Research Article

On the Minimum Variable Connectivity Index of Unicyclic Graphs with a Given Order

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The variable connectivity index, introduced by the chemist Milan Randić in the first quarter of 1990s, for a graph G is defined as $\sum_{vw \in E(G)} ((d_v + \gamma)(d_w + \gamma))^{-1/2}$, where γ is a non-negative real number and d_w is the degree of a vertex w in G . We call this index as the variable Randić index and denote it by ${}^vR_\gamma$. In this paper, we show that the graph created from the star graph of order n by adding an edge has the minimum ${}^vR_\gamma$ value among all unicyclic graphs of a fixed order n , for every $n \geq 4$ and $\gamma \geq 0$.

1. Introduction

All graphs that we discuss in the present study are simple, connected, undirected, and finite. For a graph $G = (V, E)$, the number $|V(G)|$ is called its order and $|E(G)|$ is called its size. For a vertex $v_1 \in V(G)$, denoted by $N(v_1)$, the set of all those vertices of G are adjacent with v_1 . The number $d_{v_1} = |N(v_1)|$ is called degree of v_1 . If $d_{v_1} = 1$, then v_1 is called a pendent vertex or a leaf. A graph of order n is known as an n -vertex graph. As usual, the n -vertex path and star graphs are denoted by P_n and S_n , respectively. An n -vertex graph containing exactly one cycle is called a unicyclic graph. The class of all n -vertex unicyclic graphs is denoted by U_n . The graph obtained from S_n by adding an edge is denoted by S_n^1 . For the (chemical) graph theoretical notation and terminology that are not defined in this paper, we refer the reader to some standard books, such as [1–3].

To model the heteroatoms molecules, it is better to use the vertex-weighted graphs, which are the graphs whose one or more vertices are distinguished in some way from the rest of the vertices [4]. Let G be a vertex-weighted graph with the vertex set $\{v_1, v_2, \dots, v_n\}$ and the vertex weight w_i of the

vertex v_i is for $i = 1, 2, \dots, n$. The augmented vertex-adjacency matrix of G is an $n \times n$ matrix denoted by ${}^{av}\mathbf{A}(G)$ and is defined as ${}^{av}\mathbf{A}(G) = [a_{i,j}]_{n \times n}$, where

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i v_j \in E(G), \\ w_i, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The variable connectivity index [5, 6], proposed by Randić, for the graph G is defined as

$${}^1\chi^f(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{(d_{v_i} + w_i)(d_{v_j} + w_j)}}. \quad (2)$$

We associate this index's name with its inventor Randić by calling it as the variable Randić index. This index was actually introduced within the QSPR/QSAR (quantitative structure-property/activity relationship) studies of heteroatoms molecules. If G is the molecular graph of a homoa-tomic molecule, then $w_1 = w_2 = \dots = w_n = \gamma$ (say) and hence the variable Randić index ${}^1\chi^f(G)$ becomes

$${}^1\chi^f(G) = \sum_{v_i, v_j \in E(G)} \frac{1}{\sqrt{(d_{v_i} + \gamma)(d_{v_j} + \gamma)}} \quad (3)$$

In the rest of this paper, we denote this index by ${}^vR_\gamma$ instead of ${}^1\chi^f$. Clearly, if we take $\gamma = 0$ then the invariant ${}^vR_\gamma(G)$ is the classical Randić index [7, 8]. Details about the chemical applications of the variable Randić index can be found in [4, 7, 9–17] and related references listed therein. In [18], a mathematical study of the variable Randić index was initiated and it was proved that the star graph has the minimum variable Randić index among all trees of a fixed order n , where $n \geq 4$. It needs to be mentioned here that the variable Randić index seems to have more chemical applications than the several well-known variable indices, see, for example, the variable indices considered in the papers [19–25].

