Research Article

Continuous Dependence on a Parameter of Exponential Attractors for Nonclassical Diffusion Equations

Gang Wang and Chaozhu Hu

School of Science, Hubei University of Technology, Wuhan, Hubei 430068, China

Correspondence should be addressed to Gang Wang; wgfeiyu@sina.com

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In this paper, a new abstract result is given to verify the continuity of exponential attractors with respect to a parameter for the underlying semigroup. We do not impose any compact embedding on the main assumptions in the abstract result which is different from the corresponding result established by Efendiev et al. in 2004. Consequently, it can be used for equations whose solutions have no higher regularity. As an application, we prove the continuity of exponential attractors in H_0^1 for a class of nonclassical diffusion equations with initial datum in H_0^1 .

1. Introduction

In this paper, we study the existence and robustness of exponential attractors for the following nonclassical diffusion equation:

$$u_t^\epsilon - \epsilon \Delta u_t^\epsilon - \Delta u^\epsilon + f(u^\epsilon) = g, \quad x \in \Omega, t > 0, \quad (1)$$

with the initial-boundary value conditions

$$\begin{cases} u^\epsilon(x, 0) = u_0(x), & x \in \Omega, \\ u^\epsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $\epsilon \in [0, 1]$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded open set with smooth boundary $\partial\Omega$. When $\epsilon = 0$, it turns out to be the classical reaction-diffusion equation. We assume that $g \in L^2(\Omega)$ and the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following (see, e.g., [1]):

- (F1): there exists $\nu > 0$ such that $f'(s) \geq -\nu, \forall s \in \mathbb{R}$
- (F2): there exists $\kappa_1 > 0$ such that $|f'(s)| \leq \kappa_1(1 + |s|^{2/(N-2)}), \forall s \in \mathbb{R}$
- (F3): $\liminf_{|s| \rightarrow \infty} F(s)/s^2 \geq 0$, where $F(s) = \int_0^s f(r)dr$
- (F4): there exists $\kappa_2 > 0$ such that $\liminf_{|s| \rightarrow \infty} s f(s) - \kappa_2 F(s)/s^2 \geq 0$

Nonclassical diffusion equations appear in fluid mechanics, soil mechanics, and heat conduction theory (see, e.g., [2]). The long-time behavior of solutions to nonclassical diffusion equations has been extensively studied by many authors for both autonomous and nonautonomous cases [3–9].

The global attractor plays an important role in the study of long-time behavior of infinite dimension systems arising from physics and mechanics. It is a compact invariant set and attracts uniformly the bounded sets of the phase space. However, the rate of attraction may be arbitrary, and it may be sensible to perturbations. These drawbacks can be overcome by creating the notion of the exponential attractor [10], which is a compact, positively invariant set of finite dimension and exponentially attracts each bounded set. The existence of the exponential attractor has been extensively studied since 1994, see e.g., [5, 10–17].

As discussed in [12], exponential attractors are to be more robust objects under perturbations than global attractors. In general, global attractors are only upper semicontinuous with respect to perturbations, and the lower semicontinuity property is much more delicate and can be established only for some particular cases. However, one can prove the continuity of exponential attractors under perturbations in many cases [5, 13]. In particular, for problems (1) and (2), the existence of a pullback attractor was shown

by Anh and Bao in [3] for the subcritical case in $H_0^1(\Omega)$. They also proved the upper semicontinuity of the pullback attractors. However, this upper semicontinuity of the pullback attractor was established only in $L^2(\Omega)$, and the upper semicontinuity in $H_0^1(\Omega)$ remains an open problem. In this paper, we not only prove the upper and lower semicontinuity of the exponential attractor but also show these continuities in a stronger space, i.e., $H_0^1(\Omega)$ when the initial value only belongs to $H_0^1(\Omega)$.

In [12], see also [17], Efendiev et al. gave an abstract result about the robustness of exponential attractors (Theorem 4.4 in [12]). A main assumption called “compact Lipschitz condition” was proposed in that theorem. The main difficulty when we apply this result to problems (1) and (2) is that the solution to problems (1) and (2) has no regularity as $\epsilon > 0$ [3]. For example, if the initial datum u_0 belongs to $H_0^1(\Omega)$, the solution with initial $u(0) = u_0$ is always in $H_0^1(\Omega)$ and has no higher regularity. Thus, it is impossible to verify the “compact Lipschitz condition” when we want to prove the continuity of exponential attractors in $H_0^1(\Omega)$. Motivated by [18], we modify the result in [12] to adapt it to our case. Moreover, some of the coefficients are allowed to be dependent on the parameter ϵ which relax the conditions, see Theorem 1 in the following.

The rest of this paper is organized as follows. In Section 2, we formulate and prove the main abstract result, i.e., Theorem 1. In Section 3, we apply Theorem 1 to the dynamical system generated by (1) and (2) to prove the continuity of the exponential attractors, and we consider two cases according to the constant ν in this section.

Throughout this paper, we denote by $\|\cdot\|_X$ the norm of a Banach space X . The inner product and norm of $L^2(\mathbb{R}^n)$ are written as $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. We also use $\|u\|_r$ to denote the norm of $u \in L^r(\mathbb{R}^n)$ ($r \geq 1, r \neq 2$) and $|u|$ to denote the modular of u . Letter c is a generic positive constant independent of ϵ which may change its values from line to line even in the same line (sometimes for the special case, we also denote different positive constants by c_i ($i = 1, 2, \dots$)).

2. The Abstract Result and Its Proof

In this section, we modify Theorem 4.4 in [12] to adapt to problems (1) and (2). We start with the definition of exponential attractors.

Definition 1. Let E be a metric space, B is a bounded set in E , and let $S: B \rightarrow B$ be a map. We define a discrete semigroup $\{S^n, n \in \mathbb{Z}^+\}$ by $S^n x := S \circ \dots \circ S$ (n times). A set $\mathcal{M} \subset B$ is an exponential attractor for the map if the following properties hold:

- (1) The set $\mathcal{M}$ is compact in E and has finite fractal dimension.
- (2) The set $\mathcal{M}$ is positively invariant with respect to S , i.e., $S(\mathcal{M}) \subset \mathcal{M}$.
- (3) There exist positive constants α_0 and β_0 such that

$$\text{dist}_E(S^n B, \mathcal{M}) \leq \alpha_0 e^{-\beta_0 n}, \quad (3)$$

where $\text{dist}_E(C_1, C_2)$ denotes the Hausdorff semidistance between C_1 and C_2 in E given by

$$\text{dist}_E(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} \|x - y\|_E, \quad \text{for } C_1, C_2 \subset E. \quad (4)$$

Definition 2. Let X be a complete metric space endowed with the metric d and M be a bounded closed set in X . Assume that ϱ is a pseudometric [19] defined on M . Let $B \subset M$ and $\epsilon > 0$.

- (i) A subset $\mathcal{U}$ in B is said to be (ϵ, ϱ) -distinguishable if $\varrho(x, x') > \epsilon$ for any $x, x' \in \mathcal{U}, x \neq x'$. We denote by $m_\varrho(B, \epsilon)$ the maximal cardinality of an (ϵ, ϱ) -distinguishable subset of B .
- (ii) Pseudometric ϱ is said to be compact on M iff $m_\varrho(M, \epsilon)$ is finite for every $\epsilon > 0$.
- (iii) For any $r > 0$, we define a local (r, ϵ, ϱ) -capacity of set M by the formula

$$\mathcal{C}_\varrho(M; r, \epsilon) = \sup \{ \ln m_\varrho(B, \epsilon) : B \subset M, \text{diam} B \leq 2r \}. \quad (5)$$

We now state and prove the main abstract theorem. The proof is essentially a combination of that in [18] and that of [12].

Theorem 1. Let X be a Banach space and B be a bounded set in X . We assume that there exists a family of operators $S_\epsilon : B \rightarrow B$, $\epsilon \in [0, \epsilon_0]$, which satisfy, for any $\epsilon \in [0, \epsilon_0]$,

- (i) S_ϵ is Lipschitz on B , i.e., there exists $L_\epsilon > 0$ such that

$$\|S_\epsilon x_1 - S_\epsilon x_2\|_X \leq L_\epsilon \|x_1 - x_2\|_X, \quad \forall x_1, x_2 \in B. \quad (6)$$

- (ii) There exist constants η and K_ϵ and compact seminorms $n_1^\epsilon(x)$ and $n_2^\epsilon(x)$ on B such that

$$\|S_\epsilon x_1 - S_\epsilon x_2\|_X \leq \eta \|x_1 - x_2\|_X + K_\epsilon [n_1^\epsilon(x_1 - x_2) + n_2^\epsilon(S_\epsilon x_1 - S_\epsilon x_2)], \quad (7)$$

for any $x_1, x_2 \in B$, where $0 < \eta < 1$ is independent of ϵ and $K_\epsilon > 0$ is a constant which may be dependent on ϵ (seminorm $n(x)$ on B is said to be compact iff for any subset $B' \subset B$ if there exists a sequence $\{x_n\} \subset B'$ such that $n(x_m - x_n) \rightarrow 0$ as $n, m \rightarrow \infty$).

- (iii) $\forall \epsilon \in [0, \epsilon_0], \forall i \in \mathbb{N}, \forall x \in B$,

$$\|S_\epsilon^i x - S_0^i x\|_X \leq \kappa^i \epsilon^\alpha, \quad (8)$$

where $\kappa > 0$ and $\alpha \in (0, 1)$ are constants independent of ϵ and for mapping V , $V^i = V \circ \dots \circ V$ (i times).

