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Research Article

Lyapunov-Type Inequalities for Second-Order Boundary Value Problems with a Parameter

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In this paper, we will establish some new Lyapunov-type inequalities for a class of second-order boundary value problems with a parameter. The inequalities generalize some early results in the literature.

1. Introduction

Up until now, integral inequalities have attracted the attention of many researchers, due to its wide applications in the research of qualitative and quantitative properties such as global existence, boundedness, and stability of differential and integral equations (see [1–26] and the references therein). Among these inequalities, one important kind is the Lyapunov-type inequality, which was originally presented by Lyapunov in [27] as follows.

If $u(t)$ is a solution of

$$u'' + q(t)u = 0, \tag{1}$$

satisfying $u(a) = u(b) = 0$ $(a < b)$ and $u(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b |q(t)| \mathrm{d}t > \frac{4}{b-a}, \tag{2}$$

and afterwards by Wintner [28] as

$$\int_a^b q^+(t) \mathrm{d}t > \frac{4}{b-a}, \tag{3}$$

where $q^+(t): = \max\{q(t), 0\}$.

Following Lyapunov's landmark work, there have been plenty of references focused on the Lyapunov-type inequality and its generalizations which are widely used in various problems such as asymptotic theory, disconjugacy, and eigenvalue problems of differential equations and difference equations (see [29–41] and the references therein).

For example, in 2003, Yang [29] obtained the following result for the second-order half-linear equation:

$$\begin{cases} \left(r(t)|u'(t)|^{p-1}u'(t)\right)' + q(t)|u(t)|^{p-1}u(t) = 0, \\ u(a) = u(b) = 0, \quad u(t) \neq 0, t \in (a, b), \end{cases} \tag{4}$$

where $q, r \in C([a, b], \mathbb{R})$ such that $r(t) > 0$, for $t \in [a, b]$, and $p > 0$.

Theorem 1 (see [29]). *Assume boundary value problem (4) has a solution $u(t)$; then, the following inequality holds:*

$$\int_a^b q_+(t) \mathrm{d}t \geq \frac{2^{p+1}}{\left(\int_a^b r^{-1/p}(t) \mathrm{d}t\right)^p}, \tag{5}$$

where $q_+(t) := max\{q(t), 0\}$.

In 2012, Tiryaki et al. [34] established an inequality for boundary value problem of the form

$$\begin{cases} \left(r(t)|u'(t)|^{\alpha-2}u'(t)\right)' + q(t)|u(t)|^{\alpha_*-2}, \quad u(t) = 0, \\ u(a) = u(b) = 0, \quad u(t) \neq 0, \ t \in (a, b), \end{cases} \tag{6}$$

where $\alpha > 1$ and $\alpha_* = (\alpha/\alpha - 1)$. Their result is as follows.

Theorem 2 (see [34]). *Assume boundary value problem (6) has a solution $u(t)$; then, the following inequality holds:*

$$\int_a^b \frac{q_+(t)}{h_a^{1-\alpha}(t)+h_b^{1-\alpha}(t)}\mathrm{d}t \geq 1, \tag{7}$$

where $h_a(t)=\int_a^t r^{1-\alpha_*}(s)\mathrm{d}s$, $h_b(t)=\int_t^b r^{1-\alpha_*}(s)\mathrm{d}s$, *and* $q_+(t):=max\{q(t),0\}$.

In 2015, Agarwal et al. [36] established a Lyapunov-type inequality for the second-order forced boundary value problem of the form:

$$\begin{cases} \left(r(t)|u'(t)|^{\beta-1}u'(t)\right)' + q(t)|u(t)|^{\gamma-1}, \quad u(t)=f(t), \\ u(a)=u(b)=0, \quad u(t)\neq 0,\ t\in(a,b), \end{cases} \tag{8}$$

in the subhalf-linear $(0<\gamma<\beta)$ and the super-half-linear $(0<\beta<\gamma<2\beta)$ cases, where $r(t)$ and $q(t)$ are integrable on $[a,b]$ with $r(t)>0$ on $[a,b]$. Their result is as follows.

