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Research Article

Global Dynamics of a 3 × 6 System of Difference Equations

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In the proposed work, global dynamics of a 3 × 6 system of rational difference equations has been studied in the interior of $\mathbb{R}_+^3$. It is proved that system has at least one and at most seven boundary equilibria and a unique $+ve$ equilibrium under certain parametric conditions. By utilizing method of Linearization, local dynamical properties about equilibria have been investigated. It is shown that every $+ve$ solution of the system is bounded, and equilibrium P_0 becomes a globally asymptotically stable if $\alpha_1 < \alpha_2, \alpha_4 < \alpha_5, \alpha_7 < \alpha_8$. It is also shown that every $+ve$ solution of the system converges to P_0. Finally theoretical results are verified numerically.

1. Introduction

The importance of difference equations cannot be overemphasized. These equations model discrete physical phenomena on one hand and on the other hand these integral parts of numerical schemes used to solve differential equations. This widens the applicability of such equations to many branches of scientific knowledge. Discrete dynamical systems are described by difference equations and have applications in many branches of modern science like population dynamics, ecology, psychology, economics, queuing problems, genetics in biology, statistical problems, electrical networks, number theory, neural networks, quanta in radiation, sociology, physics, engineering, economics, probability theory, combinatorial analysis, stochastic time series, geometry, and resource management [1, 2]. Though the study of the global dynamics of higher-order nonlinear difference equations or system of difference equations is a challenging task but it is rewarding. These results pave path towards the development of basic theory of difference equations of higher-order. In recent years several authors have explored the behavior of solution of such difference equations or system of difference equations by studying equilibrium point, local and global dynamics about equilibria, boundedness and persistence, periodicity nature, prime period 2-solution, semicycle analysis, forbidden set, and many more (see [3–19] and references cited therein). For instance, Gibbons et al. [20] have explored the global dynamical properties of following difference equation:

$$x_{n+1} = \frac{\alpha_1 + \alpha_2 x_{n-1}}{\alpha_3 + x_n}, \quad n = 0, 1, \ldots, \tag{1}$$

where $\alpha_j (j = 1, 2, 3)$ and $x_{-j} (j = 1, 0)$ are $+ve$ real numbers. Çinar [21] has explored the $+ve$ solution of following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{1 + \prod_{i=0}^{1} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{2}$$

where $x_{-j} (j = 1, 0)$ are $+ve$ real numbers. Amleh et al. [22, 23] have explored boundedness, periodicity nature, and global dynamical properties of following difference equation:

$$x_{n+1} = \frac{\alpha_1 + \alpha_2 x_n x_{n-1} + \alpha_3 x_{n-1}}{\alpha_4 + \alpha_5 x_n x_{n-1} + \alpha_6 x_{n-1}}, \quad n = 0, 1, \ldots, \tag{3}$$

where $\alpha_j (j = 1, \ldots, 6)$ and $x_{-j} (j = 1, 0)$ are $+ve$ real numbers. Amleh et al. [22, 23] have pointed that difference equation depicted in (3) consists of numerous challenging and inspiring special cases having order 2, that can also arise from the following planer system when it is reduced to a difference equation (see [24]):

$$x_{n+1} = \frac{\alpha_1 + \alpha_2 y_n}{y_n},$$

$$y_{n+1} = \frac{\alpha_3 + \alpha_4 x_n + \alpha_5 y_n}{\alpha_6 + \alpha_7 x_n + \alpha_8 y_n}, \quad n = 0, 1, \ldots, \tag{4}$$

where $\alpha_j (j = 1, \ldots, 8)$ and x_0, y_0 are $+ve$ real numbers. Amleh et al. [22, 23] have also pointed that when one or more $\alpha_j (j = 1, \ldots, 6)$ allowed to zero then (3) contains 49 special cases in which 19 are riccati, linear, trivial, reducible to riccati, or linear difference equations while remaining 30 are posed to conjectures and open problems. Shojaei et al. [25] have extended the work of Çinar [21] to explore the dynamical properties of following difference equation:

$$x_{n+1} = \frac{\alpha_1 x_{n-2}}{\alpha_2 + \alpha_3 x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots, \tag{5}$$

where $\alpha_j (j = 1, 2, 3)$ and $x_{-j} (j = 2, 1, 0)$ are $+ve$ real numbers. Later Bajo and Liz [26] have explored the dynamical properties of following difference equation which is special case of (3):

$$x_{n+1} = \frac{x_{n-1}}{\alpha_1 + \alpha_2 \prod_{i=0}^{1} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $\alpha_j (j = 1, 2)$ and $x_{-j} (j = 1, 0)$ are $+ve$ real numbers. Zhang et al. [27] have extended the work studied by several authors [21, 25, 26], to explore the dynamical properties of following system of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-2}}{\alpha_2 + \prod_{i=0}^{2} y_{n-i}}, \\ y_{n+1} &= \frac{y_{n-1}}{\alpha_1 + \prod_{i=0}^{2} x_{n-i}}, \\ &\qquad n = 0, 1, \ldots, \end{aligned} \tag{7}$$

where $\alpha_j (j = 1, 2)$ and $x_j, y_{-j} (j = 2, 1, 0)$ are $+ve$ real numbers. Motivated from aforementioned studies, our aim is to extend the work studied by [21, 25–27], to explore the global dynamical properties of the following 3×6 difference equations system:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_{n-1}}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}}, \\ y_{n+1} &= \frac{\alpha_4 y_{n-1}}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}}, \\ z_{n+1} &= \frac{\alpha_7 z_{n-1}}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}}, \\ &\qquad n = 0, 1, \ldots, \end{aligned} \tag{8}$$

where α_j $(j = 1, 2, \ldots, 9)$ and x_{-j}, y_{-j}, z_{-j} $(j = 1, 0)$ are $+ve$ real numbers.

The flow pattern of the remaining paper, which is our main contribution and meaningful addition towards discrete dynamical systems described by the 3×6 system of difference equations, is as follows: Section 2 deals with the study of existence of equilibria in $\mathbb{R}_+^3$ whereas linearized form of the under consideration system is given in Section 3. Section 4 deals with the study of local dynamical properties about equilibria. Boundeness of the solution of the system is discussed in Section 5 whereas global dynamics about P_0 of the system is studied in Section 6. In Section 7, it is proved that every $+ve$ solution of under consideration system converges to P_0. Discussion and simulations are given in the last section.

2. Existence of Equilibria

This section deals with the study of existence of equilibria of the system (8) in $\mathbb{R}_+^3$. The result about the existence of equilibria can be summarized as follows.