Then, $\forall \epsilon \in [0, \epsilon_0]$, and the discrete dynamical system generated by the iterations of S_ϵ possesses an exponential attractor $\mathcal{M}_\epsilon$ on B such that

(1) The fractal dimension of $\mathcal{M}_\epsilon$ is bounded in X :

$$\dim_{f,X} \mathcal{M}_\epsilon \leq c(\epsilon) := \left[\ln \frac{2}{1+\eta} \right]^{-1} \ln m_\epsilon \left(\frac{4K_\epsilon(1+L_\epsilon^2)^{1/2}}{1-\eta} \right), \quad (9)$$

where $\dim_{f,X} \mathcal{A}$ denotes the fractal dimension of $\mathcal{A}$ in X and $m_\epsilon(R)$ is the maximal number of pairs (x_i, y_i) in $X \times X$ possessing the properties

$$\|x_i\|_X^2 + \|y_i\|_X^2 \leq R^2, n_1^\epsilon(x_i - x_j) + n_2^\epsilon(y_i - y_j) > 1, \quad i \neq j. \quad (10)$$

(2) $\mathcal{M}_\epsilon$ attracts B in X , uniformly with respect to ϵ ,

$$\text{dist}_X(S_\epsilon^i B, \mathcal{M}_\epsilon) \leq c_1 e^{-c_2 i}, \quad c_2 > 0, i \in \mathbb{N}, \quad (11)$$

where c_1 and c_2 are independent of ϵ .

(3) The family $\{\mathcal{M}_\epsilon, \epsilon \in [0, \epsilon_0]\}$ is continuous at 0:

$$\text{dist}_{\text{sym},X}(\mathcal{M}_\epsilon, \mathcal{M}_0) \leq c_3 \epsilon^{c_4}, \quad (12)$$

where c_3 and $c_4 \in (0, 1)$ are independent of ϵ and $\text{dist}_{\text{sym},X}$ denotes the symmetric Hausdorff distance in X between sets defined by

$$\text{dist}_{\text{sym},X}(A, B) := \max[\text{dist}_X(A, B), \text{dist}_X(B, A)]. \quad (13)$$

Proof. For any fixed $\epsilon \in [0, \epsilon_0]$, we set $\varrho_\epsilon(x, y) = K_\epsilon[n_1^\epsilon(x - y) + n_2^\epsilon(S_\epsilon x - S_\epsilon y)]$; then, ϱ_ϵ is compact on B in the sense of Definition 2. From [18], we see that the local $(r, \rho, \varrho_\epsilon)$ -capacity of the set B admits the estimate

$$\mathcal{C}_{\varrho_\epsilon}(B; r, \rho) \leq \ln m_\epsilon \left[\frac{2K_\epsilon(1+L_\epsilon^2)^{1/2}r}{\rho} \right], \quad (14)$$

where $m_\epsilon(R)$ is the maximal number of pairs (x_i, y_i) in $X \times X$ possessing the properties

$$\|x_i\|_X^2 + \|y_i\|_X^2 \leq R^2, n_1^\epsilon(x_i - x_j) + n_2^\epsilon(y_i - y_j) > 1, \quad i \neq j. \quad (15)$$

We assume $\text{diam } B = 2R$. Let $\delta = 1 - \eta/2$ and $\{x_{i_1} : i_1 = 1, \dots, n_1\}$ be a maximal $(\delta R, \varrho_\epsilon)$ -distinguishable subset of B . Then, from (14), we have

$$\begin{aligned} n_1 &= m_{\varrho_\epsilon}(B, \delta R) \leq \exp\{\mathcal{C}_{\varrho_\epsilon}(B; R, \delta R)\} \\ &\leq m_\epsilon \left(\frac{2K_\epsilon(1+L_\epsilon^2)^{1/2}}{\delta} \right) := P_\epsilon, \end{aligned} \quad (16)$$

$$B = \bigcup_{i_1=1}^{n_1} B_{i_1},$$

$$B_{i_1} = \{v \in B : \varrho_\epsilon(v, x_{i_1}) \leq \delta R\}.$$

Therefore,

$$S_\epsilon B = \bigcup_{i_1=1}^{n_1} S_\epsilon B_{i_1}. \quad (17)$$

If $y_1, y_2 \in B_{i_1}$, then from (7), we have

$$\begin{aligned} \|S_\epsilon y_1 - S_\epsilon y_2\|_X &\leq \eta \|y_1 - y_2\|_X + \varrho_\epsilon(y_1, x_{i_1}) \\ &\quad + \varrho_\epsilon(y_2, x_{i_1}) \leq 2\eta R + 2\delta R := 2qR, \end{aligned} \quad (18)$$

where $q = \eta + \delta = 1 + \eta/2 < 1$. We set $E_\epsilon^0 = \{x_{i_1}\}_{i_1=1}^{n_1}$ and $E_\epsilon^1 = S_\epsilon E_\epsilon^0$; then,

$$\begin{aligned} E_\epsilon^1 &\subset S_\epsilon B, S_\epsilon E_\epsilon^0 \subset E_\epsilon^1, \\ \#E_\epsilon^1 &\leq P_\epsilon \leq P_\epsilon^2, \end{aligned} \quad (19)$$

$$\text{dist}_X(S_\epsilon B, E_\epsilon^1) \leq 2qR.$$

Next, for any fixed $i_1 \in \{1, \dots, n_{i_1}\}$, we assume that $\{x_{i_1, i_2} : i_2 = 1, \dots, n_{i_1, 2}\}$ be a maximal $(\delta qR, \varrho_\epsilon)$ -distinguishable subset of $S_\epsilon B_{i_1}$. Then,

$$\begin{aligned} n_{i_1, 2} &= m_{\varrho_\epsilon}(S_\epsilon B_{i_1}, \delta qR) \leq \exp\{\mathcal{C}_{\varrho_\epsilon}(S_\epsilon B; qR, \delta qR)\} \\ &\leq \exp\{\mathcal{C}_{\varrho_\epsilon}(B; qR, \delta qR)\} \leq P_\epsilon, \end{aligned}$$

$$S_\epsilon B = \bigcup_{i_1=1}^{n_1} \bigcup_{i_2=1}^{n_{i_1, 2}} B_{i_1, i_2}, B_{i_1, i_2} = \{v \in S_\epsilon B_{i_1} : \varrho_\epsilon(v, x_{i_1, i_2}) \leq \delta qR\}. \quad (20)$$

Therefore,

$$S_\epsilon^2 B = \bigcup_{i_1=1}^{n_1} \bigcup_{i_2=1}^{n_{i_1, 2}} S_\epsilon B_{i_1, i_2}. \quad (21)$$

If $y_1, y_2 \in B_{i_1, i_2}$, then from (7), we have

$$\begin{aligned} \|S_\epsilon y_1 - S_\epsilon y_2\|_X &\leq \eta \|y_1 - y_2\|_X + \varrho_\epsilon(y_1, x_{i_1, i_2}) + \varrho_\epsilon(y_2, x_{i_1, i_2}) \\ &\leq 2\eta qR + 2\delta qR = 2q^2 R. \end{aligned} \quad (22)$$

We set $E_\epsilon^2 = S_\epsilon E_\epsilon^1 \cup \{S_\epsilon x_{i_1, i_2}\}$, and then we have

$$\begin{aligned} E_\epsilon^2 &\subset S_\epsilon^2 B, \\ S_\epsilon E_\epsilon^1 &\subset E_\epsilon^2, \\ \#E_\epsilon^2 &\leq P_\epsilon^2 + P_\epsilon^2 \leq P_\epsilon^3, \\ \text{dist}_X(S_\epsilon^2 B, E_\epsilon^2) &\leq 2q^2 R. \end{aligned} \quad (23)$$

By the induction procedure, we can find sets E_ϵ^i , $i \in \mathbb{N}$, enjoy the following properties:

$$E_\epsilon^i \subset S_\epsilon^i B, \quad (24)$$

$$S_\epsilon E_\epsilon^i \subset E_\epsilon^{i+1}, \quad (25)$$

$$\#E_\epsilon^i \leq P_\epsilon^{i+1}, \quad (26)$$

$$\text{dist}_X(S_\epsilon^i B, E_\epsilon^i) \leq 2q^i R. \quad (27)$$

We now define the exponential attractor for the map $S_0 : B \longrightarrow B$ as follows:

$$\begin{aligned}\mathcal{M}'_0 &= \bigcup_{i=1}^{\infty} E_0^i, \\ \mathcal{M}_0 &= \overline{\mathcal{M}'_0}^X.\end{aligned}\quad (28)$$

Then, from (24)–(27), we see that $\mathcal{M}_0$ is indeed an exponential attractor for the map $S_0: B \rightarrow B$ (see [12]). For $\epsilon \in (0, \epsilon_0]$, one can also construct exponential attractors for $S_\epsilon: B \rightarrow B$ as above. However, they are not the ones we needed. Totally similar to [12], one can construct exponential attractors $\mathcal{M}_\epsilon$ based on $\{E_0^i\}$. We note that the only difference between our construction procedure and that of in [12] is that the number P_ϵ here may be dependent on ϵ . However, (26) only contributes to the fractal dimension of $\mathcal{M}_\epsilon$. Therefore, (11) and (12) in Theorem 1 hold true. Finally, assertion (9) is a direct result of [18]. The proof is completed.

Remark 1. Similar to [12], we can give an explicit expression for c_4 , i.e., $c_4 = -\alpha \ln q / \ln \kappa - \ln q$. If $L_\epsilon, K_\epsilon, n_1^\epsilon(x)$, and $n_2^\epsilon(x)$ are all independent of ϵ , we can obtain the uniform bound with respect to ϵ of the fractal dimension, and we will apply this abstract result in the last section.

