Fibonacci numbers. Fibonacci sequence is famous for their amazing properties (see [1–5]). Many research articles talk about these numbers. The third-order linear recurrence of these sequences are what we call *Tribonacci* sequences $\{T_n\}$ by the recurrence relation $T_{n+2} = T_{n+1} + T_n + T_{n-1}$ with $T_0 = a$, $T_1 = b$, $T_2 = c$, $a, b, c \in \mathbb{N}$ and n as an integer. Han, Kim, and Neggers studied Fibonacci numbers [6–8] and introduced the concept of Fibonacci functions with Fibonacci numbers in [8] which were later extended by B. Sroysang [9] to Fibonacci functions with period p. In the same order as in [8], Parizi and Gordji [10] studied Tribonacci functions. They gave some properties of Tribonacci functions: a function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Tribonacci function if $\psi(x+3) = \psi(x+2)+\psi(x+1)+\psi(x)$, for all $x \in \mathbb{R}$. They also showed that if ψ is a Tribonacci function, then $\lim_{x\longrightarrow\infty}(\psi(x+1)/\psi(x)) = \beta = (1/3)[(19+3\sqrt{33})^{1/3} + (19-3\sqrt{33})^{1/3} + 1]$ which is a root of $x^3 - x^2 - x - 1 = 0$.

In this paper, for any positive integer p, a function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Tribonacci function with period p if $\psi(x+3p) = \psi(x+2p) + \psi(x+p) + \psi(x)$, for all $x \in \mathbb{R}$. In Sections 2 and 3, we present some properties of these functions. In Section 4, we develop the notions of these functions using the concept of even and odd functions discussed in [8, 10]. We also show that if ψ is a Tribonacci function with period p, then $\lim_{x\longrightarrow\infty}(\psi(x+p)/\psi(x)) = \beta$.

2. Tribonacci Functions with Period p

In this section, we present some properties of Tribonacci functions with period p.

Definition 1. Let p be a positive integer. A function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Tribonacci function with period p if $\psi(x+3p) = \psi(x+2p) + \psi(x+p) + \psi(x)$, for all $x \in \mathbb{R}$.

Example 2. Let β be a positive root of the equation $x^3 - x^2 - x - 1 = 0$. Then, $\beta^3 = \beta^2 + \beta + 1$. Define a map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(x) = \beta^{x/p}$, where $p \in \mathbb{N}$. Then, φ is a Tribonacci function with period p.

Proposition 3. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Tribonacci function with period $p \in \mathbb{N}$ and define $h_t(x) = \psi(x+t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, h_t is also a Tribonacci function with period p.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} h_t(x+3p) &= \psi(x+3p+t) = \psi(x+t+3p) \\ &= \psi(x+t+2p) + \psi(x+t+p) \\ &\quad + \psi(x+t) \\ &= \psi(x+2p+t) + \psi(x+p+t) \\ &\quad + \psi(x+t) \\ &= h_t(x+2p) + h_t(x+p) + h_t(x). \end{aligned} \quad (1)$$

□

Theorem 4. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Tribonacci function with period p and let $\{T_n\}$, $\{T'_n\}$, and $\{T''_n\}$ be the sequences of Tribonacci numbers with $T_1 = 1, T_2 = 2, T_3 = 3, T_{n+3} = T_{n+2}+T_{n+1}+T_n$ and $T'_1 = 0, T'_2 = T'_3 = 1, T'_{n+3} = T'_{n+2}+T'_{n+1}+T'_n$ and $T''_1 = T''_2 = 0$ $T''_3 = 1, T''_{n+3} = T''_{n+2} + T''_{n+1} + T''_n$. Then, $\psi(x + np) = T'_n\psi(x + 2p) + T_{n-2}\psi(x + p) + T''_n\psi(x)$ for any $x \in \mathbb{R}$ and $n \geq 3$ an integer.*

Proof. $\psi(x+3p) = \psi(x+2p)+\psi(x+p)+\psi(x)$. The assertion holds for $n = 3$.