Lemma 1. *System (8) has at least one and at most eight equilibria in $\mathbb{R}_+^3$. More precisely*

(i) *system (8) has a unique boundary equilibrium $P_0 = (0, 0, 0) \forall \alpha_j (j = 1, 2, \ldots, 9)$;*

(ii) *system (8) has a boundary equilibrium $P_1 = (\sqrt{(\alpha_7 - \alpha_8)/\alpha_9}, 0, \sqrt{(\alpha_4 - \alpha_5)/\alpha_6})$ if $\alpha_1 = \alpha_2, \alpha_4 > \alpha_5$ and $\alpha_7 > \alpha_8$;*

(iii) *system (8) has a boundary equilibrium $P_2 = (\sqrt{(\alpha_7 - \alpha_8)/\alpha_9}, \sqrt{(\alpha_1 - \alpha_2)/\alpha_3}, 0)$ if $\alpha_1 > \alpha_2, \alpha_4 = \alpha_5$ and $\alpha_7 > \alpha_8$;*

(iv) *system (8) has a boundary equilibrium $P_3 = (0, \sqrt{(\alpha_1 - \alpha_2)/\alpha_3}, \sqrt{(\alpha_4 - \alpha_5)/\alpha_6})$ if $\alpha_1 > \alpha_2, \alpha_4 > \alpha_5$ and $\alpha_7 = \alpha_8$;*

(v) *system (8) has a boundary equilibrium $P_4 = (\sqrt{(\alpha_7 - \alpha_8)/\alpha_9}, 0, 0)$ if $\alpha_1 = \alpha_2, \alpha_4 = \alpha_5$ and $\alpha_7 > \alpha_8$;*

(vi) *system (8) has a boundary equilibrium $P_5 = (0, 0, \sqrt{(\alpha_4 - \alpha_5)/\alpha_6})$ if $\alpha_1 = \alpha_2, \alpha_4 > \alpha_5$ and $\alpha_7 = \alpha_8$;*

(vii) *system (8) has a boundary equilibrium $P_6 = (0, \sqrt{(\alpha_1 - \alpha_2)/\alpha_3}, 0)$ if $\alpha_1 > \alpha_2, \alpha_4 = \alpha_5$ and $\alpha_7 = \alpha_8$;*

(viii) *system (8) has a $+ve$ equilibrium $P_7 = (\sqrt{(\alpha_7 - \alpha_8)/\alpha_9}, \sqrt{(\alpha_1 - \alpha_2)/\alpha_3}, \sqrt{(\alpha_4 - \alpha_5)/\alpha_6})$ if $\alpha_1 > \alpha_2, \alpha_4 > \alpha_5$ and $\alpha_7 > \alpha_8$. More specifically, $P_7 = (\sqrt{(\alpha_7 - \alpha_8)/\alpha_9}, \sqrt{(\alpha_1 - \alpha_2)/\alpha_3}, \sqrt{(\alpha_4 - \alpha_5)/\alpha_6})$ is a unique $+ve$ equilibrium of the system (8) if $\alpha_1 > \alpha_2, \alpha_4 > \alpha_5$ and $\alpha_7 > \alpha_8$.*

3. Linearized Form of (8)

We have the following map in order to construct the corresponding linearized form of (8):

$$\begin{aligned} &(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6) \\ &\quad \longrightarrow (x_{n+1}, x_n, y_{n+1}, y_n, z_{n+1}, z_n), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Omega_1 &= \frac{\alpha_1 x_{n-1}}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}}, \\ \Omega_2 &= x_n, \\ \Omega_3 &= \frac{\alpha_4 y_{n-1}}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}}, \end{aligned}$$

$$\Omega_4 = y_n,$$
$$\Omega_5 = \frac{\alpha_7 z_{n-1}}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}},$$
$$\Omega_6 = z_n. \tag{10}$$

Moreover $J|_\Lambda$ about Λ under the map depicted in (9) is

$$J|_\Lambda = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2+\alpha_3 y^2} & -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2} & -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5+\alpha_6 z^2} & -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2} & -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2} & -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2} & 0 & 0 & 0 & \frac{\alpha_7}{\alpha_8+\alpha_9 x^2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{11}$$

where $\Lambda = (x, y, z)$.

4. Local Stability of System (8)

Hereafter we will study the local dynamical properties about equilibria of the system (8) by utilizing Theorem 1.5 of [1].

4.1. Local Stability about Boundary Equilibria. This subsection is purely dedicated to the study of local dynamical properties about boundary equilibria: $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ of the system (8), respectively.

Theorem 2. *For P_0, the following hold:*

(i) *P_0 of the system (8) is a sink if*

$$\alpha_1 < \alpha_2, \quad \alpha_4 < \alpha_5, \quad \alpha_7 < \alpha_8; \tag{12}$$

(ii) *P_0 of the system (8) is unstable if*

$$\alpha_1 > \alpha_2 \quad \text{or } \alpha_4 > \alpha_5 \quad \text{or } \alpha_7 > \alpha_8. \tag{13}$$

Proof. (i) We have the following linearized system of (8) about P_0:

$$\Upsilon_{n+1} = B\Upsilon_n, \tag{14}$$

where

$$\Upsilon_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \\ z_n \\ z_{n-1} \end{pmatrix},$$

$$B = J|_{P_0} = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_7}{\alpha_8} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{15}$$

Let the eigenvalues of B are $\nu_1, \nu_2, \cdots, \nu_6$. Also let

$$D = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 \end{pmatrix}, \tag{16}$$

denoting the diagonal matrix where

$$d_1 = d_3 = d_5 = 1, \quad d_2 = d_4 = d_6 = 1 - \epsilon \quad \text{for } 0 < \epsilon < 1, \tag{17}$$

and

$$0 < \epsilon < \min\left\{1 - \frac{\alpha_1}{\alpha_2},\ 1 - \frac{\alpha_4}{\alpha_5},\ 1 - \frac{\alpha_7}{\alpha_8}\right\}. \tag{18}$$

By computing DBD^{-1} one gets

$$DBD^{-1} = \begin{pmatrix} 0 & d_1 d_2^{-1}\frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ d_2 d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 d_4^{-1}\frac{\alpha_4}{\alpha_5} & 0 & 0 \\ 0 & 0 & d_4 d_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_5 d_6^{-1}\frac{\alpha_7}{\alpha_8} \\ 0 & 0 & 0 & 0 & d_6 d_5^{-1} & 0 \end{pmatrix}. \tag{19}$$

Moreover,

$$\begin{aligned} d_1 &> d_2 > 0, \\ d_3 &> d_4 > 0, \\ d_5 &> d_6 > 0. \end{aligned} \tag{20}$$

Equation (20) then implies that

$$\begin{aligned} d_2 d_1^{-1} &< 1, \\ d_4 d_3^{-1} &< 1, \\ d_6 d_5^{-1} &< 1. \end{aligned} \tag{21}$$