Theorem 3 (see [36]). *Suppose that a,b, $a<b$, are consecutive zeros of a nontrivial solution of the first part of equation (8), then the inequality*

$$2\Gamma_{\gamma\beta}\int_a^b q_+(t)\mathrm{d}t + \int_a^b |f(t)|\mathrm{d}t > 2^{\beta+1}\sqrt{\Gamma_{\gamma\beta}}\left(\int_a^b r^{-1/\beta}(t)\mathrm{d}t\right)^{-\beta} \tag{9}$$

holds, where $\gamma\in(0,2\beta)$ *and* $\Gamma_{\gamma\beta}=(2\beta-\gamma)\gamma^{\gamma/(2\beta-\gamma)}\beta^{-2\beta/(2\beta-\gamma)}2^{-2\beta/(2\beta-\gamma)}>0$.

Agarwal and Özbekler [36] also established a Lyapunov-type inequality for the second-order forced boundary value problem with mixed nonlinearities:

$$\begin{cases} u''(t)+p(t)|u(t)|^{\alpha-1}u(t)+q(t)|u(t)|^{\gamma-1}u(t)=f(t), \\ u(a)=u(b)=0, \quad u(t)\neq 0,\ t\in(a,b), \end{cases} \tag{10}$$

where $0<\gamma<1<\alpha<2$. The result is as follows.

Theorem 4 (see [36]). *Suppose that a,b, $a<b$, are consecutive zeros of a nontrivial solution of the first part of equation (10), then the inequality*

$$\left(\int_a^b (p_++q_+)(t)\mathrm{d}t\right)\left(\int_a^b \{\alpha_0 p_+(t)+\gamma_0 q_+(t)+|f(t)|\}\mathrm{d}t\right) > \frac{4}{(b-a)^2}, \tag{11}$$

holds, where $\alpha_0=(2-\alpha)\alpha^{\alpha/(2-\alpha)}2^{2/(\alpha-2)}>0$ *and* $\gamma_0=(2-\gamma)\gamma^{\gamma/(2-\gamma)}2^{2/(\gamma-2)}>0$.

We find that in [36], the authors studied the case $0<\gamma<1<\alpha<2$ of equation (10). It will be interesting to prove Lyapunov-type inequalities for equation (10) or other equation when α and γ have other relation. Motivated by [36], in this paper, we will establish a Lyapunov-type inequality for the nonlinear second-order boundary value problem of the form

$$\left(p(t)|u'|^{\alpha-2}u'\right)' - \lambda q(t)|u|^{\beta-2}u + \lambda r(t)|u|^{\gamma-2}u + h(t)|u|^{\alpha-2}u = 0, \quad t\in[a,b], \tag{12}$$

$$u(a)=u(b)=0, \quad u(t)\neq 0,\ t\in(a,b), \tag{13}$$

where $p,q,r,h\in C([a,b],\mathbb{R})$ such that $p(t)>0$, $q(t)>0$, $r(t)>0$ for $t\in[a,b]$, $1<\alpha<\gamma<\beta$, and $\lambda\geq 0$ is a real parameter, and

$$\begin{aligned} &\left(p(t)|u'|^{\alpha-2}u'\right)' + \lambda\left(q(t)|u'|^{\beta-2}u'\right)' + \lambda r(t)|u|^{\gamma-2}u \\ &\quad + h(t)|u|^{\alpha-2}\quad u=0,\ t\in[a,b], \end{aligned} \tag{14}$$

with the boundary condition (13), where $p,q,r,h\in C([a,b],\mathbb{R})$ such that $p(t)>0$, $q(t)>0$, $r(t)>0$ for $t\in[a,b]$, $1<\alpha<\gamma<\beta$, and $\lambda\geq 0$ is a real parameter. Our results extend and compliment the results of [29, 36].

