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Research Article

H^m Convergence of the Second-Grade Fluid Equations to Euler Equations in $\mathbb{R}^d$

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In this paper, we investigate an approximation of the Euler equation by the second-grade fluid equations in $\mathbb{R}^d\,(d=2,3)$. The convergence in H^m of a sequence of solutions to the second-grade fluid equations in a uniform interval is proven as both the viscosity coefficient (ν) and filter parameter (α) tend to zero with an initial velocity in H^m.

1. Introduction

Let $\Omega\subset\mathbb{R}^d\,(d=2,3)$ be a simply connected domain and $T>0$. The second-grade fluid equations are as follows:

$$\begin{cases} v_t-\nu\Delta u+u\cdot\nabla v+\sum_{i=1}^{d}v_i\nabla u_i+\nabla p=0, & \text{in}\,\Omega\times(0,T),\\ v=u-\alpha^2\Delta u, & \\ \nabla\cdot u=0, & \text{in}\,\Omega\times(0,T),\\ u|_{t=0}=u_0^{\alpha,\nu}, & \text{in},\Omega, \end{cases} \tag{1}$$

where u is the velocity, p is the pressure, ν is the viscosity coefficient, and α is the filter parameter. We impose the nonslip boundary conditions, i.e.,

$$u=0,\quad \text{in}\,\partial\Omega\times(0,T). \tag{2}$$

It is believed that equation (1) is an interpolation between the Navier–Stokes $(\alpha=0)$ and Euler-α equation $(\nu=0)$. When the filter parameter (α) and viscosity (ν) are very small, we expect the second-grade fluid system (1) to behave like the Euler system:

$$\begin{cases} \partial_t\overline{u}+\overline{u}\cdot\nabla\overline{u}+\nabla\overline{p}=0, & \text{in}\,\Omega\times(0,T),\\ \nabla\cdot\overline{u}=0, & \text{in}\,\Omega\times(0,T),\\ \overline{u}|_{t=0}=u_0, & \text{in}\,\Omega, \end{cases} \tag{3}$$

with the following no-penetration boundary condition:

$$\overline{u}\cdot\widehat{n}=0,\quad \text{in}\,\partial\Omega\times(0,T), \tag{4}$$

where $\widehat{n}$ is the outward-pointing normal unit vector of $\partial\Omega$.

It is well known that the zero-viscosity limit of the incompressible Navier–Stokes equations, which is expressed by (1) with a vanishing filter parameter (i.e., $\alpha=0$), is one of the most challenging open problems in fluid mechanics (see [1–4] and the references therein). This is because of the formation of a boundary layer that appears due to the different boundary conditions between the Navier–Stokes and Euler equations. Fortunately, Lopes Filho et al. [5] proved that the solution of the 2D Euler-α equation $(\nu=0)$ given by (1) with a no-slip boundary condition converges to the solution of the Euler equations (3) with no-penetration boundary conditions, despite the presence of a boundary layer. Lopes Filho et al. also obtained an approximation of the 2D Euler equation (3) using the second-grade fluid equation (1) [6]. They expected their work to shed light into the contrast between the vanishing α limit of the Euler-α and the vanishing viscosity limit of the Navier–Stokes system in the presence of a boundary layer. In their paper [7], Su and Zang selected a radical symmetric example such that the solution of the second-grade fluid equation converged to the solution of the Euler equation as α and ν tended to zero in L^2-space.

In this paper, we only examine with the vanishing α and vanishing viscosity limits in the whole space, i.e., $\Omega = \mathbb{R}^d$ $(d = 2, 3)$. For the case of $\alpha = 0$ in (1), the inviscid limit in the whole space has been examined by several authors, e.g., Swann [8], Kato [9], Constantin [10], and Masmoudi [4]. In these previous results, they proved the convergence in H^m space, as long as a solution of the Euler system exists. For the case of Euler-α (i.e., $\nu = 0$), Linshiz and Titi [11] verified that the convergence of the Euler-α in the whole space was guaranteed using the same technique used by Masmoudi [4]. Zang [12] extended these results of the Euler-α equation to periodic boundary conditions using a different method. In this paper, we focus on the uniform estimates on the viscosity and filter parameter. It is easy to see these estimates on the viscosity for Navier–Stokes equations by Swann [8] and Masmoudi [4]. However, we discuss on a priori estimates on the filter parameter α by the special structure of second-grade fluid equations.