From (17) and (18) one gets

$$\begin{aligned} d_1 d_2^{-1} \frac{\alpha_1}{\alpha_2} &= d_2^{-1} \frac{\alpha_1}{\alpha_2} = \frac{1}{1-\epsilon} \frac{\alpha_1}{\alpha_2} < 1, \\ d_3 d_4^{-1} \frac{\alpha_4}{\alpha_5} &= d_4^{-1} \frac{\alpha_4}{\alpha_5} = \frac{1}{1-\epsilon} \frac{\alpha_4}{\alpha_5} < 1, \\ d_5 d_6^{-1} \frac{\alpha_7}{\alpha_8} &= d_6^{-1} \frac{\alpha_7}{\alpha_8} = \frac{1}{1-\epsilon} \frac{\alpha_7}{\alpha_8} < 1. \end{aligned} \tag{22}$$

Since eigenvalues of B are same as DBD^{-1}, so

$$\begin{aligned} \max_{1 \le m \le 6} |\nu_m| &= \left\| DBD^{-1} \right\| \\ &= \max \left\{ d_2 d_1^{-1}, d_4 d_3^{-1}, d_6 d_5^{-1}, d_1 d_2^{-1} \frac{\alpha_1}{\alpha_2}, d_3 d_4^{-1} \right. \\ &\quad \left. \cdot \frac{\alpha_4}{\alpha_5}, d_5 d_6^{-1} \frac{\alpha_7}{\alpha_8} \right\} < 1. \end{aligned} \tag{23}$$

Thus P_0 of (8) is a sink.

(ii) Similarly it is easy to prove that P_0 of (8) is unstable if $\alpha_1 > \alpha_2$ or $\alpha_4 > \alpha_5$ or $\alpha_7 > \alpha_8$. □

Theorem 3. *P_1 of (8) is locally unstable.*

Proof. The corresponding linearized form of (8) about P_1 is

$$\Upsilon_{n+1} = B_1 \Upsilon_n, \tag{24}$$

where

$$B_1 = J|_{P_1} = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{25}$$

Again let the eigenvalues of B_1 be $\nu_1, \nu_2, \cdots, \nu_6$ and the diagonal matrix D which is depicted in (16) where (17) holds. On computing DB_1D^{-1} one gets

$$DB_1D^{-1} = \begin{pmatrix} 0 & d_1 d_2^{-1} \frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ d_2 d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 d_4^{-1} & 0 & 0 \\ 0 & 0 & d_4 d_3^{-1} & 0 & 0 & 0 \\ -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_1^{-1} & -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_2^{-1} & 0 & 0 & 0 & d_5 d_6^{-1} \\ 0 & 0 & 0 & 0 & d_6 d_5^{-1} & 0 \end{pmatrix}. \tag{26}$$

In view of (20) and (21) one has

$$d_1 d_2^{-1} \frac{\alpha_1}{\alpha_2} = d_2^{-1} \frac{\alpha_1}{\alpha_2} = \frac{1}{1-\epsilon} \frac{\alpha_1}{\alpha_2} < 1, \tag{27}$$

which implies that $\epsilon < 1 - \alpha_1/\alpha_2$. Also

$$d_3 d_4^{-1} = d_4^{-1} = \frac{1}{1-\epsilon} > 1. \tag{28}$$

Hence P_1 of the system (8) is unstable. □

Theorem 4. *P_2 of (8) is locally unstable.*

Proof. The corresponding linearized form of (8) about P_2 is

$$\Upsilon_{n+1} = B_2 \Upsilon_n, \tag{29}$$

where

$$B_2 = J_{P_2} = \begin{pmatrix} 0 & 1 & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{30}$$

Using same arrangement as in the proof of Theorems 2–3 and compute DB_2D^{-1}; one gets

$$DB_2D^{-1} = \begin{pmatrix} 0 & d_1 d_2^{-1} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_3^{-1} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_4^{-1} & 0 & 0 \\ d_2 d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} d_3 d_4^{-1} & 0 & 0 \\ 0 & 0 & d_4 d_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_5 d_6^{-1} \\ 0 & 0 & 0 & 0 & d_6 d_5^{-1} & 0 \end{pmatrix}. \tag{31}$$

In view of (17), (20), and (21) one has

$$\begin{aligned} & d_1 d_2^{-1} + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_3^{-1} \\ & \quad + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_4^{-1} \\ & \quad = \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} \\ & \quad + \left(1 + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}}\right) \frac{1}{1-\epsilon} > 1. \end{aligned} \tag{32}$$

And

$$\frac{\alpha_4}{\alpha_5} d_3 d_4^{-1} = \frac{\alpha_4}{\alpha_5} \frac{1}{1-\epsilon} < 1, \tag{33}$$

which implies that $\epsilon < 1 - \alpha_4/\alpha_5$. Also

$$d_5 d_6^{-1} = d_6^{-1} = \frac{1}{1-\epsilon} > 1. \tag{34}$$

Hence P_2 of (8) is locally unstable. □

Theorem 5. *P_3 of the system (8) is locally unstable.*

Proof. Its proof is similar to Theorems 3–4. □

Theorem 6. *P_4 of (8) is locally unstable.*

Proof. The corresponding linearized system is

$$\Upsilon_{n+1} = B_3 \Upsilon_n, \tag{35}$$

where

$$B_3 = J_{P_4} = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{36}$$

Again using same arrangements as in the proof of Theorems 2–4 and computing DB_3D^{-1}, one gets

$$DB_3D^{-1} = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2} d_1 d_2^{-1} & 0 & 0 & 0 & 0 \\ d_2 d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} d_3 d_4^{-1} & 0 & 0 \\ 0 & 0 & d_4 d_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_5 d_6^{-1} \\ 0 & 0 & 0 & 0 & d_6 d_5^{-1} & 0 \end{pmatrix}. \tag{37}$$

Further,

$$\frac{\alpha_1}{\alpha_2} d_1 d_2^{-1} = \frac{1}{1-\epsilon} > 1, \tag{38}$$

$$\frac{\alpha_4}{\alpha_5} d_3 d_4^{-1} = \frac{1}{1-\epsilon} > 1, \tag{39}$$

and

$$d_6 d_5^{-1} = \frac{1}{1-\epsilon} > 1. \tag{40}$$

Hence P_4 of the system (8) is locally unstable. □

Theorem 7. *P_5 and P_6 of the system (8) are locally unstable.*

Proof. Similar to Theorem 6. □

4.2. Local Stability about the Unique Positive Equilibrium. Hereafter local dynamical properties about P_7 of (8) are explored. The following result shows that P_7 of system (8) is locally unstable.