2. Main Results

Lemma 1. *If u is differential on $[a,\ b]$ satisfying $u(a)=u(b)=0$ and $u(t)\neq 0$ for $t\in(a,b)$, then*

$$\sup_{a\leq t\leq b}|u(t)| \leq \frac{1}{2}\int_a^b |u'(t)|\mathrm{d}t. \tag{15}$$

Proof. Since u is differential on $[a,\ b]$ satisfying $u(a)=u(b)=0$, then we have

$$u(t)=\frac{1}{2}\int_a^t u'(s)\mathrm{d}s - \frac{1}{2}\int_t^b u'(s)\mathrm{d}s, \quad t\in[a,b]. \tag{16}$$

So,

$$|u(t)| \leq \frac{1}{2}\int_a^t |u'(s)|\mathrm{d}s + \frac{1}{2}\int_t^b |u'(s)|\mathrm{d}s = \frac{1}{2}\int_a^b |u'(s)|\mathrm{d}s, \quad t\in[a,b]. \tag{17}$$

Therefore, (15) holds. □

Lemma 2 (see [13]). *Let $A>0, B>0$, and $1<\alpha<\gamma<\beta$ be given. Then, for each $x\geq 0$,*

$$Ax^\gamma - Bx^\beta \leq \frac{A(\beta-\gamma)}{\beta-\alpha}\left(\frac{(\beta-\alpha)B}{(\gamma-\alpha)A}\right)^{(\gamma-\alpha)/(\gamma-\beta)} x^\alpha, \tag{18}$$

holds.

Theorem 5. *Assume u is a solution of equation (12) satisfying the boundary conditions (13). Then,*

$$\frac{\lambda}{2^\alpha}\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha}\int_a^b\left[r^{\beta-\alpha}(t)q^{\alpha-\gamma}(t)\right]^{1/(\beta-\gamma)}\mathrm{d}t + \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t \geq \left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{1-\alpha}, \tag{19}$$

where $h^+(t) := max\{h(t),0\}$.

Proof. Multiplying (12) by $u(t)$ and integrating over $[a,b]$, yields:

$$\int_a^b\left(p(t)\left|u'(t)\right|^{\alpha-2}u'(t)\right)'u(t)\mathrm{d}t - \lambda\int_a^b q(t)|u(t)|^\beta\mathrm{d}t + \lambda\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t + \int_a^b h(t)|u(t)|^\alpha\mathrm{d}t = 0. \tag{20}$$

Using integration by parts to the first integral on the left-hand side of (20) and from (13), we have

$$-\int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t - \lambda\int_a^b q(t)|u(t)|^\beta\mathrm{d}t + \lambda\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t + \int_a^b h(t)|u(t)|^\alpha\mathrm{d}t = 0. \tag{21}$$

Then, we obtain

$$-\int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t - \lambda\int_a^b q(t)|u(t)|^\beta\mathrm{d}t + \lambda\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t \geq 0, \tag{22}$$

i.e.,

$$\int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t \leq \lambda\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t - \lambda\int_a^b q(t)|u(t)|^\beta\mathrm{d}t + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t. \tag{23}$$

By using Hölder's inequality with indices $(1/\tau) + (1/\rho) = 1$,

$$\int_a^b|f(t)g(t)|\mathrm{d}t \leq \left(\int_a^b|f(t)|^\tau\mathrm{d}t\right)^{1/\tau}\left(\int_a^b|g(t)|^\rho\mathrm{d}t\right)^{1/\rho}, \tag{24}$$

with $f(t) = p^{1/\alpha}(t)|u'(t)|$, $g(t) = p^{-1/\alpha}(t)$, $\tau = \alpha$, and $\rho = (\alpha/\alpha-1)$, we obtain that

$$\int_a^b\left|u'(t)\right|\mathrm{d}t \leq \left(\int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t\right)^{1/\alpha}\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{(\alpha-1/\alpha)}, \tag{25}$$

i.e.,

$$\frac{\left(\int_a^b\left|u'(t)\right|\mathrm{d}t\right)^\alpha}{\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{\alpha-1}} \leq \int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t. \tag{26}$$