The remainder of this article is divided into two sections. In Section 2, we introduce basic notation and the existence theorem of the second-grade fluid equations, and we present a technique lemma. In Section 3, we present the main results of this paper and provide a proof.

2. Existence Theorem and Technique Lemma

First, we let $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\mathbb{R}^d)$, respectively, and let $(\cdot,\cdot)_m$ and $\|\cdot\|_m$ be the scalar product and the norm in $H^m(\mathbb{R}^d)$, respectively. For any $f, g \in H^m(\mathbb{R}^d)$,

$$(f, g)_m = \sum_{|\alpha|\le m} (D^\alpha f, D^\alpha g), \tag{5}$$

where D^α is a multi-index derivation, $\alpha = (\alpha_1, \dots, \alpha_d)$.

We denote by H and V_m the closures of $C_0^\infty(\mathbb{R}^d)$ that are divergence free in $L^2(\mathbb{R}^d)$ and $H^m(\mathbb{R}^d)$ $(m \ge 1)$, respectively. By the Sobolev Theorem, V_m is embedded in H.

The well-posedness of (1) has been established previously [13, 14], and the existence and well-posedness of (3) have been determined by many mathematicians (see [15–17] and the references therein). The following theorem can be obtained.

Theorem 1. *Let $u_0 \in V_m$, $(m > s_0 = (d/2) + 1)$. There exists a $T^* = T^*(\|u_0\|_{s_0}) > 0$ such that, for any $T < T^*$, there exists a unique solution $u \in C([0,T]; V_m) \cap AC([0,T]; V_{m-1})$ of problem (3) with an initial velocity u_0, where $AC[0,T]$ represents the class of the absolutely continuous functions on $[0,T]$. In two dimensions, the solution exists globally in time $(T^* = \infty)$. Similar results hold for the second-grade fluid equations (1) with a maximal interval of existence of the three-dimensional second-grade fluid equations, which are also dependent on α and ν, and the solution $u^{\alpha,\nu}$ satisfies*

$$\left\|u^{\alpha,\nu}(t)\right\|^2 + \alpha^2\left\|\nabla u^{\alpha,\nu}(t)\right\|^2 + \nu\int_0^t \left\|\nabla u^{\alpha,\nu}(\tau)^2 d\tau\right\| = \left\|u_0\right\|^2 + \alpha^2\left\|\nabla u_0\right\|^2, \tag{6}$$

for every $t \in [0,T]$.

Throughout this paper, the following lemma plays a crucial role in proving the convergence of the second-grade fluid equations to the Euler equations in H^m.

Lemma 1. *Let $m > s_0$ and both $u \in H^{m+3}$ and $v \in H^{m+3}$ be divergence free vectors. We can obtain the following estimates:*

$$\left|((u\cdot\nabla)\Delta u, v)_m\right| \le C\|u\|_m\|\nabla u\|_m\|v\|_{m+3}, \tag{7}$$

$$\left|\sum_{j=1}^d \left(\Delta u_j \Delta u_j, v\right)_m\right| \le C\|u\|_m\|\nabla u\|_m\|v\|_{m+2}. \tag{8}$$

Assuming that $m > s_0 + 1$, the following estimates can be obtained:

$$\left|((u\cdot\nabla)\Delta u, u)_m\right| + \left|\sum_{j=1}^d \left(\Delta u_j \Delta u_j, u\right)_m\right| \le C\|u\|_{H^m}\|\nabla u\|_m^2. \tag{9}$$