For convenience, we introduce some further notation and terminology. Let $A_{n,l,j}^k$ be the unicyclic graph constructed from a path $P = u_0 u_1 \dots u_j$ ($j \geq 1$) by connecting k pendant vertices to u_0 and a cycle C_l , $l = n - j - k$ to u_j , respectively. Let $A_{n,l}^k$ be the unicyclic graph constructed from cycle C_l , $l = n - k$ by connecting k pendant vertices to one vertex of C_l and $A_{n,l}^{i,j,k}$ be the unicyclic graph created from cycle C_l , $n = i + j + k + l$ by attaching $K_{1,i}$, $K_{1,j}$, and $K_{1,k}$ to the vertices of C_l . Let $\mathbf{D}$ be the such class of U_n whose every member has unique 3-cycle, and the vertices apart from 3-cycle are pendent vertices. Let $\mathbf{F}$ be the such class of U_n whose every member has unique 4-cycle, and the vertices apart from 4-cycle are pendent vertices and are joined to nonadjacent vertices of the unique 4-cycle.

In this paper, we characterize the collection of unicyclic graphs on n vertices that minimize variable Randić index. We further show that, for $\gamma \geq 0$, S_n^1 has minimum variable Randić index among the collection U_n and

$${}^vR_\gamma(G) \geq \frac{n-3}{\sqrt{(1+\gamma)(n-1+\gamma)}} + \frac{2}{\sqrt{(2+\gamma)(n-1+\gamma)}} + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}}, \quad (4)$$

where equality holds if and only if $G \cong S_n^1$.

2. Transformations Which Decrease the Variable Randić Index

We introduce some transformations and prove some lemmas to establish main results.

Lemma 1. *If $\gamma \geq 0$ and $s, t \geq 1$, then the function Ψ defined as*

$$\begin{aligned} \Psi(\gamma, s, t) = & \frac{s}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{s+2+\gamma}} - \frac{1}{\sqrt{s+t+3+\gamma}} \right) \\ & + \frac{t}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{t+2+\gamma}} - \frac{1}{\sqrt{s+t+3+\gamma}} \right) \\ & + \frac{1}{\sqrt{(s+2+\gamma)(t+2+\gamma)}} \\ & - \frac{1}{\sqrt{(1+\gamma)(s+t+3+\gamma)}}, \end{aligned} \quad (5)$$

is positive valued.

Proof. We note that the function $\partial\Psi/\partial s$ is strictly increasing in t on the interval $(1, \infty)$ because

$$\begin{aligned} \frac{\partial\Psi}{\partial s} = & \frac{1}{\sqrt{\gamma+1}} \left(\frac{1}{\sqrt{s+2+\gamma}} - \frac{1}{\sqrt{s+t+3+\gamma}} \right) \\ & + \frac{s}{2\sqrt{\gamma+1}} \left(\frac{1}{(s+t+3+\gamma)^{3/2}} - \frac{1}{(s+2+\gamma)^{3/2}} \right) \\ & + \frac{t}{2\sqrt{\gamma+1}(s+t+3+\gamma)^{3/2}} \\ & - \frac{1}{2\sqrt{t+2+\gamma}(s+2+\gamma)^{3/2}} \\ & + \frac{1}{2\sqrt{\gamma+1}(s+t+3+\gamma)^{3/2}}, \\ \frac{\partial}{\partial t} \left(\frac{\partial\Psi}{\partial s} \right) = & \frac{1}{4} \left(\frac{1}{(s+2+\gamma)^{3/2}(t+2+\gamma)^{3/2}} \right. \\ & \left. + \frac{s+t+9+4\gamma}{\sqrt{1+\gamma}(s+t+3+\gamma)^{5/2}} \right) > 0. \end{aligned} \quad (6)$$

Also, it can be seen that the value of the function $\partial\Psi/\partial s$ at $t = 1$ is > 0 , which implies that the function $\partial\Psi/\partial s$ is positive valued for $t > 1$, and hence the function Ψ is strictly increasing w.r.t. s on the interval $(1, \infty)$. In our case $\Psi(\gamma, s, t) = \Psi(\gamma, s, t)$, the function Ψ is also strictly increasing w.r.t. t on the interval $(1, \infty)$. Hence, $\Psi(\gamma, s, t) > 0$ for all $s, t > 1$ and $\gamma \geq 0$. $\square$

Lemma 2. *If $\gamma \geq 0$ and $s, t \geq 1$, then the function Φ defined as*

$$\begin{aligned} \Phi(\gamma, s, t) = & \frac{s}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{s+2+\gamma}} - \frac{1}{\sqrt{s+t+4+\gamma}} \right) \\ & + \frac{t}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{t+2+\gamma}} - \frac{1}{\sqrt{s+t+4+\gamma}} \right) \\ & + \frac{1}{\sqrt{(s+2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(s+t+4+\gamma)}} \\ & + \frac{1}{\sqrt{(t+2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(s+t+4+\gamma)}}, \end{aligned} \quad (7)$$

is positive valued.