On the contrary, from Lemma 1, we obtain

$$\begin{aligned}\int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t &\leq \left(\sup_{a\leq t\leq b}|u(t)|\right)^\alpha\int_a^b h^+(t)\mathrm{d}t\\ &\leq \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t\left(\int_a^b\left|u'(t)\right|\mathrm{d}t\right)^\alpha.\end{aligned} \tag{27}$$

Then, from (23), (26), and (27), we obtain

$$\begin{aligned}\frac{\left(\int_a^b\left|u'(t)\right|\mathrm{d}t\right)^\alpha}{\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{\alpha-1}} &\leq \int_a^b p(t)\left|u'(t)\right|^\alpha\mathrm{d}t\\ &= \lambda\left[\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t - \int_a^b q(t)|u(t)|^\beta\mathrm{d}t\right] + \int_a^b h^+(t)|u(t)|^2\mathrm{d}t\\ &\leq \lambda\int_a^b\left[r(t)|u(t)|^\gamma - q(t)|u(t)|^\beta\right]\mathrm{d}t\\ &\quad + \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t\left(\int_a^b|u'(t)|\mathrm{d}t\right)^\alpha.\end{aligned} \tag{28}$$

For the first integral on the right-hand side of (28), inequality (18) in Lemma 2 with $A = r(t)$, $B = q(t)$, and $x = |u(t)| \geq 0$ for $t \in [a,b]$ implies that

$$r(t)|u(t)|^{\gamma} - q(t)|u(t)|^{\beta} \leq r^{(\beta-\alpha)/(\beta-\gamma)}(t) q^{(\alpha-\gamma)/(\beta-\gamma)}(t) \cdot \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)} \frac{\beta-\gamma}{\beta-\alpha} |u(t)|^{\alpha}. \tag{29}$$

By (28) and (29), we obtain

$$\frac{\left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha}}{\left(\int_a^b p^{1/(1-\alpha)}(t) \mathrm{d}t\right)^{\alpha-1}} \leq \lambda \int_a^b \left[\left(r^{\beta-\alpha}(t) q^{\alpha-\gamma}(t)\right)^{1/(\beta-\gamma)} \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)} \frac{\beta-\gamma}{\beta-\alpha} |u(t)|^{\alpha}\right] \mathrm{d}t + \frac{1}{2^{\alpha}} \int_a^b h^{+}(t) \mathrm{d}t \left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha}. \tag{30}$$

From Lemma 1, we have

$$\left(\sup_{a \leq t \leq b} |u(t)|\right)^{\alpha} \leq \frac{1}{2^{\alpha}} \left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha}. \tag{31}$$

In view of (30) and (31), we obtain that

$$\begin{aligned} &\frac{\left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha}}{\left(\int_a^b p^{1/(1-\alpha)}(t) \mathrm{d}t\right)^{\alpha-1}} \\ &\quad \leq \frac{\lambda}{2^{\alpha}} \int_a^b \left[\left(r^{\beta-\alpha}(t) q^{\alpha-\gamma}(t)\right)^{1/(\beta-\gamma)} \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)} \frac{\beta-\gamma}{\beta-\alpha}\right] \mathrm{d}t \left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha} \\ &\quad + \frac{1}{2^{\alpha}} \int_a^b h^{+}(t) \mathrm{d}t \left(\int_a^b |u'(t)| \mathrm{d}t\right)^{\alpha}. \end{aligned} \tag{32}$$