Proof. First, we examine estimates (7) and (8). Since $m \ge s_0$,

$$\begin{aligned}
&\left|\sum_{j=1}^d \left(\Delta u_j \nabla u_j, v\right)_m\right| \\
&\le \left|\sum_{j=1}^d \sum_{|\beta|=2} \left(\Delta u_j \nabla u_j, \partial^{2\beta} v\right)_{m-2}\right| + \left|\sum_{j=1}^d \left(\Delta u_j \nabla u_j, v\right)_1\right| \\
&\le \sum_{j=1}^d \left\|\Delta u_j \nabla u_j\right\|_{m-2}\|v\|_{m+2} + \|\nabla u\|_1\|\Delta u\|_1\|v\|_2 + \|u\|_2^2\|v\|_1 \\
&\le C\|\nabla u\|_m\|\Delta u\|_{m-2}\|v\|_{m+2} + \|\nabla u\|_1\|\Delta u\|_1\|v\|_2 + \|u\|_2^2\|v\|_1 \\
&\le C\|u\|_m\|\nabla u\|_m\|v\|_{m+2}.
\end{aligned} \tag{10}$$

Integrating by parts and using a similar approach as in the proof above, the following is obtained:

$$\begin{aligned}
\left|(u\cdot\nabla(\Delta u), v)_m\right| &= \left|(u\cdot\nabla v, \Delta u)_m\right| \\
&\le \|u\Delta u\|_{m-2}\|\nabla v\|_{m+2} + |u\Delta u|_2\|\nabla v\|_1 \\
&\le C\|u\|_m\|\nabla u\|_m\|\nabla v\|_{m+2}.
\end{aligned} \tag{11}$$

Thus, it is sufficient to prove that (9) is true, since

$$\begin{aligned}
((u\cdot\nabla)\Delta u, u)_{H^m} &= \sum_{|\alpha|=0}^m \int_{\mathbb{R}^d} D^\alpha(u\cdot\nabla(\Delta u))D^\alpha u\,dx \\
&= -\sum_{|\alpha|=0}^m \sum_{\beta\le\alpha} \binom{\alpha}{\beta}\left(\int \sum_{l=1}^d \left(D^\beta \partial_l u \cdot \nabla\left(D^{\alpha-\beta}\partial_l u\right) D^\alpha u\,dx\right)\right. \\
&\quad - \sum_{|\alpha|=0}^m \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \int \sum_{l=1}^d D^\beta u \cdot \nabla\left(D^{\alpha-\beta}\partial_l u\right)\right) D^\alpha \partial_l u\,dx \\
&= I_1 + I_2.
\end{aligned} \tag{12}$$

We divide I_1 into four parts, as follows:

$$
\begin{aligned}
I_1 = & \sum_{|\alpha|>0}^{m} \sum_{0<|\beta|\le[m/2]} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \sum_{l=1}^{d} \left(D^{\beta}\partial_l u \cdot \left(D^{\alpha-\beta}\partial_l u (\nabla D^{\alpha} u \right) \mathrm{d}x \\
& + \sum_{|\alpha|>0}^{m} \sum_{|\beta|\le[m/2]+1}^{m-1} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \sum_{l=1}^{d} \left(D^{\beta}\partial_l u \cdot \left(D^{\beta-\alpha} D_l u (\nabla D^{\alpha} u \right) \mathrm{d}x \\
& + \sum_{|\alpha|>0}^{m} \sum_{l=1}^{d} \int_{\mathbb{R}^d} \partial_l u \left(D^{\alpha}\partial_l u (\nabla D^{\alpha} u \right) \mathrm{d}x \\
& + \sum_{l=1}^{d} \sum_{|\beta|=m} \int_{\mathbb{R}^d} D^{\beta}\partial_l u \partial_l u (\nabla D^{\alpha} u) \mathrm{d}x \\
= & I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{13}
$$