The proof of Lemma 2 is analogous to the proof of Lemma 1.

Lemma 3. *If $\gamma \geq 0$, $t > 2$, and $r > 1$, it holds that*

$$\begin{aligned}\Omega(\gamma, r, t) &= \frac{r-1}{\sqrt{1+\gamma}} \left(\frac{1}{r+\gamma} - \frac{1}{\sqrt{r+t+\gamma}} \right) \\ &\quad + \frac{1}{\sqrt{(r+\gamma)(t+1+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(r+t+\gamma)}} > 0.\end{aligned}\tag{8}$$

Proof. Let

$$\begin{aligned}\Omega(\gamma, r, t) &= \frac{(r-1)(t-1)}{\sqrt{(1+\gamma)(r+\gamma)(t+\gamma)(r+t+\gamma)}} \\ &\quad \times \left(\frac{\sqrt{t+\gamma}}{\sqrt{r+\gamma} + \sqrt{r+t-1+\gamma}} - \frac{1}{\sqrt{(r+\gamma)(t+\gamma)} + \sqrt{(1+\gamma)(r+t-1+\gamma)}} \right) \\ &= \left(\frac{(r-1)(t-1)}{\sqrt{(1+\gamma)(r+\gamma)(t+\gamma)(r+t-1+\gamma)}} \right) \\ &\quad \times \left[\left(\frac{\sqrt{t+\gamma}(\sqrt{(r+\gamma)(t+\gamma)} + \sqrt{(1+\gamma)(r+t-1+\gamma)})}{(\sqrt{r+\gamma} + \sqrt{r+t-1+\gamma})(\sqrt{(r+\gamma)(t+\gamma)} + \sqrt{(1+\gamma)(r+t-1+\gamma)})} \right) \right. \\ &\quad \left. - \left(\frac{(\sqrt{r+\gamma} + \sqrt{r+t-1+\gamma})}{(\sqrt{r+\gamma} + \sqrt{r+t-1+\gamma})(\sqrt{(r+\gamma)(t+\gamma)} + \sqrt{(1+\gamma)(r+t-1+\gamma)})} \right) \right] \\ &= \frac{(r-1)(t-1)}{\sqrt{(1+\gamma)(r+\gamma)(t+\gamma)(r+t-1+\gamma)}} \\ &\quad \times \left(\frac{\sqrt{r+\gamma}(t+\gamma-1) + \sqrt{r+t-1+\gamma}(\sqrt{1+\gamma}\sqrt{t+\gamma}-1)}{(\sqrt{r+\gamma} + \sqrt{r+t-1+\gamma})(\sqrt{(r+\gamma)(t+\gamma)} + \sqrt{(1+\gamma)(r+t-1+\gamma)})} \right).\end{aligned}\tag{9}$$

Since $\gamma \geq 0$, $t > 2$ so $t + \gamma > 0$, also $(\sqrt{1+\gamma}\sqrt{t+\gamma}-1) > 0$. Hence, $\Omega(\gamma, r, t) > 0$. $\square$

The proof of Lemma 4 is analogous to the proof of Lemma 3.

Lemma 4. For $\gamma \geq 0$, it holds that

$$\begin{aligned}f(\gamma) &= \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{\gamma+2}} - \frac{1}{\sqrt{\gamma+3}} \right) \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(\gamma+3)}} > 0.\end{aligned}\tag{10}$$

Lemma 5. For $\gamma \geq 0$, it holds that

$$\begin{aligned}f(\gamma) &= \frac{1}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{\gamma+3}} - \frac{1}{\sqrt{\gamma+4}} \right) + \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{\gamma+3}} - \frac{1}{\sqrt{\gamma+4}} \right) \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(\gamma+4)}} > 0.\end{aligned}\tag{11}$$

The proof of Lemma 5 is analogous to the proof of Lemma 3.