Theorem 8. *P_7 of (8) is unstable.*

Proof. The corresponding linearized form of (8) about P_7 is

$$\Upsilon_{n+1} = B_4 = J_{P_7} \Upsilon_n, \tag{41}$$

where

$$B_4 = \begin{pmatrix} 0 & 1 & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} & -\frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{42}$$

Also

$$DB_4D^{-1} = \begin{pmatrix} 0 & d_1 d_2^{-1} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_3^{-1} & -\frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_4^{-1} & 0 & 0 \\ d_2 d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 d_4^{-1} & -\frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} d_3 d_5^{-1} & -\frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} d_3 d_6^{-1} \\ 0 & 0 & d_4 d_3^{-1} & 0 & 0 & 0 \\ -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_1^{-1} & -\frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_2^{-1} & 0 & 0 & 0 & d_5 d_6^{-1} \\ 0 & 0 & 0 & 0 & d_6 d_5^{-1} & 0 \end{pmatrix}. \tag{43}$$

Moreover

$$\begin{aligned} & d_1 d_2^{-1} + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_3^{-1} \\ & \quad + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_1 d_4^{-1} \\ & \quad = \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} \\ & \quad + \left(1 + \frac{\alpha_3}{\alpha_1}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}}\right)\frac{1}{1-\epsilon} > 1, \end{aligned} \tag{44}$$

$$\begin{aligned} & d_3 d_4^{-1} + \frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} d_3 d_5^{-1} \\ & \quad + \frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} d_3 d_6^{-1} \\ & \quad = \frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}} \\ & \quad + \left(1 + \frac{\alpha_6}{\alpha_4}\sqrt{\frac{\alpha_1-\alpha_2}{\alpha_3}}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\right)\frac{1}{1-\epsilon} > 1, \end{aligned} \tag{45}$$

and

$$\begin{aligned} & \frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_1^{-1} \\ & \quad + \frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} d_5 d_2^{-1} + d_5 d_6^{-1} \\ & \quad = \frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}} \\ & \quad + \left(1 + \frac{\alpha_9}{\alpha_7}\sqrt{\frac{\alpha_4-\alpha_5}{\alpha_6}}\sqrt{\frac{\alpha_7-\alpha_8}{\alpha_9}}\right)\frac{1}{1-\epsilon} > 1. \end{aligned} \tag{46}$$

Hence P_7 of the system (8) is locally unstable. □

5. Boundedness

Theorem 9. *Any positive solution $\{(x_n, y_n, z_n)\}$ of (8) satisfies following inequalities for $m \geq 0$:*

$$\begin{aligned} & 0 < x_n \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{m+1} x_{-1}, \quad \text{if } n = 2m+1, \\ & 0 < x_n \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{m+1} x_0, \quad \text{if } n = 2m+2. \end{aligned} \tag{47}$$

And

$$\begin{aligned} & 0 < y_n \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{m+1} y_{-1}, \quad \text{if } n = 2m+1, \\ & 0 < y_n \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{m+1} y_0, \quad \text{if } n = 2m+2. \end{aligned} \tag{48}$$

Finally,

$$\begin{aligned} & 0 < z_n \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{m+1} z_{-1}, \quad \text{if } n = 2m+1, \\ & 0 < z_n \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{m+1} z_0, \quad \text{if } n = 2m+2. \end{aligned} \tag{49}$$

Proof. Obviously (47), (48), and (49) hold for $m = 0$. Now suppose that for $m = p \geq 1$ inequalities (47), (48), and (49) are true, *i.e.*,

$$\begin{aligned} & 0 < x_n \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1} x_{-1}, \quad \text{if } n = 2p+1, \\ & 0 < x_n \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1} x_0, \quad \text{if } n = 2p+2. \end{aligned} \tag{50}$$

And

$$\begin{aligned} & 0 < y_n \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{p+1} y_{-1}, \quad \text{if } n = 2p+1, \\ & 0 < y_n \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{p+1} y_0, \quad \text{if } n = 2p+2. \end{aligned} \tag{51}$$

Finally,

$$\begin{aligned} & 0 < z_n \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{p+1} z_{-1}, \quad \text{if } n = 2p+1, \\ & 0 < z_n \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{p+1} z_0, \quad \text{if } n = 2p+2. \end{aligned} \tag{52}$$

Now, for $m = p+1$ using (8) one has

$$\begin{aligned} & 0 < x_{2p+3} = \frac{\alpha_1 x_{2p+1}}{\alpha_2 + \alpha_3 y_{2p+2} y_{2p+1}} \leq \frac{\alpha_1}{\alpha_2} x_{2p+1} \\ & \quad \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{p+2} x_{-1}, \\ & 0 < x_{2p+4} = \frac{\alpha_1 x_{2p+2}}{\alpha_2 + \alpha_3 y_{2p+3} y_{2p+2}} \leq \frac{\alpha_1}{\alpha_2} x_{2p+2} \\ & \quad \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{p+2} x_0. \end{aligned} \tag{53}$$

And

$$\begin{aligned} & 0 < y_{2p+3} = \frac{\alpha_4 y_{2p+1}}{\alpha_5 + \alpha_6 z_{2p+2} z_{2p+1}} \leq \frac{\alpha_4}{\alpha_5} y_{2p+1} \\ & \quad \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{p+2} y_{-1}, \end{aligned}$$

$$0 < y_{2p+4} = \frac{\alpha_4 y_{2p+2}}{\alpha_5 + \alpha_6 z_{2p+3} z_{2p+2}} \leq \frac{\alpha_4}{\alpha_5} y_{2p+2} \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{p+2} y_0. \tag{54}$$

Finally,

$$\begin{aligned} 0 < z_{2p+3} &= \frac{\alpha_7 z_{2p+1}}{\alpha_8 + \alpha_9 x_{2p+2} x_{2p+1}} \leq \frac{\alpha_7}{\alpha_8} z_{2p+1} \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{p+2} z_{-1}, \\ 0 < z_{2p+3} &= \frac{\alpha_7 z_{2p+2}}{\alpha_8 + \alpha_9 x_{2p+3} x_{2p+2}} \leq \frac{\alpha_7}{\alpha_8} z_{2p+2} \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{p+2} z_0. \end{aligned} \tag{55}$$

□

Lemma 10. *If the conditions depicted in (12) hold then $\{(x_n, y_n, z_n)\}$ of (8) is bounded.*

Proof. Direct consequence of Theorem 9. □

6. Global Stability about the Equilibrium P_0

Now we will explore the global dynamics of (8) about P_0.