So, fix $n \in \mathbb{N}$ and assume that the assertion is valid for every $s \in \{3, \ldots, n+2\}$. Then,

$$\begin{aligned}
\psi(x+(n+3)p) &= \psi((x+np)+2p) \\
&\quad + \psi((x+np)+p) + \psi(x+np). \\
&= \psi((x+(n+2)p) \\
&\quad + \psi((x+(n+1)p) \\
&\quad + \psi(x+np). \\
&= T'_{n+2}\psi(x+2p) + T_n\psi(x+p) \\
&\quad + T''_{n+2}\psi(x) + T'_{n+1}\psi(x+2p) \\
&\quad + T_{n-1}\psi(x+p) + T''_{n+1}\psi(x) \\
&\quad + +T'_n\psi(x+2p) + T_{n-2}\psi(x+p) \\
&\quad + T''_n\psi(x). \\
\psi(x+(n+3)p) &= \left(T'_{n+2} + T'_{n+1} + T'_n\right)\psi(x+2p) \\
&\quad + (T_n + T_{n-1} + T_{n-2})\psi(x+p) \\
&\quad + \left(T''_{n+2} + T''_{n+1} + T''_n\right)\psi(x) \\
&= T'_{n+3}\psi(x+2p) + T_{n+1}\psi(x+p) \\
&\quad + T''_{n+3}\psi(x).
\end{aligned} \tag{2}$$

□

Corollary 5. *Let $\{T_n\}$, $\{T'_n\}$, and $\{T''_n\}$ be the sequences of Tribonacci numbers with $T_1 = 1, T_2 = 2, T_3 = 3, T_{n+3} = T_{n+2}+T_{n+1}+T_n$ and $T'_1 = 0, T'_2 = T'_3 = 1, T'_{n+3} = T'_{n+2}+T'_{n+1}+T'_n$ and $T''_1 = T''_2 = 0$ $T''_3 = 1$, $T''_{n+3} = T''_{n+2} + T''_{n+1} + T''_n$. Let β be the root of the equation $x^3 - x^2 - x - 1 = 0$ such that $1 < \beta < 2$. Then, $\beta^n = T'_n\beta^2 + T_{n-2}\beta + T''_n$.*

Proof. Let $\psi(x) = \beta^{x/p}$. We have seen in Example 2 that $\psi(x) = \beta^{x/p}$ is a Tribonacci function with period p. Applying Theorem 4, we have $\beta^{x/p+n} = T'_n\beta^{x/p+2} + T_{n-2}\beta^{x/p+1} + T''_n\beta^{x/p}$, for all $x \in \mathbb{R}$. We obtain $\beta^n = T'_n\beta^2 + T_{n-2}\beta + T''_n$. □

Remark 6. Consider the Tribonacci sequenses $\{T_n\}$, $\{T'_n\}$, and $\{T''_n\}$ as in Theorem 4. We have the relations $T_n = T'_n + T'_{n+1}$ and $T''_n = T'_{n-1}$.

3. Odd Tribonacci Functions with Period p

Here we discuss odd Tribonacci functions with period p defined as follows.

Definition 7. Let p be a positive integer. A function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be an odd Tribonacci function with period p if $\psi(x + 3p) = -\psi(x + 2p) - \psi(x + p) + \psi(x)$, for all $x \in \mathbb{R}$.

Example 8. Let $\psi(x) = \alpha^{x/p}$ be an odd Tribonacci function with period $p \in \mathbb{N}$, $\alpha > 0$. It is clear that $\alpha^{x/p+3} = -\alpha^{x/p+2} - \alpha^{x/p+1} + \alpha^{x/p}$, for all $x \in \mathbb{R}$. We have $\alpha^3 = -\alpha^2 - \alpha + 1$. Then, $\alpha = \beta_1$ such that β_1 is root of the equation $x^3 + x^2 + x - 1 = 0$, $0 < \beta_1 < 1$. Thus, $\psi(x) = \beta_1^{x/p}$ is an odd Tribonacci function with period p on $\mathbb{R}$.