Integrating by parts, we observe that

$$
\begin{aligned}
I_{11} & \le C \sum_{|\alpha|>0}^{m} \sum_{0<|\beta|\le[m/2]} \left\| D^{\beta}\partial_l u \right\|_{L^4} \left\| D^{\alpha-\beta}\partial_l u \right\|_{L^4} \|\nabla u\|_m \\
& \le C \sum_{|\alpha|>0}^{m} \sum_{0<|\beta|\le[m/2]} \left\| D^{\beta+1} u \right\|_{L^4} \left\| D^{\alpha-\beta}\nabla u \right\|_{L^4} \|\nabla u\|_m .
\end{aligned} \tag{14}
$$

As $m \ge s_0 + 1$ and $\beta + 2 \le [m/2] + 2 \le (m/2) + 2 \le m$, it follows that

$$
I_{11} \le C \|u\|_m \|\nabla u\|_m^2 . \tag{15}
$$

Similarly, $\alpha - \beta + 2 \le m$ for $\beta \ge [m/2] + 1$, and thus,

$$
I_{12} \le C \|u\|_m \|\nabla u\|_m^2 . \tag{16}
$$

Now, we determine the bounds of I_{13} and I_{14}, as follows:

$$
\begin{aligned}
I_{13} & = \sum_{|\alpha|=0}^{m} \sum_{l=1}^{d} \int_{\mathbb{R}^d} \partial_l u \left(D^{\alpha}\partial_l u (\nabla D^{\alpha} u \right) \mathrm{d}x \\
& \le C \|\nabla u\|_{L^\infty} \|\nabla u\|_{H^m}^2 \le C \|u\|_{H^3} \|\nabla u\|_{H^m}^2 \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{17}
$$

Along with the above estimate, we have the following:

$$
\begin{aligned}
I_{14} & = \sum_{l=1}^{d} \sum_{|\beta|=m} \int_{\mathbb{R}^d} D^{\beta}\partial_l u \partial_l u (\nabla D^{\alpha} u) \mathrm{d}x \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{18}
$$

For I_2, integrating by parts yields the following:

$$
\begin{aligned}
I_2 = & - \sum_{|\alpha|=0}^{m} \sum_{|\beta|>0} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} D^{\beta} u \cdot \nabla D^{\alpha-\beta}\partial_l u \cdot D^{\alpha}\partial_l u \mathrm{d}x \\
\le & \sum_{|\alpha|=0}^{m} \sum_{0<|\beta|\le[m/2]} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} D^{\beta} u \cdot \nabla D^{\alpha-\beta}\partial_l u \cdot D^{\alpha}\partial_l u \mathrm{d}x \\
& + \sum_{|\alpha|=0}^{m} \sum_{0\le|\alpha-\beta|<m-3} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} D^{\beta} u \cdot \nabla D^{\alpha-\beta}\partial_l u \cdot D^{\alpha}\partial_l u \mathrm{d}x \\
& + \sum_{|\alpha|=0}^{m} \sum_{m-3<|\alpha-\beta|\le(m/2)} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} D^{\beta} u \cdot \nabla D^{\alpha-\beta}\partial_l u \cdot D^{\alpha}\partial_l u \mathrm{d}x \\
= & I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{19}
$$

Estimating the integrals one by one in (19), it is easy to verify that

$$
\begin{aligned}
I_{21} & \le C \sum_{|\alpha|=0}^{m} \sum_{0<|\beta|\le[m/2]} \left\| D^{\beta} u \right\|_{L^\infty} \left\| D^{\alpha-\beta+1}\nabla u \right\|_{L^2} \|\nabla u\|_{H^m} \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 ,
\end{aligned} \tag{20}
$$

since for $\beta < [m/2]$, $\|D^{\beta} u\|_{L^\infty} \le C \|D^{\beta} u\|_{H^2} \|u\|_{H^m}$ $(m \ge 4)$.