Now, we use three transformations, which will reduce the variable Randić index.

2.1. Transformation 1. Let $G \in U_n$ and uv be an edge of G such that $d_v = r \geq 2$. $N(v) = \{u, w_1, w_2, \dots, w_{r-1}\}$, where $w_1, w_2, \dots, w_{r-1}$ are leaves.

Define $\dot{G} = G - \{vw_1, vw_2, \dots, vw_{r-1}\} + \{uw_1, uw_2, \dots, uw_{r-1}\}$. G and $\dot{G}$ are depicted in Figure 1.

FIGURE 1: The graph $\hat{G}$ (obtained using Transformation 1) from G .

Lemma 6. Let $G, \hat{G} \in U_n$. By applying Transformation 1 and for $\gamma \geq 0$, we have

$${}^vR_\gamma(G) > {}^vR_\gamma(\hat{G}). \quad (12)$$

Proof. Let $d_u = t \geq 3$:

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\hat{G}) &> \frac{d_v - 1}{\sqrt{(1 + \gamma)(d_v + \gamma)}} + \frac{1}{\sqrt{(d_v + \gamma)(d_u + \gamma)}} - \frac{d_v}{\sqrt{(1 + \gamma)(d_v + d_u - 1 + \gamma)}}, \\ {}^vR_\gamma(G) - {}^vR_\gamma(\hat{G}) &> \frac{d_v - 1}{\sqrt{1 + \gamma}} \left(\frac{1}{\sqrt{d_v + \gamma}} - \frac{1}{\sqrt{d_v + d_u - 1 + \gamma}} \right) + \frac{1}{\sqrt{(d_v + \gamma)(d_u + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(d_v + d_u - 1 + \gamma)}}, \\ {}^vR_\gamma(G) - {}^vR_\gamma(\hat{G}) &> \frac{r - 1}{\sqrt{1 + \gamma}} \left(\frac{1}{\sqrt{r + \gamma}} - \frac{1}{\sqrt{t + r - 1 + \gamma}} \right) + \frac{1}{\sqrt{(r + \gamma)(t + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(t + r - 1 + \gamma)}} > 0. \end{aligned} \quad (13)$$

Using Lemma 3, one can see that inequality (13) holds. $\square$

Lemma 7. Let $G, \hat{G} \in U_n$ such that $|G| = |\hat{G}|$. By applying Transformation 2 and for $\gamma \geq 0$, we have

$${}^vR_\gamma(G) > {}^vR_\gamma(\hat{G}). \quad (14)$$

2.2. Transformation 2. Let $G, \hat{G} \in U_n$ such that $|G| = |\hat{G}|$ and their unique cycles are r -cycle and $(r - 1)$ -cycle, respectively, $r \geq 4$. For both graphs, the vertices apart from the cycle are leaf vertices joined to exactly one vertex of the cycle.

Proof. Since

$$\begin{aligned} {}^vR_\gamma(G) &= \frac{r - 2}{\sqrt{(\gamma + 2)(\gamma + 2)}} + \frac{n - r}{\sqrt{(\gamma + 1)(n - r + \gamma + 2)}} + \frac{2}{\sqrt{(\gamma + 2)(n - r + \gamma + 2)}}, \\ {}^vR_\gamma(\hat{G}) &= \frac{r - 3}{\sqrt{(\gamma + 2)(\gamma + 2)}} + \frac{n - r + 1}{\sqrt{(\gamma + 1)(n - r + \gamma + 3)}} + \frac{2}{\sqrt{(\gamma + 2)(n - r + \gamma + 3)}}, \\ {}^vR_\gamma(G) - {}^vR_\gamma(\hat{G}) &= \frac{n - r}{\sqrt{1 + \gamma}} \left(\frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) \\ &\quad + \frac{2}{\sqrt{2 + \gamma}} \left(\frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) \\ &\quad + \frac{1}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(n - r + \gamma + 3)}}. \end{aligned} \quad (15)$$

Now, there will be three cases.