Theorem 11. *If the conditions depicted in (12) hold then P_0 of (8) is globally asymptotically stable.*

Proof. Since P_0 of (8) is a sink by Theorem 2 and hence it is enough to prove

$$\lim_{n\longrightarrow\infty} (x_n, y_n, z_n) = P_0. \tag{56}$$

From (8), one gets

$$\begin{aligned} 0 < x_{n+1} &= \frac{\alpha_1 x_{n-1}}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}} \leq \frac{\alpha_1}{\alpha_2} x_{n-1}, \\ 0 < y_{n+1} &= \frac{\alpha_4 y_{n-1}}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}} \leq \frac{\alpha_4}{\alpha_5} y_{n-1}, \\ 0 < z_{n+1} &= \frac{\alpha_7 z_{n-1}}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}} \leq \frac{\alpha_7}{\alpha_8} z_{n-1}. \end{aligned} \tag{57}$$

By induction, one gets

$$\begin{aligned} x_{2n-1} &\leq \left(\frac{\alpha_1}{\alpha_2}\right)^n x_{-1}, \\ x_{2n} &\leq \left(\frac{\alpha_1}{\alpha_2}\right)^n x_0, \\ y_{2n-1} &\leq \left(\frac{\alpha_4}{\alpha_5}\right)^n y_{-1}, \\ y_{2n} &\leq \left(\frac{\alpha_4}{\alpha_5}\right)^n y_0, \\ z_{2n-1} &\leq \left(\frac{\alpha_7}{\alpha_8}\right)^n z_{-1}, \\ z_{2n} &\leq \left(\frac{\alpha_7}{\alpha_8}\right)^n z_0. \end{aligned} \tag{58}$$

Finally if (12) holds then from (58) one gets $\lim_{n\longrightarrow\infty}(x_n, y_n, z_n) = P_0$. □

7. Rate of Convergence

Theorem 12. *If (12) holds then error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_n^3 \\ e_{n-1}^3 \end{pmatrix} = \begin{pmatrix} x_n - x \\ x_{n-1} - x \\ y_n - y \\ y_{n-1} - y \\ z_n - z \\ z_{n-1} - z \end{pmatrix}, \tag{59}$$

of every +ve solution $\{(x_n, y_n, z_n)\}$ of (8) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n\longrightarrow\infty} \sqrt[n]{\|e_n\|} &= \left|\nu_{1,\ldots,6}\, J|_{P_0}\right|, \\ \lim_{n\longrightarrow\infty} \frac{\|e_{n+1}\|}{\|e_n\|} &= \left|\nu_{1,\ldots,6}\, J|_{P_0}\right|, \end{aligned} \tag{60}$$

where $|\nu_{1,\ldots,6} J|_{P_0}|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at P_0.

Proof. Let $\{(x_n, y_n, z_n)\}$ be an arbitrary solution of (8), *s.t.*, $\lim_{n\longrightarrow\infty} x_n = x, \lim_{n\longrightarrow\infty} y_n = y$ and $\lim_{n\longrightarrow\infty} z_n = z$. To find error term one has from the system (8)

$$\begin{aligned} x_{n+1} - x &= \frac{\alpha_1 x_{n-1}}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}} - \frac{\alpha_1 x}{\alpha_2 + \alpha_3 y^2}, \\ &= \frac{\alpha_1}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}} (x_{n-1} - x) \\ &\quad - \frac{\alpha_1 \alpha_3 x y}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)} (y_n - y) \\ &\quad - \frac{\alpha_1 \alpha_3 x y_n}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)} (y_{n-1} - y), \\ y_{n+1} - y &= \frac{\alpha_4 y_{n-1}}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}} - \frac{\alpha_4 y}{\alpha_5 + \alpha_6 z^2}, \end{aligned}$$

$$= \frac{\alpha_4}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}} (y_{n-1} - y) - \frac{\alpha_4 \alpha_6 yz}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)} (z_n - z) - \frac{\alpha_4 \alpha_6 y z_n}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)} (z_{n-1} - z), \tag{61}$$

and

$$\begin{aligned} z_{n+1} - z &= \frac{\alpha_7 z_{n-1}}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}} - \frac{\alpha_7 y}{\alpha_8 + \alpha_9 x^2}, \\ &= -\frac{\alpha_7 \alpha_9 xz}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)} (x_n - x) \\ &\quad - \frac{\alpha_7 \alpha_9 x_n z}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)} (x_{n-1} - x) \\ &\quad + \frac{\alpha_7}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}} (z_{n-1} - z). \end{aligned} \tag{62}$$

So

$$\begin{aligned} x_{n+1} - x &= \frac{\alpha_1}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}} (x_{n-1} - x) - \frac{\alpha_1 \alpha_3 xy}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)} (y_n - y) - \frac{\alpha_1 \alpha_3 x y_n}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)} (y_{n-1} - y), \\ y_{n+1} - y &= \frac{\alpha_4}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}} (y_{n-1} - y) - \frac{\alpha_4 \alpha_6 yz}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)} (z_n - z) - \frac{\alpha_4 \alpha_6 y z_n}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)} (z_{n-1} - z), \\ z_{n+1} - z &= -\frac{\alpha_7 \alpha_9 xz}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)} (x_n - x) - \frac{\alpha_7 \alpha_9 x_n z}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)} (x_{n-1} - x) + \frac{\alpha_7}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}} (z_{n-1} - z). \end{aligned} \tag{63}$$

Set

$$\begin{aligned} e_n^1 &= x_n - x, \\ e_n^2 &= y_n - y, \\ e_n^3 &= z_n - z. \end{aligned} \tag{64}$$

In view of (64), (63) becomes

$$\begin{aligned} e_{n+1}^1 &= \Lambda_n^1 e_{n-1}^1 + \Lambda_n^2 e_n^2 + \Lambda_n^3 e_{n-1}^2, \\ e_{n+1}^2 &= \Lambda_n^4 e_{n-1}^2 + \Lambda_n^5 e_n^3 + \Lambda_n^6 e_{n-1}^3, \\ e_{n+1}^3 &= \Lambda_n^7 e_n^1 + \Lambda_n^8 e_{n-1}^1 + \Lambda_n^9 e_{n-1}^3, \end{aligned} \tag{65}$$

where

$$\begin{aligned} \Lambda_n^1 &= \frac{\alpha_1}{\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}}, \\ \Lambda_n^2 &= -\frac{\alpha_1 \alpha_3 xy}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)}, \\ \Lambda_n^3 &= -\frac{\alpha_1 \alpha_3 x y_n}{\left(\alpha_2 + \alpha_3 \prod_{i=0}^{1} y_{n-i}\right)\left(\alpha_2 + \alpha_3 y^2\right)}, \\ \Lambda_n^4 &= \frac{\alpha_4}{\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}}, \\ \Lambda_n^5 &= -\frac{\alpha_4 \alpha_6 yz}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)}, \\ \Lambda_n^6 &= -\frac{\alpha_4 \alpha_6 y z_n}{\left(\alpha_5 + \alpha_6 \prod_{i=0}^{1} z_{n-i}\right)\left(\alpha_5 + \alpha_6 z^2\right)}, \\ \Lambda_n^7 &= -\frac{\alpha_7 \alpha_9 xz}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)}, \\ \Lambda_n^8 &= -\frac{\alpha_7 \alpha_9 x_n z}{\left(\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}\right)\left(\alpha_8 + \alpha_9 x^2\right)}, \\ \Lambda_n^9 &= \frac{\alpha_7}{\alpha_8 + \alpha_9 \prod_{i=0}^{1} x_{n-i}}. \end{aligned} \tag{66}$$