Proposition 9. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be an odd Tribonacci function with period $p \in \mathbb{N}$ and define $h_t(x) = \psi(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, h_t is also an odd Tribonacci function with period p.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned}
h_t(x+3p) &= \psi(x+3p+t) = \psi(x+t+3p) \\
&= -\psi(x+t+2p) - \psi(x+t+p) \\
&\quad + \psi(x+t) \\
&= -\psi(x+2p+t) - \psi(x+p+t) \\
&\quad + \psi(x+t) \\
&= -h_t(x+2p) - h_t(x+p) + h_t(x).
\end{aligned} \tag{3}$$

□

Theorem 10. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be an odd Tribonacci function with period p and let $\{T_{-n}\}$, $\{T'_{-n}\}$, and $\{T''_{-n}\}$ be the sequences of Tribonacci numbers with $T_0 = 0, T_1 = 1, T_2 = 2, T_{-n-1} = -T_{-n} - T_{-n+1} + T_{-n+2}$ and $T'_0 = T'_1 = 0, T'_2 = 1, T'_{-n-1} = -T'_{-n} - T'_{-n+1} + T'_{-n+2}$ and $T''_0 = 1, T''_1 = T''_2 = 0, T''_{-n-1} = -T''_{-n} - T''_{-n+1} + T''_{-n+2}$. Then, $\psi(x + np) = T'_{-n+1}\psi(x + 2p) + T_{-n}\psi(x + p) + T''_{-n+3}\psi(x)$ for any $x \in \mathbb{R}$.*

Proof. $\psi(x+3p) = -\psi(x+2p)-\psi(x+p)+\psi(x)$. The assertion holds for $n = 3$.

So, fix $n \in \mathbb{N}$ and assume that the assertion is valid for every $s \in \{3, \ldots, n+2\}$. Then,

$$\begin{aligned}
&\psi(x+(n+3)p) \\
&\quad = \psi((x+np)+2p) + \psi((x+np)+p) \\
&\qquad + \psi(x+np). \\
&\quad = \psi((x+(n+2)p) + \psi((x+(n+1)p) \\
&\qquad + \psi(x+np).
\end{aligned}$$

$$\begin{aligned}
&= T'_{-n-1}\psi(x+2p) + T_{-n-2}\psi(x+p) + T''_{-n+1}\psi(x)\\
&\quad + T'_{-n}\psi(x+2p) + T_{-n-1}\psi(x+p) + T''_{-n+2}\psi(x)\\
&\quad + +T'_{-n+1}\psi(x+2p) + T_{-n}\psi(x+p)\\
&\quad + T''_{-n+3}\psi(x).\\
&\psi(x+(n+3)p)\\
&= \left(-T'_{-n-1} - T'_{-n} + T'_{-n+1}\right)\psi(x+2p)\\
&\quad + \left(-T_{-n-2} - T_{-n-1} + T_{-n}\right)\psi(x+p)\\
&\quad + +\left(-T''_{-n+1} - T''_{-n+2} + T''_{-n+3}\right)\psi(x)\\
&= T'_{-(n+3)+1}\psi(x+2p) + T_{-(n+3)}\psi(x+p)\\
&\quad + T''_{-(n+3)+3}\psi(x).
\end{aligned} \tag{4}$$

□

4. Even and Odd Functions with Period p

In this section, we will talk about the notion of Tribonacci functions using even and odd functions. Here we get results obtained in [10] with third-order linear recurrence. We give the limit of the quotient of a Tribonacci function with period p, extending the results of [11] in third-order linear recurrence.

Definition 11. Let p be a positive integer and $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ a function such that the preimage of 0 by λ has the empty interior. The function λ is said to be an even (resp., odd) function with period p if $\lambda(x+p) = \lambda(x)$ (resp., $\lambda(x+p) = -\lambda(x)$), for all $x \in \mathbb{R}$.

Example 12. If $\lambda(x) = x - \lfloor x \rfloor$, then $\lambda(x)h(x) \equiv 0$ implies that $h(x) \equiv 0$ if $x \notin \mathbb{Z}$. Due to the fact that $\mathbb{R} \setminus \mathbb{Z}$ is dense in $\mathbb{R}$ and h is continuous, it follows that $h = 0$. Let $p \in \mathbb{N}$ and $x \in \mathbb{R}$. Then, $\lambda(x+p) = x + p - \lfloor x+p \rfloor = x + p - \lfloor x \rfloor - p = x - \lfloor x \rfloor = \lambda(x)$. Hence, λ is an even function with period p.