Case I: if $n = r$, then

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &= \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{\gamma+2}} - \frac{1}{\sqrt{\gamma+3}} \right) \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(\gamma+3)}}. \end{aligned} \quad (16)$$

Using Lemma 4, one can see that relation (16) is positive valued.

Case II: if $n - r = 1$, then

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &= \frac{1}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{\gamma+3}} - \frac{1}{\sqrt{\gamma+4}} \right) \\ &\quad + \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{\gamma+3}} - \frac{1}{\sqrt{\gamma+4}} \right) \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(\gamma+4)}}. \end{aligned} \quad (17)$$

Using Lemma 5, one can see that relation (17) is positive valued.

Case III: if $n - r \geq 2$, then

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &= \frac{n-r}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{n-r+\gamma+2}} - \frac{1}{\sqrt{n-r+\gamma+3}} \right) \\ &\quad + \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{n-r+\gamma+2}} - \frac{1}{\sqrt{n-r+\gamma+3}} \right) \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(n-r+\gamma+3)}} \\ {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &> \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}} - \frac{1}{\sqrt{(1+\gamma)(n-r+\gamma+3)}} > 0. \end{aligned}$$

$$\begin{aligned} \text{Since } \frac{n-r}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{n-r+\gamma+2}} \right. \\ \left. - \frac{1}{\sqrt{n-r+\gamma+3}} \right) > 0 \text{ and} \end{aligned}$$

$$\begin{aligned} \frac{2}{\sqrt{2+\gamma}} \left(\frac{1}{\sqrt{n-r+\gamma+2}} \right. \\ \left. - \frac{1}{\sqrt{n-r+\gamma+3}} \right) > 0. \end{aligned}$$

(18)
□

2.3. Transformation 3. Let $G \in U_n$ with $|G| = n$ and $u, v \in V(G)$ are on the unique cycle of G . Pendant neighbors of u and v are u_i , where $1 \leq i \leq s$, and v_j , where $1 \leq j \leq t$,

respectively. Assume that the path between u and v on cycle is P_{uv} such that $|E(P_{uv})| = r$, $d(V(P_{uv})) = 2$ if $r \geq 2$ and $|E(P_{uv})| \geq 3$. Construct $\dot{G}$ from G by removing the edges uu_i and vv_j , reducing the path P_{uv} into one vertex $u(v)$ and attaching a star $K_{1,r+s+t+1}$ to $u(v)$, by making sure that $|\dot{G}| = |G|$. G and $\dot{G}$ are depicted in Figure 2.

Lemma 8. Let $G, \dot{G} \in U_n$ such that $|\dot{G}| = |G|$. By applying Transformation 3 and for $\gamma \geq 0$, $s, t \geq 1$, we have

$${}^vR_\gamma(G) > {}^vR_\gamma(\dot{G}). \quad (19)$$

Proof

Case I: let $r = 1$ and $uv \in E(G)$, then

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &> \frac{s}{\sqrt{(1+\gamma)(s+2+\gamma)}} \\ &\quad + \frac{t}{\sqrt{(1+\gamma)(t+2+\gamma)}} \\ &\quad + \frac{1}{\sqrt{(s+2+\gamma)(t+2+\gamma)}} \\ &\quad - \frac{s+t+1}{\sqrt{(1+\gamma)(s+t+3+\gamma)}} \\ {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &> \frac{s}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{s+2+\gamma}} - \frac{1}{\sqrt{s+t+3+\gamma}} \right) \\ &\quad + \frac{t}{\sqrt{1+\gamma}} \left(\frac{1}{\sqrt{t+2+\gamma}} - \frac{1}{\sqrt{s+t+3+\gamma}} \right) \\ &\quad + \frac{1}{\sqrt{(s+2+\gamma)(t+2+\gamma)}} \\ &\quad - \frac{1}{\sqrt{(1+\gamma)(s+t+3+\gamma)}}. \end{aligned} \quad (20)$$

Using Lemma 1, one can see that relation (20) is positive valued.