Moreover

$$\begin{aligned} \lim_{n\to\infty} \Lambda_n^1 &= \frac{\alpha_1}{\alpha_2 + \alpha_3 y^2}, \\ \lim_{n\to\infty} \Lambda_n^2 &= -\frac{\alpha_1 \alpha_3 xy}{\left(\alpha_2 + \alpha_3 y^2\right)^2}, \\ \lim_{n\to\infty} \Lambda_n^3 &= -\frac{\alpha_1 \alpha_3 xy}{\left(\alpha_2 + \alpha_3 y^2\right)^2}, \\ \lim_{n\to\infty} \Lambda_n^4 &= \frac{\alpha_4}{\alpha_5 + \alpha_6 z^2}, \\ \lim_{n\to\infty} \Lambda_n^5 &= -\frac{\alpha_4 \alpha_6 yz}{\left(\alpha_5 + \alpha_6 z^2\right)^2}, \end{aligned}$$

$$\lim_{n\longrightarrow\infty}\Lambda_n^6 = -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2},$$
$$\lim_{n\longrightarrow\infty}\Lambda_n^7 = -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2},$$
$$\lim_{n\longrightarrow\infty}\Lambda_n^8 = -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2},$$
$$\lim_{n\longrightarrow\infty}\Lambda_n^9 = \frac{\alpha_7}{\alpha_8+\alpha_9 x^2}, \tag{67}$$

that is,

$$\Lambda_n^1 = \frac{\alpha_1}{\alpha_2+\alpha_3 y^2}+\varrho_1,$$
$$\Lambda_n^2 = -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2}+\varrho_2,$$
$$\Lambda_n^3 = -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2}+\varrho_3,$$
$$\Lambda_n^4 = \frac{\alpha_4}{\alpha_5+\alpha_6 z^2}+\varrho_4,$$
$$\Lambda_n^5 = -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2}+\varrho_5,$$
$$\Lambda_n^6 = -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2}+\varrho_6,$$
$$\Lambda_n^7 = -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2}+\varrho_7,$$
$$\Lambda_n^8 = -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2}+\varrho_8,$$
$$\Lambda_n^9 = \frac{\alpha_7}{\alpha_8+\alpha_9 x^2}+\varrho_9, \tag{68}$$

where $\varrho_1,\ldots,\varrho_9 \longrightarrow 0$ as $n \longrightarrow \infty$. Thus we have the system [28]:

$$e_{n+1} = [A+B(n)]\,e_n, \tag{69}$$

where

$$A = \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2+\alpha_3 y^2} & -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2} & -\frac{\alpha_1\alpha_3 xy}{(\alpha_2+\alpha_3 y^2)^2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5+\alpha_6 z^2} & -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2} & -\frac{\alpha_4\alpha_6 yz}{(\alpha_5+\alpha_6 z^2)^2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2} & -\frac{\alpha_7\alpha_9 xz}{(\alpha_8+\alpha_9 x^2)^2} & 0 & 0 & 0 & \frac{\alpha_7}{\alpha_8+\alpha_9 x^2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$
$$B(n) = \begin{pmatrix} 0 & \varrho_1 & \varrho_2 & \varrho_3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varrho_4 & \varrho_5 & \varrho_6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \varrho_7 & \varrho_8 & 0 & 0 & 0 & \varrho_9 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{70, 71}$$

and

$$\|B(n)\| \longrightarrow 0, \quad n\longrightarrow\infty. \tag{72}$$

Therefore limiting system of error terms takes the following form:

$$\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n+1}^2 \\ e_n^2 \\ e_{n+1}^3 \\ e_n^3 \end{pmatrix} = J|_\Lambda \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_n^3 \\ e_{n-1}^3 \end{pmatrix}. \tag{73}$$

Hence (73) is same as linearized system of (8) about (x, y, z). Particularly about P_0 it becomes

$$\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n+1}^2 \\ e_n^2 \\ e_{n+1}^3 \\ e_n^3 \end{pmatrix}$$