Since $\int_a^b |u'(t)| \mathrm{d}t > 0$ (in fact, if $\int_a^b |u'(t)| \mathrm{d}t = 0$, we have $u'(t) = 0$ for $t \in [a,b]$. By condition (13), we obtain $u(t) = 0$ for $t \in [a,b]$, which contradicts to $u(t) \neq 0$, $t \in [a,b]$), dividing both sides of (32) by $(\int_a^b |u'(t)| \mathrm{d}t)^{\alpha}$, we obtain

$$\begin{aligned} &\frac{\lambda}{2^{\alpha}} \int_a^b \left[\left(r^{\beta-\alpha}(t) q^{\alpha-\gamma}(t)\right)^{1/(\beta-\gamma)} \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)} \frac{\beta-\gamma}{\beta-\alpha}\right] \mathrm{d}t \\ &\quad + \frac{1}{2^{\alpha}} \int_a^b h^{+}(t) \mathrm{d}t \geq \left(\int_a^b p^{1/(1-\alpha)}(t) \mathrm{d}t\right)^{1-\alpha}, \end{aligned} \tag{33}$$

which also leads to (19). The proof is complete.

If we take $\alpha = 2$ and $\lambda = 1$ in inequality (19), we obtain the following result. □

Corollary 1. *Assume u is a solution of equation*

$$(p(t)u')' - q(t)|u|^{\beta-2}u + r(t)|u|^{\gamma-2}u + h(t)u = 0, \quad t \in [a,b], \tag{34}$$

satisfying boundary condition (13). Then,

$$\begin{aligned} &\frac{1}{4}\left(\frac{\gamma-2}{\beta-2}\right)^{(\gamma-2)/(\beta-\gamma)} \frac{\beta-\gamma}{\beta-2} \int_a^b \left[r^{\beta-2}(t) q^{2-\gamma}(t)\right]^{1/(\beta-\gamma)} \mathrm{d}t \\ &\quad + \frac{1}{4} \int_a^b h^{+}(t) \mathrm{d}t \geq \left(\int_a^b p^{-1}(t) \mathrm{d}t\right)^{-1}, \end{aligned} \tag{35}$$

where $h^{+}(t) := max\{h(t), 0\}$.

Theorem 6. *Assume u is a solution of equation (14) satisfying boundary condition (13). Then,*

$$\lambda\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha},$$
$$\cdot\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(1-\beta)(\gamma-\alpha)/(\gamma-\beta)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)}+\frac{1}{2^{\alpha}}\int_a^b h^{+}(t)\mathrm{d}t \quad (36)$$
$$\geq\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{1-\alpha},$$

where $h^{+}(t):=max\{h(t),0\}$.

Proof. Multiplying (14) by $u(t)$ and integrating over $[a,b]$ yields

$$\int_a^b\left(p(t)|u'(t)|^{\alpha-2}u'(t)\right)'u(t)\mathrm{d}t+\lambda\int_a^b\left(q(t)|u'(t)|^{\beta-2}u'(t)\right)'u(t)\mathrm{d}t+\lambda\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t+\int_a^b h(t)|u(t)|^{\alpha}\mathrm{d}t=0. \quad (37)$$

Using integration by parts to the first and second integrals on the left-hand side of (37) and from (13), we have

$$-\int_a^b p(t)|u'(t)|^{\alpha}\mathrm{d}t-\lambda\int_a^b q(t)|u'(t)|^{\beta}\mathrm{d}t+\lambda\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t+\int_a^b h(t)|u(t)|^{\alpha}\mathrm{d}t=0, \quad (38)$$

i.e.,

$$\int_a^b p(t)|u'(t)|^{\alpha}\mathrm{d}t$$
$$=\lambda\left[\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t-\int_a^b q(t)|u'(t)|^{\beta}\mathrm{d}t\right]+\int_a^b h(t)|u(t)|^{\alpha}\mathrm{d}t \quad (39)$$
$$\leq\lambda\left[\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t-\int_a^b q(t)|u'(t)|^{\beta}\mathrm{d}t\right]+\int_a^b h^{+}(t)|u(t)|^{\alpha}\mathrm{d}t.$$