Case II: now let $r \geq 2$, then

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &> \frac{s}{\sqrt{(1+\gamma)(s+2+\gamma)}} + \frac{t}{\sqrt{(1+\gamma)(t+2+\gamma)}} \\ &\quad + \frac{1}{\sqrt{(s+2+\gamma)(2+\gamma)}} + \frac{1}{\sqrt{(2+\gamma)(t+2+\gamma)}} \\ &\quad + \frac{r-2}{\sqrt{(2+\gamma)(2+\gamma)}} \\ &\quad - \frac{s+t+r}{\sqrt{(1+\gamma)(s+t+r+2+\gamma)}} \end{aligned} \quad (21)$$

Since $r \geq 2$ and keeping in mind the fact that

FIGURE 2: The graph G and G (obtained using Transformation 3).

$$\Psi(r, s, t) = \frac{s + t + r}{\sqrt{(1 + \gamma)(s + t + r + 2 + \gamma)}}, \quad (22)$$

is an increasing function because

$$\frac{\partial \Psi}{\partial r} = \frac{s + t + r + 4 + 2\gamma}{2\sqrt{\gamma + 1}(s + t + r + 2 + \gamma)^{3/2}} > 0. \quad (23)$$

We have

$$\begin{aligned} {}^vR_\gamma(G) - {}^vR_\gamma(\dot{G}) &> \frac{s}{\sqrt{1 + \gamma}} \left(\frac{1}{\sqrt{s + 2 + \gamma}} - \frac{1}{\sqrt{s + t + 4 + \gamma}} \right) \\ &+ \frac{t}{\sqrt{1 + \gamma}} \left(\frac{1}{\sqrt{t + 2 + \gamma}} - \frac{1}{\sqrt{s + t + 4 + \gamma}} \right) \\ &+ \frac{1}{\sqrt{(s + 2 + \gamma)(2 + \gamma)}} \\ &- \frac{1}{\sqrt{(1 + \gamma)(s + t + 4 + \gamma)}} \\ &+ \frac{1}{\sqrt{(t + 2 + \gamma)(2 + \gamma)}} \\ &- \frac{1}{\sqrt{(1 + \gamma)(s + t + 4 + \gamma)}} \end{aligned} \quad (24)$$

Using Lemma 2, one can see that relation (24) is positive valued.

Let $\mathbf{D}_{n,3}^{i,j,k}$ be the such class of $\mathbf{D}$ that is obtained from cycle $C_l, l = 3$ by attaching $K_{1,i}, K_{1,j}$, and $K_{1,k}$ to the vertices of $C_l, l = 3$. Let $\mathbf{F}_{n,4}^{r,s}$ be the such class of $\mathbf{F}$ that is obtained from cycle $C_l, l = 4$ by attaching $K_{1,r}$ and $K_{1,s}$ to the non-adjacent vertices of $C_l, l = 4$. $\square$

Lemma 9. Let $\mathbf{D}_{n,3}^{i,j,k}, \mathbf{D}_{n,3}^{i,j-1,k}$, and $\mathbf{D}_{n,3}^{i+1,j-1,k}$ be three unicyclic graphs in $\mathbf{D}$ as in Figure 3. If $i \geq j \geq 1$, then

$${}^vR_\gamma(\mathbf{D}_{n,3}^{i,j,k}) > {}^vR_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}), \quad (25)$$

for $\gamma \geq 0$.

Proof. Let $\mathbf{D}_{n,3}^{i,j,k}, \mathbf{D}_{n,3}^{i,j-1,k}$, and $\mathbf{D}_{n,3}^{i+1,j-1,k}$ be three unicyclic graphs in $\mathbf{D}$ as in Figure 3.