By the interpolation and Hölder inequalities, the following can be inferred:

$$
\begin{aligned}
I_{22} & \le C \sum_{m-3<|\alpha-\beta|<(m/2)} \left\| D^{\beta} u \right\|_{L^4} \left\| \nabla D^{\alpha-\beta+1} u \right\|_{L^4} \|\nabla u\|_{H^m} \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{21}
$$

As in the proof of I_{21}, the following can be verified:

$$
\begin{aligned}
I_{23} & \le C \sum_{|\alpha-\beta|\le m-3} \left\| D^{\beta} u \right\|_{L^2} \left\| \nabla D^{\alpha-\beta+1} u \right\|_{L^\infty} \|\nabla u\|_{H^m} \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{22}
$$

Therefore, we have shown that

$$
\left| ((u \cdot \nabla)\Delta u, u)_{H^m} \right| \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 . \tag{23}
$$

Finally, integrating by parts yields the following:

$$
\begin{aligned}
& \left| \sum_{j=1}^{d} \left(\Delta u_j \nabla u_j, u \right)_{H^m} \right| \\
& \le \sum_{j=1}^{d} \sum_{|\beta|=1} \left(\Delta u_j \nabla u_j, \partial^{\beta+1} u \right)_{H^{m-1}} + \sum_{j=1}^{d} \left(\Delta u_j \nabla u_j, u \right)_{L^2} \\
& \le \sum_{j=1}^{d} \left\| \Delta u_j \nabla u_j \right\|_{H^{m-1}} \left\| \nabla^2 u \right\|_{H^{m-1}} + \|u\|_{L^\infty} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2} \\
& \le C \|\nabla u\|_{H^{m-1}} \|\Delta u\|_{H^{m-1}} \left\| \nabla^2 u \right\|_{H^{m-1}} + \|u\|_{H^3} \|\nabla u\|_{H^1}^2 \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{24}
$$

Therefore,

$$
\begin{aligned}
& \left| ((u \cdot \nabla)\Delta u, u)_{H^m} \right| + \left| \sum_{j=1}^{d} \left(\Delta u_j \nabla u_j, u \right)_{H^m} \right| \\
& \le C \|u\|_{H^m} \|\nabla u\|_{H^m}^2 .
\end{aligned} \tag{25}
$$

□

3. Main Theorem and Its Proof

We state the main result, which is as follows: the classical results of (1) will exist in the maximal interval of the solution of the Euler equation (3) and converge to the solutions of Euler equations under suitable assumptions of the initial velocity. The main results of this paper are described in detail in this section.

Theorem 2. *Let $m > (d/2) + 2$ and $u_0 \in H^m(\mathbb{R}^d)$ be a divergence-free and solenoidal vector $u_0^{\alpha,\nu} \in H^{m+1}(\mathbb{R}^d)$, which can be approximated by u_0 as follows:*

$$\|u_0^{\alpha,\nu} - u_0\|_m \longrightarrow 0, \quad as\ \alpha, \nu \longrightarrow 0$$
$$\alpha\|\nabla u_0^{\alpha,\nu}\|_m \longrightarrow 0, \quad as\ \alpha, \nu \longrightarrow 0$$

Let T^* be the time of existence and $\overline{u} \in C([0,T^*];H^m)$ be the solution of the Euler system (3). Then, for all $0 < T_0 < T^*$, there exists $\alpha_0, \nu_0 > 0$ such that, for all $\alpha < \alpha_0, \nu < \nu_0$, the second-grade fluid equation (1) has a unique solution $u^{\alpha,\nu} \in C([0,T_0];H^{m+1}(\mathbb{R}^d))$. Moreover,

$$\alpha\left\|\nabla u^{\alpha,\nu}\right\|_{L^\infty(0,T_0;H^m)} + \left\|u^{\alpha,\nu} - \overline{u}\right\|_{L^\infty(0,T_0;H^m)} \longrightarrow 0, \quad \text{as}\ \alpha, \nu \longrightarrow 0,$$
$$\alpha\left\|u^{\alpha,\nu}\right\|_{L^\infty(0,T;L^2)} + \left\|u^{\alpha,\nu} - \overline{u}\right\|_{L^\infty(0,T_0;L^2)}$$
$$\leq C\left((\nu t)^{(1/2)} + \alpha t^{(1/2)} + \left\|u_0^{\alpha,\nu} - u_0\right\|_{L^2} + \alpha\left\|\nabla u_0^{\alpha,\nu}\right\|_{L^2}\right),$$
$$\alpha\left\|\nabla u^{\alpha,\nu}\right\|_{L^\infty(0,T;H^{S_0})} + \left\|u^{\alpha,\nu} - \overline{u}\right\|_{L^\infty(0,T;H^{S_0})}$$
$$\leq C\left((\nu t)^{(m-S_0/2)} + \alpha t^{(m-S_0/2)} + \left\|u_0^{\alpha,\nu} - u_0\right\|_{H^m} + \alpha\left\|u_0^{\alpha,\nu}\right\|_{H^m}\right), \tag{26}$$