By using Hölder's inequality (24) with $f(t)=q^{1/\beta}(t)|u'(t)|$, $g(t)=q^{-1/\beta}(t)$, $\tau=\beta$, and $\rho=(\beta/\beta-1)$, we obtain that

$$\int_a^b|u'(t)|\mathrm{d}t\leq\left(\int_a^b q(t)|u'(t)|^{\beta}\mathrm{d}t\right)^{1/\beta}\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(\beta-1/\beta)}, \quad (40)$$

i.e.,

$$\frac{\left(\int_a^b|u'(t)|\mathrm{d}t\right)^{\beta}}{\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{\beta-1}}\leq\int_a^b q(t)|u'(t)|^{\beta}\mathrm{d}t. \quad (41)$$

From (26), (39), (41), and Lemma 1, we obtain

$$\begin{aligned}
&\frac{\left(\int_a^b \left|u'(t)\right| \mathrm{d}t\right)^{\alpha}}{\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{\alpha-1}} \leq \int_a^b p(t)\left|u'(t)\right|^{\alpha}\mathrm{d}t \\
&\quad = \lambda\left[\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t - \int_a^b q(t)\left|u'(t)\right|^{\beta}\mathrm{d}t\right] + \int_a^b h^{+}(t)|u(t)|^{\alpha}\mathrm{d}t \\
&\quad \leq \lambda\left[\int_a^b r(t)|u(t)|^{\gamma}\mathrm{d}t - \frac{\int_a^b \left(\left|u'(t)\right|\mathrm{d}t\right)^{\beta}}{\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{\beta-1}}\right] + \int_a^b h^{+}(t)|u(t)|^{\alpha}\mathrm{d}t \\
&\quad \leq \lambda\left[\left(\sup_{a\leq t\leq b}|u(t)|\right)^{\gamma}\int_a^b r(t)\mathrm{d}t - \frac{\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\beta}}{\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{\beta-1}}\right] + \int_a^b h^{+}(t)|u(t)|^{\alpha}\mathrm{d}t \\
&\quad \leq \lambda\left[\left(\frac{1}{2}\right)^{\gamma}\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\gamma}\int_a^b r(t)\mathrm{d}t - \frac{\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\beta}}{\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{\beta-1}}\right] \\
&\qquad + \int_a^b h^{+}(t)|u(t)|^{\alpha}\mathrm{d}t.
\end{aligned} \tag{42}$$

For the right-hand side of (42), inequality (18) in Lemma 2 with $A = (1/2)^{\gamma}\int_a^b r(t)\mathrm{d}t$, $B = (\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t)^{1-\beta}$, and $x = \int_a^b |u'(t)|\mathrm{d}t > 0$ implies that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\gamma}\int_a^b r(t)\mathrm{d}t\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\gamma} - \left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{1-\beta} \\
&\quad \cdot \left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\beta} \\
&\leq \left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)} \\
&\quad \cdot \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha} \cdot \\
&\quad \cdot \left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(1-\beta)(\gamma-\alpha)/(\gamma-\beta)}\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\alpha}.
\end{aligned} \tag{43}$$

From (27), (42), and (43), we have

$$\begin{aligned}
&\frac{\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\alpha}}{\left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{\alpha-1}} \\
&\leq \lambda\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)} \\
&\quad \cdot \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha} \cdot \\
&\quad \cdot \left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(1-\beta)(\gamma-\alpha)/(\gamma-\beta)}\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\alpha} \\
&\quad \cdot \frac{1}{2^{\alpha}}\int_a^b h^{+}(t)\mathrm{d}t\left(\int_a^b \left|u'(t)\right|\mathrm{d}t\right)^{\alpha}.
\end{aligned} \tag{44}$$

Thus, dividing both sides of (29) by $(\int_a^b |u'(t)|\mathrm{d}t)^{\alpha}$, we obtain

$$\lambda\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha},$$
$$\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(1-\beta)(\gamma-\alpha)/(\gamma-\beta)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)}$$
$$+\frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t \geq \left(\int_a^b p^{1/(1-\alpha)}(t)\mathrm{d}t\right)^{1-\alpha}, \tag{45}$$

which also leads to (36). The proof is complete. □

Remark 1. We note that when $\lambda = 0$, $\alpha = p+1$, and (5) can be obtained from Theorems 5 and 6, respectively.