$$\begin{aligned} {}^vR_\gamma(\mathbf{D}_{n,3}^{i,j,k}) - {}^vR_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) &= \frac{j}{\sqrt{(1 + \gamma)(j + 2 + \gamma)}} - \frac{j - 1}{\sqrt{(1 + \gamma)(j + 1 + \gamma)}} \\ &+ \frac{1}{\sqrt{(i + 2 + \gamma)(j + 2 + \gamma)}} - \frac{1}{\sqrt{(i + 2 + \gamma)(j + 1 + \gamma)}} \\ &+ \frac{1}{\sqrt{(k + 2 + \gamma)(j + 2 + \gamma)}} - \frac{1}{\sqrt{(k + 2 + \gamma)(j + 1 + \gamma)}}, \\ {}^vR_\gamma(\mathbf{D}_{n,3}^{i,j,k}) - {}^vR_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) &= \frac{1}{\sqrt{1 + \gamma}} \left(\sqrt{j + 2 + \gamma} - \sqrt{j + 1 + \gamma} \right) \\ &+ \left(\frac{1}{\sqrt{i + 2 + \gamma}} + \frac{1}{\sqrt{k + 2 + \gamma}} - \frac{2 + \gamma}{\sqrt{1 + \gamma}} \right) \left(\frac{1}{\sqrt{j + 2 + \gamma}} - \frac{1}{\sqrt{j + 1 + \gamma}} \right), \\ {}^vR_\gamma(\mathbf{D}_{n,3}^{i,j,k}) - {}^vR_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) &= \frac{1}{\sqrt{1 + \gamma}} \left(\frac{1}{2} \xi_1^{-1} \right) + \left(\frac{1}{\sqrt{i + 2 + \gamma}} + \frac{1}{\sqrt{k + 2 + \gamma}} - \frac{2 + \gamma}{\sqrt{1 + \gamma}} \right) \left(\frac{-1}{2} \xi_2^{-3} \right), \end{aligned} \quad (26)$$

FIGURE 3: The graph $\mathbf{D}_{n,3}^{i,j,k}$ (used in Lemma 9). FIGURE 4: The graph $\mathbf{F}_{n,4}^{r,s}$ (used in Lemma 10).

where $\xi_1, \xi_2 \in (j+1, j+2)$. Similarly, we have

$$\begin{aligned} {}^v R_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) - {}^v R_\gamma(\mathbf{D}_{n,3}^{i,j-1,k}) &= \frac{i+1}{\sqrt{(1+\gamma)(i+3+\gamma)}} - \frac{i}{\sqrt{(1+\gamma)(i+2+\gamma)}} \\ &+ \frac{1}{\sqrt{(i+3+\gamma)(j+1+\gamma)}} - \frac{1}{\sqrt{(i+2+\gamma)(j+1+\gamma)}} \\ &+ \frac{1}{\sqrt{(k+2+\gamma)(i+3+\gamma)}} - \frac{1}{\sqrt{(k+2+\gamma)(i+2+\gamma)}}, \end{aligned} \quad (27)$$

$$\begin{aligned} {}^v R_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) - {}^v R_\gamma(\mathbf{D}_{n,3}^{i,j-1,k}) &= \frac{1}{\sqrt{1+\gamma}} (\sqrt{i+3+\gamma} - \sqrt{i+2+\gamma}) \\ &+ \left(\frac{1}{\sqrt{j+1+\gamma}} + \frac{1}{\sqrt{k+2+\gamma}} - \frac{\gamma+2}{\sqrt{1+\gamma}} \right) \left(\frac{1}{\sqrt{i+3+\gamma}} - \frac{1}{\sqrt{i+2+\gamma}} \right) \\ {}^v R_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) - {}^v R_\gamma(\mathbf{D}_{n,3}^{i,j-1,k}) &= \frac{1}{\sqrt{1+\gamma}} \left(\frac{1}{2} \xi_3^{-1/2} \right) + \left(\frac{1}{\sqrt{j+1+\gamma}} + \frac{1}{\sqrt{k+2+\gamma}} - \frac{\gamma+2}{\sqrt{1+\gamma}} \right) \left(\frac{-1}{2} \xi_3^{-1/2} \right), \end{aligned}$$

where $\xi_3, \xi_4 \in (i+2, i+3)$.

Bearing in mind the fact that $i \geq j$, one can see that

$${}^v R_\gamma(\mathbf{D}_{n,3}^{i,j,k}) - {}^v R_\gamma(\mathbf{D}_{n,3}^{i,j-1,k}) > {}^v R_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k}) - {}^v R_\gamma(\mathbf{D}_{n,3}^{i,j-1,k}). \quad (28)$$

Hence, ${}^vR_\gamma(\mathbf{D}_{n,3}^{i,j,k}) > {}^vR_\gamma(\mathbf{D}_{n,3}^{i+1,j-1,k})$. $\square$

Lemma 10. Let $\mathbf{F}_{n,4}^{r,s}$, $\mathbf{F}_{n,4}^{r,s-1}$, and $\mathbf{F}_{n,4}^{r+1,s-1}$ be three unicyclic graphs in $\mathbf{F}$ as in Figure 4. If $r \geq s \geq 1$, then

$${}^vR_\gamma(\mathbf{F}_{n,4}^{r,s}) > {}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1}), \quad (29)$$

for $\gamma \geq 0$.