for all $0 < t < T_0$ and $(d/2) \leq S_0 \leq m-1$, and C depends only on d and T_0.

Remark 1. For $u_0 \in H^m(\mathbb{R}^d)$, from [16], if we choose a suitably small $\varepsilon > 0$ such that $(\alpha/\varepsilon) \longrightarrow 0$, then there exists a $u_0^\varepsilon \in H^{m+1}(\mathbb{R}^n)$ such that

$$\begin{aligned}&\left\|u_0 - u_0^\varepsilon\right\|_{H^m} \longrightarrow 0,\\&\alpha\left\|\nabla u_0^\varepsilon\right\|_{H^m} \longrightarrow 0, \quad \text{as}\ \alpha \longrightarrow 0, \nu \longrightarrow 0.\end{aligned} \tag{27}$$

Proof. We prove this theorem using the following three steps.

Step 1 (energy estimates): we easily obtain the uniform existence interval of the solutions to the second-grade fluid equation (1) by using the Kato–Lai Theorem [15] with the following energy estimates (dropping α and ν) from Lemma 1:

$$\frac{d}{dt}\left(\|u\|_{H^m}^2 + \alpha^2\|\nabla u\|_{H^m}^2\right) + \nu\|\nabla u\|_{H^m}^2 \leq C\|u\|_{s_0}\|u\|_{H^m}^2 + \alpha^2\|u\|_{s_0}\|\nabla u\|_{H^m}^2, \tag{28}$$

where $m+1 > s_0 > (d/2)+1$. Therefore, there exists a $T^* > 0$ that only depends on $\|u_0^{\alpha,\nu}\|_{s_0}$, and we have a uniform bound of $u^{\alpha,\nu}$ for all $t \in [0,T^*)$:

$$\|u\|_{L^\infty(0,t;H^m)} \leq K, \tag{29}$$

where K is independent of α, ν and $0 < \alpha < \alpha_0, 0 < \nu < \nu_0$.

Step 2 (convergence in L^2): the difference between (1) and (3) is given by

$$(u-\overline{u})_t - \alpha^2\Delta u_t - \nu\Delta u + (u\cdot\nabla u - \overline{u}\cdot\nabla\overline{u}) - \alpha^2(u\cdot\nabla)\Delta u - \alpha^2\sum_{j=1}^d \Delta u_j\nabla u_j = -\nabla\left(p + \frac{1}{2}|u|^2 - \overline{p}\right), \tag{30}$$

where $u = u^{\alpha,\nu}$ and $\overline{p}$ is the pressure of the Euler equation (3).

Multiplying both sides of (30) by $(u-\overline{u})$ and integrating in $\mathbb{R}^d$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u-\overline{u}\|^2 + \alpha^2\|\nabla u\|^2 - \alpha^2(\nabla u, \nabla\overline{u})\right)\\&\quad + \alpha^2(\nabla u, \nabla\overline{u}_t) + \nu\|\nabla u\|^2 - \nu(\nabla u, \nabla\overline{u})\\&\quad + \int_{\mathbb{R}^d}(u-\overline{u})\nabla\overline{u}(u-\overline{u}) - \alpha^2\int_{\mathbb{R}^d}(u\cdot\nabla)\Delta u(u-\overline{u})\\&\quad - \alpha^2\sum_{j=1}^d\int_{\mathbb{R}^d}\Delta u_j\nabla u_j(u-\overline{u})\\&= 0.\end{aligned} \tag{31}$$