3. Application to the Nonclassical Diffusion Equation

3.1. Some Useful Estimates of the Solution. Since $(-\Delta)^{-1}$ is a continuous compact operator in $L^2(\Omega)$, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$ as $j \rightarrow \infty$ and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $D(-\Delta)$, which forms an orthogonal basis in both $L^2(D)$ and $H_0^1(D)$ such that $-\Delta e_j = \lambda_j e_j, \forall j \in \mathbb{N}$. Given m , let $X_m = \text{span}\{e_1, \dots, e_m\}$ and $P_m: L^2(D) \rightarrow X_m$ be the projection operator. For any $v \in H_0^1(\Omega)$, we write $v = P_m v + (I - P_m)v =: v_1 + v_2$.

Lemma 1 (see [1]). *Let $\epsilon > 0$. Assume (F1)–(F4) hold and $g \in L^2(\Omega)$. Then, for each $u_0 \in H_0^1(\Omega)$, problems (1) and (2) have a unique solution $u^\epsilon = u^\epsilon(t) = u^\epsilon(t; u_0)$ with $u^\epsilon \in C\{[0, T], H_0^1(\Omega)\}$ and $u_t^\epsilon \in L^2[0, T; H_0^1(\Omega)]$ for any $T > 0$. Moreover, for any fixed $t > 0$, u^ϵ is continuous in u_0 .*

Lemma 2 (see [1]). *Let $\epsilon = 0$. Assume (F1)–(F4) hold and $g \in L^2(\Omega)$. Then, for each $u_0 \in H_0^1(\Omega)$, problems (1) and (2) have a unique solution $u^0 = u^0(t) = u^0(t; u_0)$ which satisfies*

$$u^0 \in C\{[0, T]; H_0^1(\Omega)\} \cap L^2[0, T; H^2(\Omega)], \quad \forall T > 0. \quad (29)$$

Also for any fixed $t > 0$, u^0 is continuous in u_0 .

From Lemmas 1 and 2, we can define a semigroup $S_\epsilon(t): H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ by the expression

$$S_\epsilon(t)u_0 = u^\epsilon(t), \quad t \geq 0, \quad (30)$$

where $u^\epsilon(t)$ is the solution of (1) and (2).

Lemma 3 (see [1]). *Assume (F1)–(F4) hold and $g \in L^2(\Omega)$. Then, for any bounded set $D \subset H_0^1(\Omega)$, there exist positive*

constants $E(D)$, c_0 , and $T_1(D)$ such that, for any solution u^ϵ of problems (1) and (2),

$$\begin{aligned}\|\nabla u^\epsilon(t)\| &\leq E(D), \quad t \geq 0, \\ \|\nabla u^\epsilon(t)\| &\leq c_0, \quad t \geq T_1(D),\end{aligned}\quad (31)$$

provided $u_0 \in D$, where $E(D)$, c_0 , and $T_1(D)$ are independent of ϵ .

Set $B = \{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq c_0\}$, where c_0 is the constant in Lemma 3; then, B is a uniformly (with respect to ϵ) bounded absorbing set for $S_\epsilon(t)$ in $H_0^1(\Omega)$. We note that the absorbing time is independent of ϵ , and we can choose $T_1 = T_1(B)$ large enough such that $S_\epsilon(t)B \subset B$ for any $t \geq T_0$ and any $\epsilon \in [0, 1]$.

The following lemma gives several priori estimates for the derivative $u_t^\epsilon(t)$ of the solution to (1) and (2).

Lemma 4. *Under the assumptions (F1)–(F4), for any $D \in H_0^1(\Omega)$, D is bounded, and there exists $T_2(D)$, which is independent of ϵ such that, for all $\epsilon \in [0, 1]$,*

$$\|u_t^\epsilon(t)\|^2 + \epsilon \|\nabla u_t^\epsilon(t)\|^2 \leq c, \quad (32)$$

$$\int_t^{t+1} \|\nabla u_t^\epsilon(s)\|^2 ds \leq c, \quad (33)$$

for any $u_0 \in D$ and any $t \geq T_2(D)$, where c is a constant independent of ϵ .

Proof. We first take the inner product of (1) with $u^\epsilon(t)$ in $L^2(\Omega)$ to get

$$\frac{1}{2} \frac{d}{dt} \left[\|\nabla u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \|\nabla u^\epsilon\|^2 + [f(u^\epsilon), u^\epsilon] = (g, u^\epsilon). \quad (34)$$

By condition (F4), for any $\theta > 0$, there exists $c_\theta > 0$ satisfying

$$[f(u), u] - \kappa_2 \int_\Omega F(u) + \theta \|u\|^2 + c_\theta \geq 0, \quad \forall u \in H_0^1(\Omega). \quad (35)$$

Putting (35) into (34), it yields that

$$\begin{aligned}\frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + 2 \|\nabla u^\epsilon\|^2 + 2 \kappa_2 \int_\Omega F(u^\epsilon) &\leq 2(g, u^\epsilon) \\ &+ 2\theta \|u^\epsilon\|^2 + 2c_\theta.\end{aligned}\quad (36)$$

By condition (F3), for any $\gamma > 0$, there exists $c_\gamma > 0$ satisfying

$$\int_\Omega F(u) + \gamma \|u\|^2 + c_\gamma \geq 0, \quad \forall u \in H_0^1(\Omega). \quad (37)$$

Combining (36) and (37), we get

$$\begin{aligned}\frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + 2 \|\nabla u^\epsilon\|^2 &\leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 \\ &+ 2c_\theta + 2\kappa_2 \gamma \|u^\epsilon\|^2 + 2\kappa_2 c_\gamma,\end{aligned}\quad (38)$$

that is,

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \|u^\epsilon\|^2 &\leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 \\ &+ 2c_\theta + 2\kappa_2 \gamma \|u^\epsilon\|^2 + 2\kappa_2 c_\gamma. \end{aligned} \quad (39)$$

We choose $\alpha_1 > 0$ small enough such that $\alpha_1 \lambda_1 < 1$ and $1 - \alpha_1 > 1/2$, and apply Poincaré inequality in (39) to get

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \alpha_1 \lambda_1 \|u^\epsilon\|^2 + (1 - \alpha_1) \|u^\epsilon\|^2 \\ \leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 + 2c_\theta + 2\kappa_2 \gamma \|u^\epsilon\|^2 + 2\kappa_2 c_\gamma. \end{aligned} \quad (40)$$

Choosing θ, γ small enough and using Hölder inequality, we can get from (40) that

$$\frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \frac{\alpha_1 \lambda_1}{2} \|u^\epsilon\|^2 + \frac{1}{2} \|\nabla u^\epsilon\|^2 \leq c. \quad (41)$$

We set $\sigma = \alpha_1 \lambda_1 / 2$; then, $0 < \sigma < 1/2$. From (41), we obtain

$$\frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \sigma \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] \leq c, \quad (42)$$

that is,

$$\frac{d}{dt} e^{\sigma t} \left[\|\nabla u^\epsilon(t)\|^2 + \epsilon \|\nabla u^\epsilon(t)\|^2 \right] \leq c e^{\sigma t}. \quad (43)$$

Integrating the above inequality from 0 to t , it yields that

$$e^{\sigma t} \left[\|\nabla u^\epsilon(t)\|^2 + \epsilon \|\nabla u^\epsilon(t)\|^2 \right] \leq \left[\|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c e^{\sigma t}, \quad t \geq 0. \quad (44)$$

Now, we consider (36) again.

If $0 < \kappa_2 \leq 1$, we can get from (36) that

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \|\nabla u^\epsilon\|^2 + \kappa_2 \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \\ \leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 + 2c_\theta. \end{aligned} \quad (45)$$

Using a similar process as (39)–(42), we deduce that, for θ small enough,

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \sigma \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] \\ + \kappa_2 \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c. \end{aligned} \quad (46)$$

Therefore,

$$\frac{d}{dt} e^{\sigma t} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \kappa_2 e^{\sigma t} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c e^{\sigma t}. \quad (47)$$

If $\kappa_2 > 1$, we can get from (36) that

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + 2 \|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \\ + (2\kappa_2 - 2) \int_\Omega F(u^\epsilon) \leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 + 2c_\theta. \end{aligned} \quad (48)$$

Applying (37) to the above, we obtain

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \|\nabla u^\epsilon\|^2 + \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \\ \leq 2(g, u^\epsilon) + 2\theta \|u^\epsilon\|^2 + 2c_\theta + (2\kappa_2 - 2) \gamma \|u^\epsilon\|^2 + (2\kappa_2 - 2) c_\gamma. \end{aligned} \quad (49)$$

Following a similar process as (39)–(42), we deduce that, for θ, γ small enough,

$$\begin{aligned} \frac{d}{dt} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + \sigma \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] \\ + \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c. \end{aligned} \quad (50)$$

Therefore,

$$\frac{d}{dt} e^{\sigma t} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + e^{\sigma t} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c e^{\sigma t}. \quad (51)$$

From (47) and (51), we see that, for any $\kappa_2 > 0$,

$$\frac{d}{dt} e^{\sigma t} \left[\|u^\epsilon\|^2 + \epsilon \|\nabla u^\epsilon\|^2 \right] + c_5 e^{\sigma t} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c e^{\sigma t}. \quad (52)$$

where $c_5 = \kappa_2$ when $0 < \kappa_2 \leq 1$ and $c_5 = 1$ when $\kappa_2 > 1$.