Proof. Let $\Psi(t) = (t/\sqrt{1+\gamma} + 2/\sqrt{2+\gamma})(1/\sqrt{t+2+\gamma})$ and keeping in mind Figure 4, we have

$$\begin{aligned} {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s}) &= \frac{r}{\sqrt{(1+\gamma)(r+2+\gamma)}} \\ &\quad + \frac{s}{\sqrt{(1+\gamma)(s+2+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(r+2+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(s+2+\gamma)}}, \\ {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s-1}) &= \frac{r}{\sqrt{(1+\gamma)(r+2+\gamma)}} \\ &\quad + \frac{s-1}{\sqrt{(1+\gamma)(s+1+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(r+2+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(s+1+\gamma)}}, \\ {}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1}) &= \frac{r+1}{\sqrt{(1+\gamma)(r+3+\gamma)}} \\ &\quad + \frac{s-1}{\sqrt{(1+\gamma)(s+1+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(r+3+\gamma)}} \\ &\quad + \frac{2}{\sqrt{(2+\gamma)(s+1+\gamma)}}. \end{aligned}$$

$$\begin{aligned} {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s}) - {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s-1}) &= \left(\frac{s}{\sqrt{1+\gamma}} + \frac{2}{\sqrt{2+\gamma}} \right) \frac{1}{\sqrt{s+2+\gamma}} \\ &\quad - \left(\frac{s-1}{\sqrt{1+\gamma}} + \frac{2}{\sqrt{2+\gamma}} \right) \frac{1}{\sqrt{s+1+\gamma}}, \\ {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s}) - {}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1}) &= \Psi(s) - \Psi(s-1) = \Psi'(\xi_1), \end{aligned} \quad (30)$$

where $\xi_1 \in (s+1, s+2)$. Similarly, we have

$$\begin{aligned} {}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1}) - {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s-1}) &= \left(\frac{r+1}{\sqrt{1+\gamma}} + \frac{2}{\sqrt{2+\gamma}} \right) \\ &\quad \cdot \frac{1}{\sqrt{r+3+\gamma}} \\ &\quad - \left(\frac{r}{\sqrt{1+\gamma}} + \frac{2}{\sqrt{2+\gamma}} \right) \\ &\quad \cdot \frac{1}{\sqrt{r+2+\gamma}}, \end{aligned}$$

$${}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1}) - {}^vR_\gamma(\mathbf{F}_{n,4}^{r,s-1}) = \Psi(r+1) - \Psi(r) = \Psi'(\xi_2), \quad (31)$$

where $\xi_2 \in (r+2, r+3)$.

Bearing in mind the fact that $r \geq s \geq 1$, $\Psi''(t) = -1/2(t+2+\gamma)^{5/2}(2(t+2+\gamma)/\sqrt{1+\gamma} - (3/2)\{t/\sqrt{1+\gamma} + 2/\sqrt{2+\gamma}\}) < 0$ for all $t > 0$. Also, $\Psi'(\xi_1) - \Psi'(\xi_2) > 0$, which implies that ${}^vR_\gamma(\mathbf{F}_{n,4}^{r,s}) > {}^vR_\gamma(\mathbf{F}_{n,4}^{r+1,s-1})$.

As a consequence of Lemmas 6–10, one has the following. $\square$

Theorem 1. Among all the unicyclic graphs of order $n \geq 4$, S_n^1 has the minimum variable Randić index for $\gamma \geq 0$ and its value is

$$\begin{aligned} {}^vR_\gamma(G) &\geq \frac{n-3}{\sqrt{(1+\gamma)(n-1+\gamma)}} + \frac{2}{\sqrt{(2+\gamma)(n-1+\gamma)}} \\ &\quad + \frac{1}{\sqrt{(2+\gamma)(2+\gamma)}}. \end{aligned} \quad (32)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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