It is evident that

$$\begin{aligned}\alpha^2\int_{\mathbb{R}^d}(u\cdot\nabla)\Delta u\overline{u}\mathrm{d}x &= -\alpha^2\sum_{i=1}^d\sum_{j=1}^d\int_{\mathbb{R}^d}u_i\Delta u_j\partial_i\overline{u}_j\\&= \alpha^2\sum_{i=1}^d\sum_{j=1}^d\sum_{l=1}^d\int_{\mathbb{R}^d}\partial_l u_i\partial_l u_j\partial_i\overline{u}_j\\&\quad + \int_{\mathbb{R}^d}u_i\partial_l u_j\partial_l\partial_i\overline{u}_j\\&\leq \alpha^2\|\nabla u\|^2\|\nabla\overline{u}\|_{L^\infty} + \alpha^2\|u\|_{L^4}\|\nabla u\|_{L^2}\left\|\nabla^2\overline{u}\right\|_{L^4}\\&\leq C\|\overline{u}\|_{H^3}\left(\alpha^2\|\nabla u\|^2\right).\end{aligned} \tag{32}$$

Similarly,

$$\begin{aligned}&\alpha^2\sum_{l=1}^d\int_{\mathbb{R}^n}\nabla u_i\nabla u_j\overline{u}\\&= \alpha^2\sum_{j=1}^d\left(-\int_{\mathbb{R}^d}\partial_l u_j\nabla\partial_l u_j\overline{u} + \int_{\mathbb{R}^d}\partial_l u_j\nabla u_j\partial_l\overline{u}\right)\\&\leq \alpha^2\sum_{l=1}^d\int_{\mathbb{R}^d}\partial_l u_j\nabla u_j\partial_l\overline{u} \leq C\|\nabla\overline{u}\|_{L^\infty}\alpha^2\|\nabla u\|^2\\&\leq C\|\overline{u}\|_{H^3}\left(\alpha^2\|\nabla u\|^2\right).\end{aligned} \tag{33}$$

Equation (31) can be rewritten as follows:

$$\frac{d}{dt}\left(\|u-\overline{u}\|^2+\alpha^2\|\nabla u\|^2-\alpha^2(\nabla u,\nabla\overline{u})\right)+\nu\|\nabla u\|^2 \leq \alpha^2\|\nabla u\|\left\|\nabla\overline{u}_t\right\|+\nu\|\nabla u\|\|\nabla\overline{u}\| + \|\nabla\overline{u}\|_{L^\infty}\|u-\overline{u}\|^2+C\alpha^2\|\overline{u}\|_{H^3}\|\nabla u\|^2. \tag{34}$$

It can be inferred that

$$\begin{aligned}&\|u-\overline{u}\|^2+\alpha^2\|\nabla u\|^2+\nu\int_0^t\|\nabla u\|^2 ds\\ &\leq\frac{\alpha^2}{2}\|\nabla u\|^2+\frac{\alpha^2}{2}\left\|\nabla u_0\right\|^2+\frac{\alpha^2}{2}\left\|\nabla u_0^{\alpha,\nu}\right\|^2+\frac{\alpha^2}{2}\|\nabla\overline{u}\|^2\\ &\quad+\frac{\nu}{2}\int_0^t\|\nabla u\|^2+\frac{\nu}{2}\int_0^t\|\nabla\overline{u}\|^2+C\int_0^t\|\nu\|_{H^3}\|u-\overline{u}\|^2\\ &\quad+\int_0^t C\left(\|\overline{u}\|_{H^3}+1\right)\alpha^2\|\nabla u\|^2+\frac{\alpha^2}{2}\int_0^t\left\|\nabla\overline{u}_t\right\|^2 ds\\ &\quad+\left\|u_0^{\alpha,\nu}-\overline{u}_0\right\|^2+\alpha^2\left\|\nabla u_0\right\|^2.\end{aligned} \tag{35}$$