For any $t > 0$, we integrate (52) over $[t, t+1]$ to get

$$\begin{aligned} \int_t^{t+1} e^{\sigma s} \left[\|\nabla u^\epsilon(s)\|^2 + 2 \int_\Omega F(u^\epsilon(s)) ds \right] \\ \leq c e^{\sigma t} \left[\|u^\epsilon(t)\|^2 + \epsilon \|\nabla u^\epsilon(t)\|^2 \right] + c e^{\sigma t}. \end{aligned} \quad (53)$$

Putting (44) into (53), we have, for any $t > 0$,

$$\int_t^{t+1} e^{\sigma s} \left[\|\nabla u^\epsilon(s)\|^2 + 2 \int_\Omega F(u^\epsilon(s)) ds \right] \leq c \left[\|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c e^{\sigma t}. \quad (54)$$

Next, we multiply (1) with u_t and integrate it in Ω to get

$$\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] = (g, u_t^\epsilon). \quad (55)$$

Applying Hölder inequality to the above we get

$$\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 + \frac{d}{dt} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \leq c, \quad t > 0. \quad (56)$$

Thus,

$$\begin{aligned}
& e^{\sigma t} \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right] + \frac{d}{dt} e^{\sigma t} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right] \\
& \leq c e^{\sigma t} + c e^{\sigma t} \left[\|\nabla u^\epsilon\|^2 + 2 \int_\Omega F(u^\epsilon) \right], \quad t > 0.
\end{aligned} \tag{57}$$

Dropping the first term in the left-hand side of (57), from (54) and using uniform Gronwall inequality, we conclude that

$$e^{\sigma t} \left\{ \|\nabla u^\epsilon(t)\|^2 + 2 \int_\Omega F(u^\epsilon(t)) \right\} \leq c \left[\|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c e^{\sigma t}, \quad t \geq 1. \tag{58}$$

We now differentiate equation (1) with respect to t to get

$$u_{tt}^\epsilon - \epsilon \Delta u_{tt}^\epsilon - \Delta u_t^\epsilon + f'(u^\epsilon) u_t^\epsilon = 0. \tag{59}$$

Taking the inner product of (59) with u_t^ϵ in $L^2(\Omega)$, we obtain

$$\frac{d}{dt} \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right] + 2 \|\nabla u_t^\epsilon\|^2 \leq c \|u_t^\epsilon\|^2. \tag{60}$$

Since $0 < \sigma < 1/2$, we have

$$\frac{d}{dt} \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right] + \sigma \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right] \leq c \|u_t^\epsilon\|^2. \tag{61}$$

Hence,

$$\frac{d}{dt} e^{\sigma t} \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right] \leq c e^{\sigma t} \left[\|u_t^\epsilon\|^2 + \epsilon \|\nabla u_t^\epsilon\|^2 \right]. \tag{62}$$

Integrating (57) over $(t, t+1)$ and using (58), we deduce that

$$\int_t^{t+1} e^{\sigma s} \left[\|u_t^\epsilon(s)\|^2 + \epsilon \|\nabla u_t^\epsilon(s)\|^2 \right] ds \leq c \left[\|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c e^{\sigma t}, \quad t \geq 1. \tag{63}$$

Combining (62) and (63) and using the uniform Gronwall inequality, we obtain that

$$\|u_t^\epsilon(t)\|^2 + \epsilon \|\nabla u_t^\epsilon(t)\|^2 \leq c e^{-\sigma t} \left[\|u_0\|^2 + \epsilon \|\nabla u_0\|^2 \right] + c, \quad t \geq 2. \tag{64}$$

Thus, for any bounded $D \subset H_0^1(\Omega)$, there exists $T_2(D) \geq 2$ which is independent of ϵ such that, for any $t \geq T_2(D)$ and any $u_0 \in D$,

$$\|u_t^\epsilon(t)\|^2 + \epsilon \|\nabla u_t^\epsilon(t)\|^2 \leq c, \tag{65}$$

where c is independent of ϵ . This proves assertion (32) in Lemma 4.

Finally, integrating (60) over $(t, t+1)$, then assertion (33) can be easily verified by using (32). The proof is completed.

Lemma 5. Under conditions (F1)–(F4), for any two solutions of (1) and (2) $u^{\epsilon_1}(t)$, $u^{\epsilon_2}(t)$ (corresponding to $\epsilon_1, \epsilon_2 \in [0, 1]$, respectively), with initials starting from B and for any $v \in L^2(\Omega)$, we have

$$\left| \int_\Omega \{f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)]\} v dx \right| \leq c_6 \|\nabla w^{\epsilon_1, \epsilon_2}(t)\| \|v\|, \quad t \geq 0, \tag{66}$$

where $w^{\epsilon_1, \epsilon_2}(t) = u^{\epsilon_1}(t) - u^{\epsilon_2}(t)$ and c_6 is a constant independent of ϵ .

Proof. By (F2), we have

$$\begin{aligned}
& \left| \int_\Omega \{f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)]\} v dx \right| \leq \int_\Omega |l(t, \epsilon_1, \epsilon_2)| |w^{\epsilon_1, \epsilon_2}(t)| |v| dx \\
& \leq c \int_\Omega \left[1 + |u^{\epsilon_1}(t)|^{2/N-2} + |u^{\epsilon_2}(t)|^{2/N-2} \right] |w^{\epsilon_1, \epsilon_2}(t)| |v| dx \leq c \int_\Omega |w^{\epsilon_1, \epsilon_2}(t)| |v| dx \\
& + c \int_\Omega |u^{\epsilon_1}(t)|^{2/N-2} |w^{\epsilon_1, \epsilon_2}(t)| |v| dx + c \int_\Omega |u^{\epsilon_2}(t)|^{2/N-2} |w^{\epsilon_1, \epsilon_2}(t)| |v| dx,
\end{aligned} \tag{67}$$

where $l(t, \epsilon_1, \epsilon_2) = \int_0^1 f'[su^{\epsilon_1}(t) + (1-s)u^{\epsilon_2}(t)] ds$.

From (31) and the Sobolev embedding $H_0^1(\Omega) \subset L^{2N/(N-2)}(\Omega)$, we have that, for $t \geq 0$,

$$\begin{aligned}
& \left| \int_\Omega \{f[u^{\epsilon_1}(t)] - f[u^{\epsilon_2}(t)]\} v dx \right| \\
& \leq c \|w^{\epsilon_1, \epsilon_2}\| \|v\| + c \|u^{\epsilon_1}\|_{2N/(N-2)}^{2/N-2} \|w^{\epsilon_1, \epsilon_2}\|_{\frac{2N}{N-2}} \|v\| + c \|u^{\epsilon_2}\|_{2N/(N-2)}^{2/N-2} \|w^{\epsilon_1, \epsilon_2}\|_{2N/(N-2)} \|v\| \\
& \leq c \|\nabla w^{\epsilon_1, \epsilon_2}\| \|v\| + c \|u^{\epsilon_1}\|^{2/N-2} \|\nabla w^{\epsilon_1, \epsilon_2}\| \|v\| + c \|\nabla u^{\epsilon_2}\|^{2/N-2} \|\nabla w^{\epsilon_1, \epsilon_2}\| \|v\| \\
& \leq c_6 \|\nabla w^{\epsilon_1, \epsilon_2}\| \|v\|.
\end{aligned} \tag{68}$$

The proof is completed.

3.2. The Main Result for a General Case: $\nu \in \mathbb{R}^+$. Now, we verify the conditions in Theorem 1 for $S_\epsilon(t)$ in this case. We first verify condition (i), i.e., the Lipschitz continuity in H_0^1 (actually uniform Lipschitz continuity in H_0^1 since the coefficient is independent of ϵ).

Lemma 6. *Under assumptions (F1)–(F4), we have, for any $\epsilon \in [0, 1]$ and any $x_1, x_2 \in B$,*

$$\|S_\epsilon(t)x_1 - S_\epsilon(t)x_2\|_{H_0^1(\Omega)} \leq e^{c_6^2 t} \|x_1 - x_2\|_{H_0^1(\Omega)}, \quad t \geq 0, \quad (69)$$

where c_6 is the constant in Lemma 5.

Proof. Assume that $u_1^\epsilon(t)$ and $u_2^\epsilon(t)$ are two solutions of (1) and (2) starting from $x_1, x_2 \in B$, respectively. We consider the difference $w^\epsilon(t) = u_1^\epsilon(t) - u_2^\epsilon(t)$, and then $w^\epsilon(t)$ satisfies

$$w_t^\epsilon(t) - \epsilon \Delta w_t^\epsilon(t) - \Delta w^\epsilon(t) + l(t, \epsilon)w^\epsilon(t) = 0. \quad (70)$$

Multiplying (70) with $w_t^\epsilon(t)$ and integrating it in Ω , we get

$$\|w_t^\epsilon(t)\|^2 + \epsilon \|\nabla w_t^\epsilon(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w^\epsilon(t)\|^2 \leq \int_\Omega |l(t, \epsilon)| w^\epsilon(t) \|w_t^\epsilon(t)\| dx. \quad (71)$$

Applying Lemma 5, we obtain

$$\|w_t^\epsilon(t)\|^2 + \epsilon \|\nabla w_t^\epsilon(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w^\epsilon(t)\|^2 \leq c_6 \|\nabla w^\epsilon(t)\| \|w_t^\epsilon(t)\|, \quad t \geq 0. \quad (72)$$

We use Hölder inequality to get

$$\frac{d}{dt} \|\nabla w^\epsilon(t)\|^2 \leq c_6^2 \|\nabla w^\epsilon(t)\|^2. \quad (73)$$

Then, the result follows immediately from Gronwall inequality. The proof is completed.