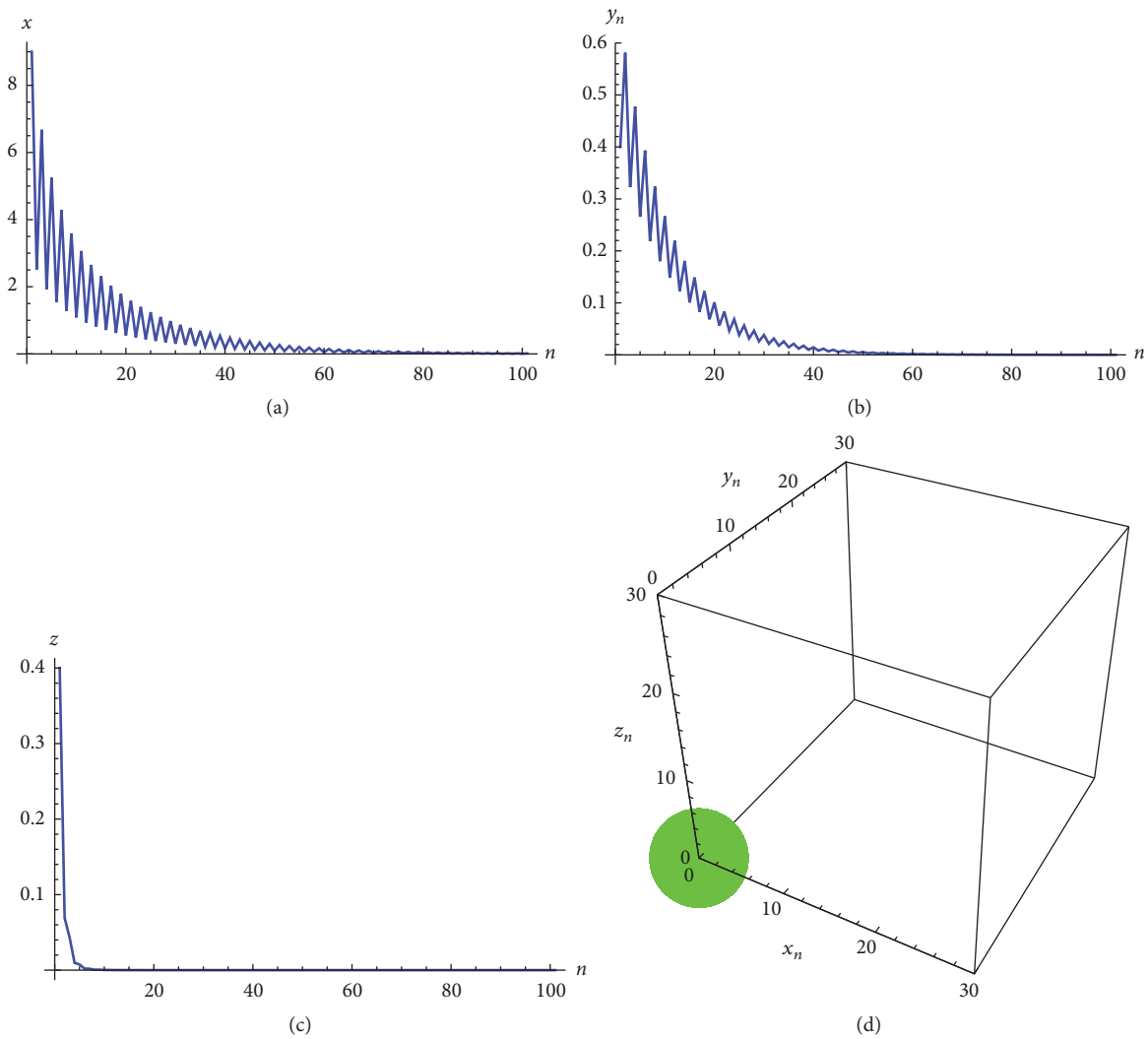

Figure 1: Phase portrait of (75).

$$= \begin{pmatrix} 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha_7}{\alpha_8} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_n^3 \\ e_{n-1}^3 \end{pmatrix}, \tag{74}$$

which is same as linearized system of (8) about P_0. □

8. Discussion and Numerical Simulations

This proposed work is about the global dynamical properties of a 3×6 system of difference equations, which is our key finding towards discrete dynamical system. We investigated that discrete-time system (8) has 8 equilibria: $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ and the unique $+ve$ equilibrium point: P_7 under restriction to the parameters α_j $(j = 1, 2, \ldots, 9)$. By method of linearization, we studied the local dynamical properties about each equilibria and conclusion are presented in Table 1. Further we proved that $\{(x_n, y_n, z_n)\}$ of (8) is bounded and P_0 is globally asymptotically stable if (12) holds. Finally, we proved that every $+ve$ solution $\{(x_n, y_n, z_n)\}$ of the system converges to P_0. The above obtained theoretical results can also be verified from following numerical simulations. Thus in the remaining section, theoretical results are verified numerically.

Example 1. If $\alpha_1 = 24, \alpha_2 = 27, \alpha_3 = 23, \alpha_4 = 14, \alpha_5 = 17, \alpha_6 = 13, \alpha_7 = 14, \alpha_8 = 16, \alpha_9 = 5$ then (8) with $x_j, y_j, z_j (j = -1, 0)$, respectively, are $3.7, 9, 0.9, 0.4, 0.9, 0.4$, the following form is taken:

$$x_{n+1} = \frac{24x_{n-1}}{27 + 23\prod_{i=0}^{1} y_{n-i}},$$

$$y_{n+1} = \frac{14y_{n-1}}{17 + 13\prod_{i=0}^{1} z_{n-i}},$$

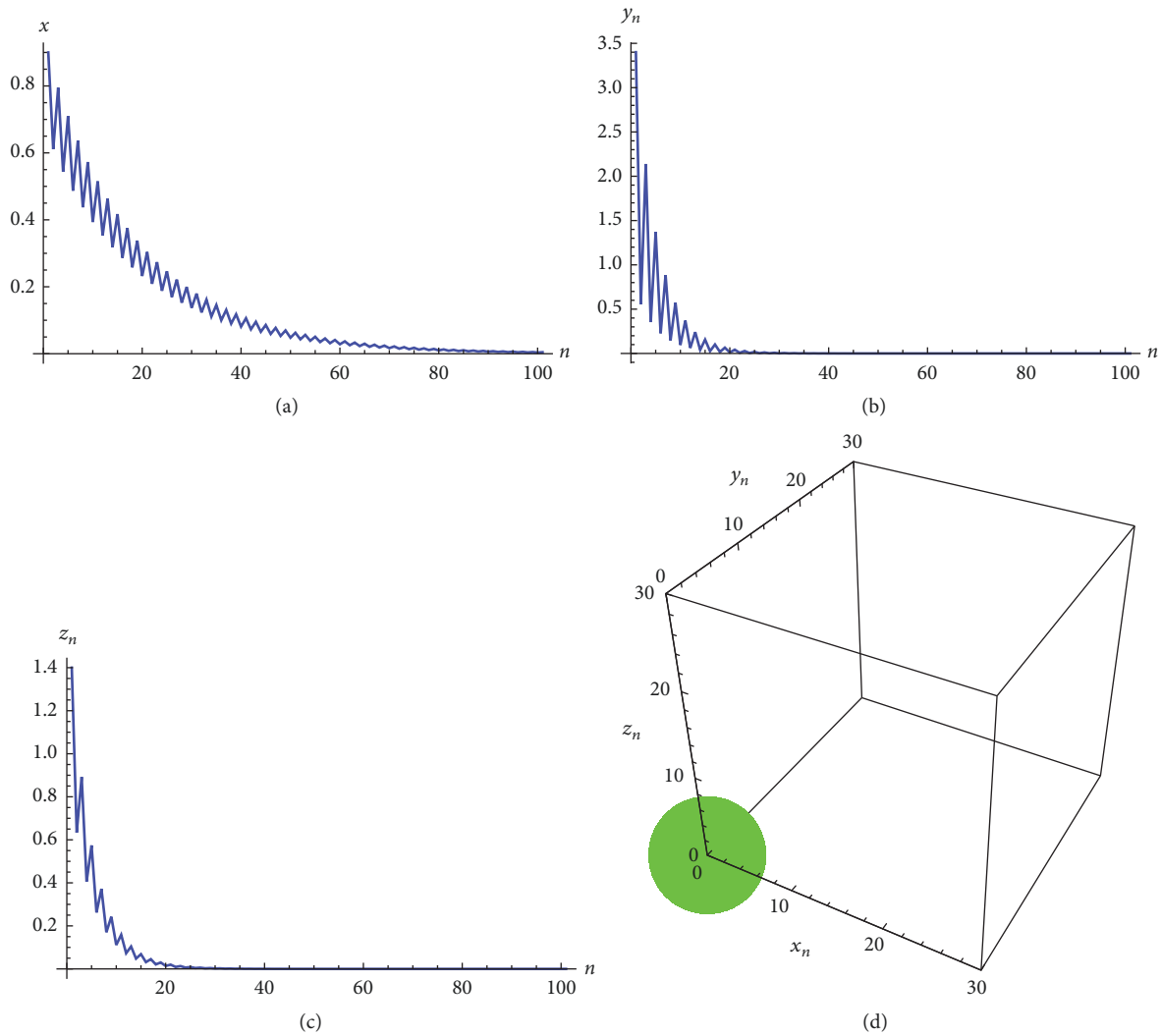

Figure 2: Phase portrait of (76).