If we take $\alpha = 2$ and $\lambda = 1$ in inequality (36), we obtain the following result.

Corollary 2. *Assume u is a solution of equation*

$$(p(t)u')' + \left(q(t)|u'|^{\beta-2}u'\right)' + r(t)|u|^{\gamma-2}u + h(t)|u|^{\alpha-2}u = 0, \quad t \in [a,b], \tag{46}$$

satisfying boundary condition (13). Then,

$$\left(\frac{1}{2}\right)^{\gamma(\beta-2)/(\beta-\gamma)}\left(\frac{\gamma-2}{\beta-2}\right)^{(\gamma-2)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-2},$$
$$\cdot\left(\int_a^b q^{1/(1-\beta)}(t)\mathrm{d}t\right)^{(1-\beta)(\gamma-2)/(\gamma-\beta)}$$
$$\cdot\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-2)/(\beta-\gamma)} + \frac{1}{4}\int_a^b h^+(t)\mathrm{d}t \tag{47}$$
$$\geq \left(\int_a^b p^{-1}(t)\mathrm{d}t\right)^{-1},$$

where $h^+(t) := max\{h(t), 0\}$.

Theorem 7. *Assume u is a solution of equation (14) satisfying boundary condition (13). Then,*

$$\lambda\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\frac{(b-a)^{\beta-1}(\gamma-\alpha)}{q(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-\alpha},$$
$$\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)} + \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t \geq p(b-a)^{1-\alpha}, \tag{48}$$

where $p := min\{p(t): t \in [a,b]\}$, $q := min\{q(t): t \in [a,b]\}$, and $h^+(t) := max\{h(t), 0\}$.

Proof. From the proof of Theorem 6, we have (39) holds. By (39) and Lemma 1, we obtain

$$\begin{aligned}
p\int_a^b |u'(t)|^\alpha \mathrm{d}t &= \min_{a\le t\le b} p(t)\cdot\int_a^b |u'(t)|^\alpha\mathrm{d}t \\
&\le \int_a^b p(t)|u'(t)|^\alpha\mathrm{d}t \\
&= \lambda\left[\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t - \int_a^b q(t)|u'(t)|^\beta\mathrm{d}t\right] + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t \\
&\le \lambda\left[\int_a^b r(t)|u(t)|^\gamma\mathrm{d}t - \min_{a\le t\le b} q(t)\int_a^b |u'(t)|^\beta\mathrm{d}t\right] + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t \\
&\le \lambda\left[\left(\sup_{a\le t\le b}|u(t)|\right)^\gamma\int_a^b r(t)\mathrm{d}t - q\int_a^b |u'(t)|^\beta\mathrm{d}t\right] + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t \\
&\le \lambda\left[\left(\frac{1}{2}\right)^\gamma\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\gamma\int_a^b r(t)\mathrm{d}t - q\int_a^b |u'(t)|^\beta\mathrm{d}t\right] \\
&\quad + \int_a^b h^+(t)|u(t)|^\alpha\mathrm{d}t.
\end{aligned} \tag{49}$$

By using Hölder's inequality (24) with $f(t) = |u'(t)|$, $g(t) = 1$, $\tau = \alpha, \beta$ and $\rho = (\alpha/\alpha - 1), (\beta/\beta - 1)$, respectively, we obtain that

$$\begin{aligned}\int_a^b |u'(t)|\mathrm{d}t &\le \left(\int_a^b |u'(t)|^\alpha \mathrm{d}t\right)^{1/\alpha} (b-a)^{(\alpha-1)/\alpha},\\ \int_a^b |u'(t)|\mathrm{d}t &\le \left(\int_a^b |u'(t)|^\beta \mathrm{d}t\right)^{1/\beta} (b-a)^{(\beta-1)/\beta}.\end{aligned} \tag{50}$$