Lemma 7. *Under assumptions (F1)–(F4), we have, for any $x_1, x_2 \in B$ and any $\eta \in (0, 1)$,*

(i) *For $\epsilon \in (0, \epsilon_0]$: there exist $T_3 = T_3(\eta)$, $K > 0$, and compact seminorm $n_1^{\epsilon, t}$ on B such that*

$$\begin{aligned} \|S_\epsilon(t)x_1 - S_\epsilon(t)x_2\|_{H_0^1(\Omega)} &\leq \eta \|x_1 - x_2\|_{H_0^1(\Omega)} \\ &+ K n_1^{\epsilon, t}(x_1 - x_2), \quad t \geq T_3. \end{aligned} \quad (74)$$

(ii) *For $\epsilon = 0$: there exist a positive integer M , $T_3 = T_3(\eta)$, and $K > 0$ such that*

$$\begin{aligned} \|S_0(t)x_1 - S_0(t)x_2\|_{H_0^1(\Omega)} &\leq \eta \|x_1 - x_2\|_{H_0^1(\Omega)} \\ &+ K n_2[S_0(t)x_1 - S_0(t)x_2], \quad t \geq T_3, \end{aligned} \quad (75)$$

for some $\epsilon_0 > 0$, where $n_2(x) = \|P_M x\|_{H_0^1(\Omega)}$, M and K are independent of ϵ , η , and t and $n_1^{\epsilon, t}$ depend on ϵ and t .

Proof. We first take the inner product of (70) with $k w^\epsilon$ in $L^2(\Omega)$ (k is a constant which will be fixed later) to get

$$k(w_t^\epsilon, w^\epsilon) + k\epsilon(\nabla w_t^\epsilon, \nabla w^\epsilon) + k\|\nabla w^\epsilon\|^2 + k \int_\Omega l(t, \epsilon) |w^\epsilon|^2 = 0. \quad (76)$$

Using (F1), we get

$$k\|\nabla w^\epsilon\|^2 \leq k\|w_t^\epsilon\| \|w^\epsilon\| + k\epsilon \|\nabla w_t^\epsilon\| \|\nabla w^\epsilon\| + k\gamma \|w^\epsilon\|^2. \quad (77)$$

Combining (72) and (77), we get

$$\begin{aligned} \|w_t^\epsilon\|^2 + \epsilon \|\nabla w_t^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w^\epsilon\|^2 + k\|\nabla w^\epsilon\|^2 \\ \leq k\|w_t^\epsilon\| \|w^\epsilon\| + k\epsilon \|\nabla w_t^\epsilon\| \|\nabla w^\epsilon\| + k\gamma \|w^\epsilon\|^2 + c_6 \|\nabla w^\epsilon\| \|w_t^\epsilon\|. \end{aligned} \quad (78)$$

For the right-hand side of (78), we use Hölder inequality to get

$$\begin{aligned} k\|w_t^\epsilon\| \|w^\epsilon\| + k\epsilon \|\nabla w_t^\epsilon\| \|\nabla w^\epsilon\| + k\gamma \|w^\epsilon\|^2 + c_6 \|\nabla w^\epsilon\| \|w_t^\epsilon\| \\ \leq \left(k\gamma + \frac{k^2}{2}\right) \|w^\epsilon\|^2 + \|w_t^\epsilon\|^2 + \frac{\epsilon}{2} \|\nabla w_t^\epsilon\|^2 + \left(\frac{k^2\epsilon}{2} + \frac{c_6^2}{2}\right) \|\nabla w^\epsilon\|^2. \end{aligned} \quad (79)$$

Putting the above inequality into (78), we obtain

$$\frac{d}{dt} \|\nabla w^\epsilon\|^2 + 2k\|\nabla w^\epsilon\|^2 \leq (2k\gamma + k^2) \|w^\epsilon\|^2 + (k^2\epsilon + c_6^2) \|\nabla w^\epsilon\|^2. \quad (80)$$

Fix $k = c_6^2$, and then choose ϵ_0 such that $\epsilon < 1/c_6^2$, $\forall \epsilon \leq \epsilon_0$. Then, we get from (80) that

$$\frac{d}{dt} \|\nabla w^\epsilon\|^2 + c_7 \|\nabla w^\epsilon\|^2 \leq c \|w^\epsilon\|^2, \quad \epsilon \in [0, \epsilon_0], c_7 > 0, \quad (81)$$

that is,

$$\frac{d}{dt} e^{c_7 t} \|\nabla w^\epsilon\|^2 \leq c e^{c_7 t} \|w^\epsilon\|^2, \quad \epsilon \in [0, \epsilon_0]. \quad (82)$$

We integrate the above inequality over $(0, t)$ to get

$$\|\nabla w^\epsilon(t)\|^2 \leq e^{-c_7 t} \|\nabla w^\epsilon(0)\|^2 + c \int_0^t \|w^\epsilon(s)\|^2 ds, \quad \epsilon \in [0, \epsilon_0]. \quad (83)$$

For any $\eta \in (0, 1)$, we can choose $T_3 = T_3(\eta) > 0$ such that $e^{-c_7 T_3} \leq \eta^2$. Therefore, from (83),

$$\|\nabla w^\epsilon(t)\|^2 \leq \eta^2 \|\nabla w^\epsilon(0)\|^2 + c \int_0^t \|w^\epsilon(s)\|^2 ds, \quad t \geq T_3, \epsilon \in [0, \epsilon_0]. \quad (84)$$

Thus,

$$\|\nabla w^\epsilon(t)\| \leq \eta \|\nabla w^\epsilon(0)\| + K \left[\int_0^t \|w^\epsilon(s)\|^2 ds \right]^{1/2}, \quad t \geq T_3, \epsilon \in [0, \epsilon_0]. \quad (85)$$

Now, we set $n_1^{\epsilon,t}(x_1 - x_2) = [\int_0^t \|w^\epsilon(s)\|^2 ds]^{1/2}$. We need to show that $n_1^{\epsilon,t}$ is compact on B for any $t > 0$ and any $\epsilon \in [0, 1]$. From condition (F2), we deduce that, for any $s \in \mathbb{R}$,

$$f(s)s \leq c(|s| + |s|^2 + |s|^{2N-2/N-2}). \quad (86)$$

Hence,

$$\begin{aligned} [f(u), u] &\leq c(\|u\|^2 + \|u\|_{2N-2/N-2}^{2N-2/N-2}) \leq c(\|u\|^2 + \|u\|_{2N-2/N-2}^{2N-2/N-2}) \\ &\leq c(\|u\|^2 + \|\nabla u\|^{2N-2/N-2}) \leq c(\|\nabla u\|^2 + \|\nabla u\|^{2N-2/N-2}), \quad \forall u \in H_0^1(\Omega), \end{aligned} \quad (87)$$

where we have used the Sobolev embedding $H_0^1(\Omega) \subset L^{2N/N-2}(\Omega)$ in the above inequality. From (35) and (87), we obtain

$$\kappa_2 \int_\Omega F(u) \leq \theta \|u\|^2 + c_\theta + c(\|\nabla u\|^2 + \|\nabla u\|^{2N-2/N-2}), \quad \forall u \in H_0^1(\Omega), \quad (88)$$

for any $\theta > 0$. For any $t \geq 0$, we can see from (32) that

$$\int_0^t \|\nabla u^\epsilon(s)\|^2 ds \leq ct, \quad \epsilon \in [0, 1], \quad (89)$$

for any $u_0 \in B$ and c is independent of ϵ . Next, we integrate (56) from 0 to t , and we get

$$\epsilon \int_0^t \|\nabla u_t^\epsilon(s)\|^2 ds \leq ct + c \left[\|\nabla u_0\|^2 + 2 \int_\Omega F(u_0) \right]. \quad (90)$$

We first fix θ small enough, and then put (88) into (90) to get, for all $u_0 \in B$,

$$\epsilon \int_0^t \|\nabla u_t^\epsilon(s)\|^2 ds \leq ct + c, \quad \epsilon \in [0, 1]. \quad (91)$$

From (89) and (91), we can easily see that, for any $t > 0$ and any fixed $\epsilon \in (0, 1]$, $\mathcal{B}_{\epsilon,t} = \{u^\epsilon(s) : s \in [0, t], u^\epsilon(0) = u_0 \in B\}$ is bounded in $W^{1,2}[0, t; H_0^1(\Omega)]$. Therefore, from the compact embedding $W^{1,2}[0, t; H_0^1(\Omega)] \subset L^2[0, t; L^2(\Omega)]$, we deduce that $n_1^{\epsilon,t}$ is compact in B . Thus, assertion (i) is proved.

Assertion (ii) can be easily proved by multiplying (70) (with $\epsilon = 0$) with $-\Delta w_2^0$, and this procedure is elementary; here, we omit it (see, e.g., [20] for details). The proof is completed.

Lemma 8. Under assumptions (F1)–(F4), we have, for any $\epsilon \in [0, 1]$, any $i \in \mathbb{N}$, and any $x \in B$, there exists a positive function $\ell(t)$ independent of ϵ such that, for all $t \geq T_2 + 1$,

$$\|S_\epsilon^i(t)x - S_0^i(t)x\|_{H_0^1(\Omega)} \leq \ell^i(t)\epsilon^{1/4}. \quad (92)$$

where $T_2 = T_2(B)$ (see Lemma 4).