Table 1: Number of Equilibria along their qualitative behavior of (8).

E.P	Corresponding behavior
P_0	locally asymptotically stable if (12) hold; unstable if (13) hold.
P_1	unstable.
P_2	unstable.
P_3	unstable.
P_4	unstable.
P_5	unstable.
P_6	unstable.
P_7	unstable.

$$z_{n+1} = \frac{14z_{n-1}}{16 + 5\prod_{i=0}^{1} x_{n-i}}. \tag{75}$$

The graphs of n *vs* x_n(respectively y_n and z_n) of (75) are shown in Figures 1(a), 1(b), and 1(c) while its attractor is shown in Figure 1(d); *i.e.*, if $\alpha_1 = 24 < \alpha_2 = 27, \alpha_4 = 14 < \alpha_5 = 17, \alpha_7 = 14 < \alpha_8 = 16$, then all solutions are attracted to P_0. This coincides with the proof of Theorem 11.

Example 2. If $\alpha_1 = 27, \alpha_2 = 30, \alpha_3 = 0.3, \alpha_4 = 24, \alpha_5 = 37, \alpha_6 = 1.3, \alpha_7 = 4, \alpha_8 = 6, \alpha_9 = 0.5$ then (8) with $x_j, y_j, z_j (j = -1, 0)$, respectively, are 0.7, 0.9, 0.9, 3.4, 0.99999, 1.4, the following form is taken:

$$\begin{aligned} x_{n+1} &= \frac{27x_{n-1}}{30 + 0.3\prod_{i=0}^{1} y_{n-i}}, \\ y_{n+1} &= \frac{24y_{n-1}}{37 + 1.3\prod_{i=0}^{1} z_{n-i}}, \\ z_{n+1} &= \frac{4z_{n-1}}{6 + 0.5\prod_{i=0}^{1} x_{n-i}}. \end{aligned} \tag{76}$$

The graphs of n *vs* x_n(respectively y_n and z_n) of (76) are shown in Figures 2(a), 2(b), and 2(c) while its attractor is shown in Figure 2(d).

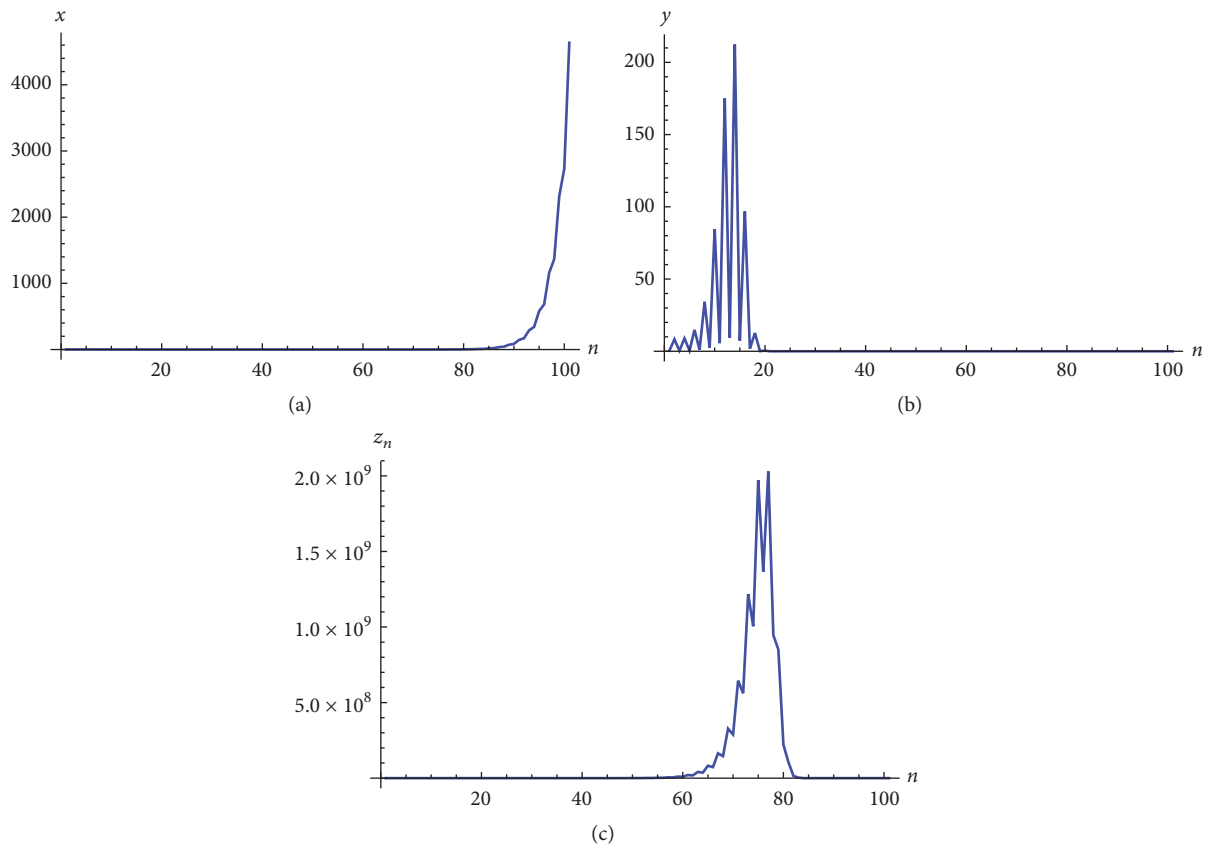

Figure 3: Phase portrait of (77).

Example 3. If $\alpha_1 = 28, \alpha_2 = 14, \alpha_3 = 3, \alpha_4 = 19, \alpha_5 = 7, \alpha_6 = 13, \alpha_7 = 4, \alpha_8 = 2, \alpha_9 = 5$ then (8) with $x_j, y_j, z_j (j = -1, 0)$, respectively, are $0.7, 0.9, 10.9, = \ 0.4, 0.99, 1.4$, the following form is taken:

$$x_{n+1} = \frac{28x_{n-1}}{14 + 3\prod_{i=0}^{1} y_{n-i}},$$
$$y_{n+1} = \frac{19y_{n-1}}{7 + 13\prod_{i=0}^{1} z_{n-i}}, \tag{77}$$
$$z_{n+1} = \frac{4z_{n-1}}{2 + 5\prod_{i=0}^{1} x_{n-i}}.$$

The graphs of n *vs* x_n(respectively y_n and z_n) of (75) are shown in Figures 3(a), 3(b), and 3(c). In this case P_0 of (77) is unstable.

Data Availability

All the data utilized in this article has been included and the sources adopted were cited accordingly.

Disclosure

The authors declare that they got no funding on any part of this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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