Therefore,

$$(b-a)^{1-\alpha}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha \le = \int_a^b |u'(t)|^\alpha \mathrm{d}t, \tag{51}$$

$$(b-a)^{1-\beta}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\beta \le \int_a^b |u'(t)|^\beta \mathrm{d}t. \tag{52}$$

From (49), (51), and (52), we have

$$\begin{aligned}&p(b-a)^{1-\alpha}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha\\ &\le \lambda\left[\left(\frac{1}{2}\right)^\gamma \int_a^b r(t)\mathrm{d}t\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\gamma - q(b-a)^{1-\beta}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\beta\right]\\ &\quad + \int_a^b h^+(t)|u(t)|^\alpha \mathrm{d}t.\end{aligned} \tag{53}$$

For the right-hand side of (53), inequality (18) in Lemma 2 with $A = (1/2)^\gamma \int_a^b r(t)\mathrm{d}t$, $B = q(b-a)^{1-\beta}$, and $x = \int_a^b |u'(t)|\mathrm{d}t > 0$ implies that

$$\begin{aligned}&\left(\frac{1}{2}\right)^\gamma \int_a^b r(t)\mathrm{d}t\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\gamma - q(b-a)^{1-\beta}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\beta\\ &\le \left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)}\left(\frac{(b-a)^{\beta-1}(\gamma-\alpha)}{q(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\\ &\quad\cdot\frac{\beta-\gamma}{\beta-\alpha}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha.\end{aligned} \tag{54}$$

From (53) and (54), we have

$$\begin{aligned}&p(b-a)^{1-\alpha}\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha\\ &\le \lambda\left[\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)}\left(\frac{(b-a)^{\beta-1}(\gamma-\alpha)}{q(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\right.\\ &\quad\left.\cdot\frac{\beta-\gamma}{\beta-\alpha}\right]\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha + \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t\left(\int_a^b |u'(t)|\mathrm{d}t\right)^\alpha.\end{aligned} \tag{55}$$

Thus, dividing both sides of (55) by $(\int_a^b |u'(t)|\mathrm{d}t)^\alpha$, we obtain

$$\lambda\left[\left(\frac{1}{2}\right)^{\gamma(\beta-\alpha)/(\beta-\gamma)}\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-\alpha)/(\beta-\gamma)}\left(\frac{(b-a)^{\beta-1}(\gamma-\alpha)}{q(\beta-\alpha)}\right)^{(\gamma-\alpha)/(\beta-\gamma)}\cdot\frac{\beta-\gamma}{\beta-\alpha}\right] + \frac{1}{2^\alpha}\int_a^b h^+(t)\mathrm{d}t \ge p(b-a)^{1-\alpha}, \tag{56}$$

which also leads to (48). The proof is complete. □

Remark 2. We note that when $\lambda = 0$, $\alpha = 2$, and $p(t) \equiv 1$, classical result (3) can be obtained from Theorems 5–7, respectively.

If we take $\alpha = 2$ and $\lambda = 1$ in inequality (48), we obtain the following result.

Corollary 3. *Assume u is a solution of equation (46), satisfying boundary condition (13). Then,*

$$\left(\frac{1}{2}\right)^{\gamma(\beta-2)/(\beta-\gamma)}\left(\frac{(b-a)^{\beta-1}(\gamma-2)}{q(\beta-2)}\right)^{(\gamma-2)/(\beta-\gamma)}\frac{\beta-\gamma}{\beta-2} \cdot\left(\int_a^b r(t)\mathrm{d}t\right)^{(\beta-2)/(\beta-\gamma)}+\frac{1}{4}\int_a^b h^+(t)\mathrm{d}t\geq\frac{p}{b-a}, \tag{57}$$

where $p := min\{p(t): t \in [a,b]\}$, $q := min\{q(t): t \in [a,b]\}$, *and* $h^+(t) := max\{h(t), 0\}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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