Proof. We assume that $u^\epsilon(t)$ ($\epsilon \in (0, 1]$) and $u^0(t)$ are the solutions for the following equations:

$$\begin{cases} u_t^\epsilon - \epsilon \Delta u_t^\epsilon - \Delta u^\epsilon + f(u^\epsilon) = g, & t > 0, x \in \Omega, \\ u^\epsilon|_{\partial\Omega} = 0, u^\epsilon(x, 0) = x_1(x) \in B, & x \in \Omega, \end{cases} \quad (93)$$

$$\begin{cases} u_t^0 - \Delta u^0 + f(u^0) = g, & t > 0, x \in \Omega, \\ u^0|_{\partial\Omega} = 0, u^0(x, 0) = x_2(x) \in B, & x \in \Omega, \end{cases}$$

respectively. Set $w = u^\epsilon - u^0$, and then w satisfies

$$w_t - \epsilon \Delta u_t^\epsilon - \Delta w + l(t)w = 0, \quad (94)$$

where $l(t) = \int_0^1 f'(su^\epsilon(t) + (1-s)u^0(t))ds$. Multiplying (94) with w_t and using Lemma 5, it yields

$$\|w_t\|^2 + \epsilon(\nabla u_t^\epsilon, \nabla w_t) + \frac{1}{2} \frac{d}{dt} \|\nabla w\|^2 \leq c_6 \|\nabla w\| \|w_t\|, \quad t \geq 0. \quad (95)$$

We choose $\sigma_1 = \max\{c_6^2, 2\nu\}$; then, the above inequality implies

$$2\epsilon \|\nabla u_t^\epsilon\|^2 + \frac{d}{dt} \|\nabla w\|^2 \leq \sigma_1 \|\nabla w\|^2 + 2\epsilon \|\nabla u_t^\epsilon\| \|\nabla u_t^0\|. \quad (96)$$

Therefore,

$$\frac{d}{dt} e^{-\sigma_1 t} \|\nabla w\|^2 \leq 2\epsilon e^{-\sigma_1 t} \|\nabla u_t^\epsilon\| \|\nabla u_t^0\|. \quad (97)$$

By Lemma 4, we see that, for $t \geq T_2$,

$$\frac{d}{dt} e^{-\sigma_1 t} \|\nabla w\|^2 \leq c\epsilon^{1/2} \|\nabla u_t^0\| \leq c\epsilon^{1/2} + c\epsilon^{1/2} \|\nabla u_t^0\|^2. \quad (98)$$

Multiplying (94) with w and integrating it in Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \epsilon(\nabla u_t^\epsilon, \nabla w) + \|\nabla w\|^2 \leq \nu \|w\|^2, \quad (99)$$

that is,

$$\frac{d}{dt} \|w\|^2 + 2\|\nabla w\|^2 \leq \sigma_1 \|w\|^2 + 2\epsilon \|\nabla u_t^\epsilon\| \|\nabla w\|. \quad (100)$$

The above inequality implies

$$\frac{d}{dt} e^{-\sigma_1 t} \|w\|^2 + 2e^{-\sigma_1 t} \|\nabla w\|^2 \leq 2\epsilon \|\nabla u_t^\epsilon\| \|\nabla w\|. \quad (101)$$

Dropping the second term in the left-hand side of (101) and integrating it over $(0, t)$, we get

$$\begin{aligned} e^{-\sigma_1 t} \|w(t)\|^2 &\leq \|w(0)\|^2 + c\epsilon \left[\int_0^t \|\nabla u_t^\epsilon(s)\|^2 ds \right]^{1/2} \\ &\quad \cdot \left[\int_0^t \|\nabla w(s)\|^2 ds \right]^{1/2}. \end{aligned} \quad (102)$$

From (91), we see that, for any $\epsilon \in (0, 1]$,

$$\int_0^t \|\nabla u_t^\epsilon(s)\|^2 ds \leq \frac{1}{\epsilon} c(t+1). \quad (103)$$

Combining (31), (102), and (103), we deduce that

$$e^{-\sigma_1 t} \|w(t)\|^2 \leq \|w(0)\|^2 + c(t+1)\epsilon^{1/2}, \quad t \geq 0. \quad (104)$$

Now, we integrate (101) over $(t, t+1)$ to get

$$\begin{aligned} \int_t^{t+1} e^{-\sigma_1 s} \|\nabla w(s)\|^2 ds &\leq c e^{-\sigma_1 t} \|w(t)\|^2 \\ &+ c \epsilon \left[\int_t^{t+1} \|\nabla u_t^\epsilon(s)\|^2 ds \right]^{1/2} \left[\int_t^{t+1} \|\nabla w(s)\|^2 ds \right]^{1/2}. \end{aligned} \quad (105)$$

By using (31) and (33), we get that, for $t \geq T_2$,

$$\begin{aligned} \int_t^{t+1} \|\nabla u_t^\epsilon(s)\|^2 ds &\leq c, \\ \int_t^{t+1} \|\nabla w(s)\|^2 ds &\leq c. \end{aligned} \quad (106)$$

Therefore, from (104)–(106), we deduce that

$$\begin{aligned} \int_t^{t+1} e^{-\sigma_1 s} \|\nabla w(s)\|^2 ds &\leq c \|w(0)\|^2 + c(t+1)\epsilon^{1/2} \\ c\epsilon &\leq c \|w(0)\|^2 + c(t+1)\epsilon^{1/2}, \quad t \geq T_2. \end{aligned} \quad (107)$$

On the contrary, using (33) again, we obtain

$$\int_t^{t+1} \left[c\epsilon^{1/2} + c\epsilon^{1/2} \|\nabla u_t^0(s)\|^2 ds \right] \leq c\epsilon^{1/2}, \quad t \geq T_2. \quad (108)$$

Combining (98), (107), and (108) and using a similar process as the proof of uniform Gronwall inequality, it yields that, for all $t \geq T_2$,

$$\begin{aligned} e^{-\sigma_1(t+1)} \|\nabla w(t+1)\|^2 &\leq c \|w(0)\|^2 + c(t+1)\epsilon^{1/2} \\ c\epsilon^{1/2} &\leq c \|w(0)\|^2 + c(t+1)\epsilon^{1/2}. \end{aligned} \quad (109)$$

This implies that there exists a positive function $\ell'(t)$ independent of ϵ such that, for $t \geq T_2 + 1$,

$$\|S_\epsilon(t)x_1 - S_0(t)x_2\|_{H_0^1(\Omega)} \leq \ell'(t)\epsilon^{1/4} + \ell'(t)\|x_1 - x_2\|_{H_0^1(\Omega)}. \quad (110)$$

When $x_1 = x_2 = x$, we get

$$\|S_\epsilon(t)x - S_0(t)x\|_{H_0^1(\Omega)} \leq \ell'(t)\epsilon^{1/4}, \quad t \geq T_2 + 1. \quad (111)$$

From (110) and (111), we obtain for $t \geq T_2 + 1$,

$$\begin{aligned} \|S_\epsilon^i(t)x - S_0^i(t)x\|_{H_0^1(\Omega)} &\leq \ell'(t)\epsilon^{1/4} + \ell'(t)\|S_\epsilon^{i-1}(t)x - S_0^{i-1}(t)x\|_{H_0^1(\Omega)} \\ &\leq \ell'(t)\epsilon^{1/4} + \ell'(t) \left[\ell'(t)\epsilon^{1/4} + \ell'(t)\|S_\epsilon^{i-2}(t)x - S_0^{i-2}(t)x\|_{H_0^1(\Omega)} \right] \\ &= \ell'(t)\epsilon^{1/4} + \ell'^2(t)\epsilon^{1/4} + \ell'^2(t)\|S_\epsilon^{i-2}(t)x - S_0^{i-2}(t)x\|_{H_0^1(\Omega)} \\ &\leq \left[\ell'(t) + \ell'^2(t) + \dots + \ell'^i(t) \right] \epsilon^{1/4} + \ell'^{i-1}(t)\|S_\epsilon(t)x - S_0(t)x\|_{H_0^1(\Omega)} \\ &\leq [2\ell'(t)]^i \epsilon^{1/4} + \ell'^i(t)\epsilon^{1/4} \leq \ell^i(t)\epsilon^{1/4}. \end{aligned} \quad (112)$$

The proof is completed.

For any fixed $\eta \in (0, 1)$, we set $T_0(\eta) = \max\{T_1, T_2 + 1, T_3(\eta)\}$ and $S_\epsilon = S_\epsilon(T_0)$, and then $S_\epsilon: B \rightarrow B$. From Lemmas 6–8, we see that the discrete system $\{S_\epsilon^n\}_n$ ($\epsilon \in [0, \epsilon_0]$) satisfies all the assumptions in Theorem 1. Therefore, we have the following.

Theorem 2. *Let assumptions (F1)–(F4) hold, and then $\forall \epsilon \in [0, \epsilon_0]$, and the discrete dynamical system $\{S_\epsilon^n\}_n$ defined above possesses an exponential attractor $\mathcal{M}_\epsilon^d$ on B such that*

(1) *The fractal dimension of $\mathcal{M}_\epsilon^d$ is bounded in $H_0^1(\Omega)$:*

$$\dim_{f, H_0^1(\Omega)} \mathcal{M}_\epsilon^d \leq c(\epsilon) := \left[\ln \frac{2}{1+\eta} \right]^{-1} \ln m_\epsilon \left(\frac{4K(1+L^2)^{1/2}}{1-\eta} \right), \quad (113)$$

where $L = e^{c_0^2 T_0}$ (see Lemma 6) and $m_\epsilon(R)$ is the maximal number of pairs (x_i, y_i) in $H_0^1(\Omega) \times H_0^1(\Omega)$ possessing the properties

$$\begin{aligned} \epsilon \in (0, \epsilon_0]: \|x_i\|_{H_0^1(\Omega)}^2 + \|y_i\|_{H_0^1(\Omega)}^2 &\leq R^2, n_1^\epsilon(x_i - x_j) > 1, \quad i \neq j, \\ \epsilon = 0: \|x_i\|_{H_0^1(\Omega)}^2 + \|y_i\|_{H_0^1(\Omega)}^2 &\leq R^2, n_2(y_i - y_j) > 1, \quad i \neq j. \end{aligned} \quad (114)$$

(2) $\mathcal{M}_\epsilon^d$ attracts B in $H_0^1(\Omega)$, uniformly with respect to ϵ ,

$$\text{dist}_{H_0^1(\Omega)}(S_\epsilon^i B, \mathcal{M}_\epsilon^d) \leq c_8 e^{-c_9 i}, \quad c_9 > 0, \quad i \in \mathbb{N}, \quad (115)$$

where c_8 and c_9 are independent of ϵ .