Theorem 13. *Let $p \in \mathbb{N}$ and $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ be an even function with period p and let $v : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then, v is a (resp., an odd) Tribonacci function with period p if and only if λv is a (resp., an odd) Tribonacci function with period p.*

Proof. Let v be a Tribonacci function with period p. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
(\lambda v)(x+3p) &= \lambda(x+3p)\,v(x+3p)\\
&= \lambda(x+2p)\,v(x+3p)\\
&= \lambda(x+2p)\left[v(x+2p) + v(x+p) + v(x)\right]\\
&= \lambda(x+2p)\,v(x+2p) + \lambda(x+p)\,v(x+p)\\
&\quad + \lambda(x)\,v(x)\\
&= (\lambda v)(x+2p) + (\lambda v)(x+p) + (\lambda v)(x).
\end{aligned} \tag{5}$$

Hence, λv is a Tribonacci function with period p.

Now assume that λv is a Tribonacci function with period p. For $x \in \mathbb{R}$, we have

$$\begin{aligned}
\lambda(x+p)\,v(x+3p) &= \lambda(x+2p)\,v(x+3p)\\
&= \lambda(x+3p)\,v(x+3p) = (\lambda v)(x+3p)\\
&= (\lambda v)(x+2p) + (\lambda v)(x+p) + (\lambda v)(x)\\
&= \lambda(x+2p)\,v(x+2p) + \lambda(x+p)\,v(x+p)\\
&\quad + \lambda(x)\,v(x)\\
&= \lambda(x+p)\,v(x+2p) + \lambda(x+p)\,v(x+p)\\
&\quad + \lambda(x+p)\,v(x)\\
&= \lambda(x+p)\left[v(x+2p) + v(x+p) + v(x)\right].
\end{aligned} \tag{6}$$

For all $x \in \mathbb{R}$, we get $v(x+3p) = v(x+2p) + v(x+p) + v(x)$. Hence, v is a Tribonacci function with period p. □

We give the proof for the case where v is a Tribonacci function with period p. The case where v is an odd Tribonacci function with period p is similar and left to the reader.

Now we give the limit of the quotient of a Tribonacci function with period p.

Theorem 14. *Let $p \in \mathbb{N}$ and $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Tribonacci function with period p. Then, the limit of the quotient $\psi(x+p)/\psi(x)$ is β, the root of the equation $x^3 - x^2 - x - 1 = 0$ such that $1 < \beta < 2$.*

Proof. Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + np$. Let us set $u = \psi(y), v = \psi(y+p)$ and $w = \psi(y+2p)$. By Theorem 4, we have $\psi(y+np) = T'_n\psi(y+2p) + T_{n-2}\psi(y+p) + T''_n\psi(y)$ and hence $\psi(y+np) = T'_n w + T_{n-2}v + T''_n u$. Hence,

$$\begin{aligned}
\frac{\psi(x+p)}{\psi(x)} &= \frac{\psi(y+(n+1)p)}{\psi(y+np)}\\
&= \frac{T'_{n+1}\psi(y+2p) + T_{n-1}\psi(y+p) + T''_{n+1}\psi(y)}{T'_n\psi(y+2p) + T_{n-2}\psi(y+p) + T''_n\psi(y)}\\
&= \frac{T'_{n+1}w + T_{n-1}v + T''_{n+1}u}{T'_n w + T_{n-2}v + T''_n u}.
\end{aligned} \tag{7}$$

By Remark 6, we have

$$\begin{aligned}
\frac{\psi(x+p)}{\psi(x)} &= \frac{T'_{n+1}w + \left(T'_{n-1} + T'_n\right)v + T'_n u}{T'_n w + \left(T'_{n-2} + T'_{n-1}\right)v + T'_{n-1}u}\\
&= \frac{T'_n}{T'_{n-1}}\,\frac{\left(T'_{n+1}/T'_n\right)w + \left(T'_{n-1}/T'_n + 1\right)v + u}{\left(T'_n/T'_{n-1}\right)w + \left(T'_{n-2}/T'_{n-1} + 1\right)v + u}.
\end{aligned} \tag{8}$$

Now, the well-known result that $\lim_{n\longrightarrow+\infty}(T'_{n+1}/T'_n) = \beta$ yields

$$\lim_{x\longrightarrow+\infty}\frac{\psi(x+p)}{\psi(x)} = \beta\frac{\beta w + (1/\beta + 1)v + u}{\beta w + (1/\beta + 1)v + u}. \quad (9)$$

Hence, we obtain

$$\lim_{x\longrightarrow+\infty}\frac{\psi(x+p)}{\psi(x)} = \beta. \quad (10)$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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