(3) The family $\{\mathcal{M}_\epsilon^d, \epsilon \in [0, \epsilon_0]\}$ is continuous at 0 in $H_0^1(\Omega)$:

$$\text{dist}_{\text{sym}, H_0^1(\Omega)}(\mathcal{M}_\epsilon^d, \mathcal{M}_0^d) \leq c_{10} \epsilon^{c_{11}}, \quad (116)$$

where $c_{11} = -1/4 \ln 1 + \eta/2 \ln \ell(T_0) - \ln 1 + \eta/2$ and c_{10} are independent of ϵ .

To obtain the corresponding result for continuous system $\{S_\epsilon(t)\}$ defined in (30), we need to show the Hölder continuity with respect to time t and the initial conditions. In general, it is difficult to verify the uniform (with respect to ϵ) Hölder continuity with respect to time t when t is small. However, when t is large enough, we have the following.

Lemma 9. *Let assumptions (F1)–(F4) hold. Then, for any $T \geq T_0$, the semigroup $S_\epsilon(t)$ defined in (30) is uniformly Hölder continuous on $[T_0, T] \times B$, i.e.,*

$$\begin{aligned} \|S_\epsilon(t_1)x_1 - S_\epsilon(t_2)x_2\|_{H_0^1(\Omega)} \\ \leq c \left(\|x_1 - x_2\|_{H_0^1(\Omega)} + |t_1 - t_2|^{1/2} \right), \quad \epsilon \in [0, 1], \end{aligned} \quad (117)$$

for $x_1, x_2 \in B$ and $T_0 \leq t_1, t_2 \leq T$ and c is independent of ϵ .

Proof. The Lipschitz continuity with respect to the initial conditions is an immediate corollary of Lemma 6. It remains to prove the continuity with respect to time t . From Lemmas 1 and 2, we know that the solution $u^\epsilon(t) \in C[(0, T), H_0^1(\Omega)]$ for the initial value belongs to B . Therefore, for any $T_0 \leq t_1 \leq t_2 \leq T$,

$$u^\epsilon(t_2) - u^\epsilon(t_1) = \int_{t_1}^{t_2} u_t^\epsilon(s) ds, \quad (118)$$

which implies that

$$\begin{aligned} \|u^\epsilon(t_1) - u^\epsilon(t_2)\|_{H_0^1(\Omega)} &= \left\| \int_{t_1}^{t_2} u_t^\epsilon(s) ds \right\|_{H_0^1(\Omega)} \\ &\leq \int_{t_1}^{t_2} \|u_t^\epsilon(s)\|_{H_0^1(\Omega)} ds \leq |t_1 - t_2|^{1/2} \left[\int_{t_1}^{t_2} \|u_t^\epsilon(s)\|_{H_0^1(\Omega)}^2 ds \right]^{1/2}. \end{aligned} \quad (119)$$

To estimate (119), we integrate (60) from t_1 to t_2 to get

$$\int_{t_1}^{t_2} \|\nabla u_t^\epsilon(s)\|^2 ds \leq c \int_{t_1}^{t_2} \|u_t^\epsilon(s)\|^2 ds + c \left[\|u_t^\epsilon(t_1)\|^2 + \epsilon \|\nabla u_t^\epsilon(t_1)\|^2 \right]. \quad (120)$$

Since $t_1 \geq T_0$, we apply (32) to (120) to get

$$\int_{t_1}^{t_2} \|\nabla u_t^\epsilon(s)\|^2 ds \leq c, \quad (121)$$

where c is independent of ϵ . Putting (121) into (119), we obtain the result. The proof is completed.

Our main result in this section reads as follows:

Theorem 3. *Let assumptions (F1)–(F4) hold. Then, for every $\epsilon \in [0, \epsilon_0]$, the semigroup $S_\epsilon(t)$ generated by equations (1) and (2) possesses an exponential attractor $\mathcal{M}_\epsilon^c$ in $H_0^1(\Omega)$. Moreover, these exponential attractors can be constructed such that*

(1) *The fractal dimension of $\mathcal{M}_\epsilon^c$ is bounded in $H_0^1(\Omega)$:*

$$\dim_{f, H_0^1(\Omega)} \mathcal{M}_\epsilon^c \leq c(\epsilon) + 2. \quad (122)$$

(2) *$\mathcal{M}_\epsilon^c$ attracts B in $H_0^1(\Omega)$, uniformly with respect to ϵ :*

$$\begin{aligned} \text{dist}_{H_0^1(\Omega)} [S_\epsilon(t)B, \mathcal{M}_\epsilon^c] &\leq c_{12} e^{-c_{13}t}, \\ c_{13} &> 0, \end{aligned} \quad (123)$$

where c_{12} and c_{13} are independent of ϵ .

(3) *The family $\{\mathcal{M}_\epsilon^c, \epsilon \in [0, \epsilon_0]\}$ is continuous at 0 in $H_0^1(\Omega)$:*

$$\text{dist}_{\text{sym}, H_0^1(\Omega)} (\mathcal{M}_\epsilon^c, \mathcal{M}_0^c) \leq c_{14} \epsilon^{c_{11}}, \quad (124)$$

where $c_{11} = -1/4 \ln 1 + \eta/2 \ln \ell(T_0) - \ln 1 + \eta/2$ and c_{14} are independent of ϵ .

Proof. Setting $\mathcal{M}_\epsilon^c := \cup_{t \in [T_0, 2T_0]} S_\epsilon(t) \mathcal{M}_\epsilon^d$, then we have

$$\begin{aligned} \mathcal{M}_\epsilon^c &:= \bigcup_{t \in [T_0, 2T_0]} S_\epsilon(t) \mathcal{M}_\epsilon^d \\ &= \bigcup_{t \in [T_0, 2T_0]} S_\epsilon(t - T_0) S_\epsilon(T_0) \mathcal{M}_\epsilon^d \\ &= \bigcup_{t \in [0, T_0]} S_\epsilon(t) S_\epsilon(T_0) \mathcal{M}_\epsilon^d. \end{aligned} \quad (125)$$

It follows from its definition that the set $S_\epsilon(T_0) \mathcal{M}_\epsilon^d$ is also an exponential attractor for the discrete dynamical system $\{S_\epsilon^n\}_n$. Thus, $\mathcal{M}_\epsilon^c$ is the set we needed (for details, see, e.g., [12]). The proof is completed.

3.3. The Main Result for a Special Case: $0 < \nu < \lambda_1$. If, in addition, $\nu < \lambda_1$, we can get the uniform bound (with respect to ϵ) for the fractal dimension. To this end, we need show that the constants L_ϵ and K_ϵ and the compact seminorms n_1^ϵ and n_2^ϵ in Theorem 1 are all independent of ϵ for the discrete system generated by problems (1) and (2). From the above, we have proved the constant L_ϵ is independent of ϵ ; thus, we only need the following lemma:

Lemma 10. *Let (F1)–(F4) hold; in addition, we assume that $\nu < \lambda_1$, and then we have for any $x_1, x_2 \in B$ and any $\eta \in (0, 1)$, there exist constants $\epsilon_0 \in (0, 1)$, $K > 0$, and $T_0'(\eta) \geq T_0(\eta)$ (T_0 is defined above) and a compact seminorm $n(x)$ on B such that*

$$\begin{aligned} &\|S_\epsilon(T_0')x_1 - S_\epsilon(T_0')x_2\|_{H_0^1(\Omega)} \\ &\leq \eta \|x_1 - x_2\|_{H_0^1(\Omega)} + K \{n(x_1 - x_2) \\ &\quad + n[S_\epsilon(T_0')x_1 - S_\epsilon(T_0')x_2]\}, \quad \epsilon \in [0, \epsilon_0], \end{aligned} \quad (126)$$

where $T_0'(\eta)$, K , and the compact seminorm $n(x)$ are all independent of ϵ .

Proof. We assume that $u_1^\epsilon(t)$ and $u_2^\epsilon(t)$ are two solutions of (1) starting from $x_1, x_2 \in B$, respectively. We consider the difference $w^\epsilon(t) = u_1^\epsilon(t) - u_2^\epsilon(t)$; then, $w^\epsilon(t)$ satisfies

$$w_t^\epsilon(t) - \epsilon \Delta w_t^\epsilon(t) - \Delta w^\epsilon(t) + l(t, \epsilon) w^\epsilon(t) = 0. \quad (127)$$

We first take the inner product of (127) with kw_2^ϵ in $L^2(\Omega)$ (k is a constant which will be fixed later) to get

$$k(w_{2,t}^\epsilon, w_2^\epsilon) + k\epsilon(\nabla w_{2,t}^\epsilon, \nabla w_2^\epsilon) + k\|\nabla w_2^\epsilon\|^2 + k \int_\Omega l(t, \epsilon) w^\epsilon w_2^\epsilon = 0. \quad (128)$$

Using Lemma 5, we get

$$k\|\nabla w_2^\epsilon\|^2 \leq k\|w_{2,t}^\epsilon\| \|w_2^\epsilon\| + k\epsilon \|\nabla w_{2,t}^\epsilon\| \|\nabla w_2^\epsilon\| + kc \|\nabla w^\epsilon\| \|w_2^\epsilon\|. \quad (129)$$

Next, we take the inner product of (127) with $w_{2,t}^\epsilon$ in $L^2(\Omega)$ to get

$$\|w_{2,t}^\epsilon\|^2 + \epsilon \|\nabla w_{2,t}^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 + \int_\Omega l(t, \epsilon) w^\epsilon w_{2,t}^\epsilon = 0. \quad (130)$$

We apply Lemma 5 again to get

$$\|w_{2,t}^\epsilon\|^2 + \epsilon \|\nabla w_{2,t}^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 \leq c_6 \|\nabla w^\epsilon\| \|w_{2,t}^\epsilon\|. \quad (131)$$

Combining (129) and (131), we get

$$\begin{aligned} & \|w_{2,t}^\epsilon\|^2 + \epsilon \|\nabla w_{2,t}^\epsilon\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 + k \|\nabla w_2^\epsilon\|^2 \\ & \leq k \|w_{2,t}^\epsilon\| \|w_2^\epsilon\| + k \epsilon \|\nabla w_{2,t}^\epsilon\| \|\nabla w_2^\epsilon\| + k c \|\nabla w^\epsilon\| \|w_2^\epsilon\| + c_6 \|\nabla w^\epsilon\| \|w_{2,t}^\epsilon\|. \end{aligned} \quad (132)$$

For the right-hand side of (132), we use Hölder inequality to get

$$\begin{aligned} & k \|w_{2,t}^\epsilon\| \|w_2^\epsilon\| + k \epsilon \|\nabla w_{2,t}^\epsilon\| \|\nabla w_2^\epsilon\| + k c \|\nabla w^\epsilon\| \|w_2^\epsilon\| + c_6 \|\nabla w^\epsilon\| \|w_{2,t}^\epsilon\| \\ & \leq c k^2 \|w_2^\epsilon\|^2 + \frac{1}{2} \|w_{2,t}^\epsilon\|^2 + \frac{\epsilon}{2} \|\nabla w_{2,t}^\epsilon\|^2 + \frac{k^2 \epsilon}{2} \|\nabla w_2^\epsilon\|^2 \\ & \quad + \frac{k}{2} \|\nabla w^\epsilon\|^2 + c k \|w_2^\epsilon\|^2 + \frac{c_6^2}{2} \|\nabla w^\epsilon\|^2 + \frac{1}{2} \|w_{2,t}^\epsilon\|^2 \\ & \leq c k^2 \|w_2^\epsilon\|^2 + \frac{1}{2} \|w_{2,t}^\epsilon\|^2 + \frac{\epsilon}{2} \|\nabla w_{2,t}^\epsilon\|^2 + \frac{k^2 \epsilon}{2} \|\nabla w_2^\epsilon\|^2 + \frac{k}{2} \|\nabla w_2^\epsilon\|^2 \\ & \quad + \frac{k}{2} \|\nabla w_1^\epsilon\|^2 + c k \|w_2^\epsilon\|^2 + \frac{c_6^2}{2} \|\nabla w_2^\epsilon\|^2 + \frac{c_6^2}{2} \|\nabla w_1^\epsilon\|^2 + \frac{1}{2} \|w_{2,t}^\epsilon\|^2. \end{aligned} \quad (133)$$

Putting the above inequality into (132), we obtain

$$\begin{aligned} & \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 + k \|\nabla w_2^\epsilon\|^2 \leq c(k^2 + k) \|w_2^\epsilon\|^2 + (c + k) \|\nabla w_1^\epsilon\|^2 \\ & \quad + (c_6^2 + k^2 \epsilon) \|\nabla w_2^\epsilon\|^2. \end{aligned} \quad (134)$$

Fix $k = 2c_6^2$, and then choose ϵ_0 such that $\epsilon < 1/4c_6^2$, $\forall \epsilon \leq \epsilon_0$. In the following, we assume $\epsilon \in [0, \epsilon_0]$. Then, we get from (134) that

$$\begin{aligned} & \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 + c_{15} \|\nabla w_2^\epsilon\|^2 \leq c \|w_2^\epsilon\|^2 + c \|\nabla w_1^\epsilon\|^2, \\ & c_{15} > 0. \end{aligned} \quad (135)$$

Using Poincaré inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \|\nabla w_2^\epsilon\|^2 + c_{15} \|\nabla w_2^\epsilon\|^2 \leq c \lambda_{m+1}^{-1} \|\nabla w_2^\epsilon\|^2 + c \|\nabla w_1^\epsilon\|^2 \\ & \leq c \lambda_{m+1}^{-1} \|\nabla w^\epsilon\|^2 + c \|\nabla w_1^\epsilon\|^2. \end{aligned} \quad (136)$$

From Lemma 6, it yields that

$$\frac{d}{dt} e^{c_{15}t} \|\nabla w_2^\epsilon\|^2 \leq c \lambda_{m+1}^{-1} e^{c_{15}t} e^{c_6^2 t} \|\nabla w^\epsilon(0)\|^2 + c e^{c_{15}t} \|\nabla w_1^\epsilon\|^2. \quad (137)$$

We integrate the above inequality over $(0, t)$ to get

$$\begin{aligned} & \|\nabla w_2^\epsilon(t)\|^2 \leq e^{-c_{15}t} \|\nabla w_2^\epsilon(0)\|^2 + c \lambda_{m+1}^{-1} e^{c_6^2 t} \|\nabla w^\epsilon(0)\|^2 \\ & \quad + c \int_0^t \|\nabla w_1^\epsilon(s)\|^2 ds \\ & \leq \left(e^{-c_{15}t} + c \lambda_{m+1}^{-1} e^{c_6^2 t} \right) \|\nabla w^\epsilon(0)\|^2 + c \int_0^t \|\nabla w_1^\epsilon(s)\|^2 ds. \end{aligned} \quad (138)$$

For any $\eta \in (0, 1)$, we first choose $T'_0 = T'_0(\eta) > T_0(\eta)$ such that $e^{-c_{15}T'_0} \leq \eta^2/2$, and then fix a positive integer $M = M(\eta)$ such that $c \lambda_{M+1}^{-1} e^{c_6^2 T'_0} \leq \eta^2/2$. Therefore, from (138),

$$\|\nabla w_2^\epsilon(T'_0)\|^2 \leq \eta^2 \|\nabla w^\epsilon(0)\|^2 + c \int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds. \quad (139)$$

Therefore,

$$\begin{aligned} & \|\nabla w^\epsilon(T'_0)\|^2 = \|\nabla w_1^\epsilon(T'_0)\|^2 + \|\nabla w_2^\epsilon(T'_0)\|^2 \\ & \leq \eta^2 \|\nabla w^\epsilon(0)\|^2 + c \left[\int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds + \|\nabla w_1^\epsilon(T'_0)\|^2 \right], \end{aligned} \quad (140)$$

that is,

$$\|\nabla w^\epsilon(T'_0)\| \leq \eta \|\nabla w^\epsilon(0)\| + K \left[\left[\int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds \right]^{1/2} + \|\nabla w_1^\epsilon(T'_0)\| \right]. \quad (141)$$

We set $n(x) = \|P_M x\|_{H_0^1}$. Then, $\|\nabla w_1^\epsilon(T'_0)\| = n[S_\epsilon(T'_0)x_1 - S_\epsilon(T'_0)x_2]$, and n is obviously compact on B . To estimate the term $(\int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds)^{1/2}$, we multiply (127) with $w_1^\epsilon(t)$ and integrate it in Ω :

$$\begin{aligned} & \frac{d}{dt} \left[\|w_1^\epsilon(t)\|^2 + \epsilon \|\nabla w_1^\epsilon(t)\|^2 \right] + 2 \|\nabla w_1^\epsilon(t)\|^2 \\ & \quad + 2[l(t, \epsilon)w^\epsilon(t), w_1^\epsilon(t)] = 0. \end{aligned} \quad (142)$$

By using (F1) and Poincaré inequality, we can get from the above inequality

$$\begin{aligned} & \frac{d}{dt} \left[\|w_1^\epsilon(t)\|^2 + \epsilon \|\nabla w_1^\epsilon(t)\|^2 \right] + 2 \|\nabla w_1^\epsilon(t)\|^2 \\ & \leq 2\nu \|w_1^\epsilon(t)\|^2 \leq 2\nu \lambda_1^{-1} \|\nabla w_1^\epsilon(t)\|^2. \end{aligned} \quad (143)$$

Therefore,

$$\frac{d}{dt} \left[\|w_1^\epsilon(t)\|^2 + \epsilon \|\nabla w_1^\epsilon(t)\|^2 \right] + \beta \|\nabla w_1^\epsilon(t)\|^2 \leq 0, \quad (144)$$

where $\beta = 2(1 - \nu \lambda_1^{-1}) > 0$. Integrating (144) from 0 to T'_0 , we get

$$\int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds \leq c \left[\|w_1^\epsilon(0)\|^2 + \epsilon \|\nabla w_1^\epsilon(0)\|^2 \right] \leq c \|\nabla w_1^\epsilon(0)\|^2. \quad (145)$$

Thus, for any $\epsilon \in [0, \epsilon_0]$,

$$\left(\int_0^{T'_0} \|\nabla w_1^\epsilon(s)\|^2 ds \right)^{1/2} \leq c n(x_1 - x_2). \quad (146)$$

Putting (146) into (141), one can obtain the result. The proof is completed.

Theorem 5. Let assumptions (F1)–(F4) hold and $\nu < \lambda_1$. Then, for every $\epsilon \in [0, \epsilon_0]$, the semigroup $S_\epsilon(t)$ generated by equations (1) and (2) possesses an exponential attractor $\mathcal{M}_\epsilon^c$ in $H_0^1(\Omega)$. Moreover, these exponential attractors can be constructed such that (2) and (3) in Theorem 3 hold and the

fractal dimension of $\mathcal{M}_\epsilon$ is uniformly (with respect to ϵ) bounded in $H_0^1(\Omega)$, i.e.,

$$\dim_{f, H_0^1(\Omega)} \mathcal{M}_\epsilon^c \leq c, \quad (147)$$

where c is a constant independent of ϵ .

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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