Letting $g(t)=\|u-\overline{u}\|^2+\alpha^2\|\nabla u\|^2$, (35) becomes

$$g(t)\leq\int_0^t g(t)ds+f\left(\alpha,\nu,u_0,\nu_0\right), \tag{36}$$

where

$$f\left(\alpha,\nu,u_0,\nu_0\right)=C\alpha^2 t+C\nu t+\left\|u_0^{\alpha,\nu}-u_0\right\|^2+\alpha^2\left\|\nabla u_0^{\alpha,\nu}\right\|^2\longrightarrow 0, \quad \text{as } \alpha\longrightarrow 0,\nu\longrightarrow 0. \tag{37}$$

By the Gronwall inequality, we obtain the following:

$$\|u-\overline{u}\|^2+\alpha^2\|\nabla u\|^2\leq C\alpha^2 t+C\nu t+\left\|u_0^{\alpha,\nu}-u_0\right\|^2+\alpha^2\left\|\nabla u_0\right\|^2. \tag{38}$$

By the energy estimate (29) and the interpolation inequality (38), the following can be obtained for any $0<s<m, 0<t<T_0$:

$$\|u-\overline{u}\|_{L^\infty(H^s)}+\alpha\|\nabla u\|_{L^\infty(H^s)}\leq C\alpha^{s/m}t^{s/2m}+C(\nu t)^{s/2m} +\left\|u_0^{\alpha,\nu}-u_0\right\|_{H^m}+\alpha\left\|\nabla u_0^{\alpha,\nu}\right\|_{H^m}, \tag{39}$$

since

$$\begin{aligned}\|u-\overline{u}\|_{H^s}&\leq C\|u-\overline{u}\|^{s/m}\|u-\overline{u}\|_{H^m}^{1-(s/m)},\\ \alpha\|\nabla u\|_{H^s}&\leq C(\alpha\|\nabla u\|)^{s/m}\left(\alpha\|\nabla u\|_{H^m}\right)^{1-(s/m)}.\end{aligned} \tag{40}$$

Step 3: (convergence in H^m): From Lemma 1, we know the following:

$$\left|((u\cdot\nabla)\nabla u,\overline{u})_{H^m}\right|\leq C\|u\|_{H^m}\|\nabla u\|_{H^m}\|\nabla\overline{u}\|_{H^{m+2}}, \tag{41}$$

$$\left|\sum_{j=1}^d\left(\Delta u_j\nabla u_j,\overline{u}\right)_{H^m}\right|\leq C\|u\|_{H^m}\|\nabla u\|_{H^m}\|\overline{u}\|_{H^{m+2}}. \tag{42}$$

From the energy estimates (41) and (42), we ensure the convergence in H^m by requiring regularization of the initial data. For all $\delta>0$, we take u_0^δ such that

$$\begin{aligned}\left\|u_0^\delta\right\|_{H^m}&\leq C\left\|u_0\right\|_{H^m},\\ \left\|u_0^\delta\right\|_{H^{m+k}}&\leq\frac{C}{\delta^k},\quad k\in\mathbb{Z}_+.\end{aligned} \tag{43}$$

For any s, such that $(d/2)<s<m-1$, we have

$$\begin{aligned}\left\|u_0^\delta-u_0\right\|_{H^s}&\leq C\delta^{m-s},\\ \left\|u_0^\delta-u_0\right\|_{H^m}&\longrightarrow 0,\end{aligned} \tag{44}$$

as $\delta\longrightarrow 0$. The existence of u_0^δ can be constructed by the following previously reported approaches [4, 16]. The maximum existence time T_0 of the Euler equations only depends on $\|u_0\|_{s_0}$ $(s_0=(d/2)+1)$. Thus, for any $\delta>0$, there exists a uniform interval $(0,T_0)$ in which the solution $\overline{u}^\delta$ of the Euler equations exists. Furthermore, $\|\overline{u}^\delta\|_{H^m}\leq C$, $\|\overline{u}^\delta\|_{H^{m+k}}\leq(C/\delta^k)(k=1,2\ldots)$, and $\|\overline{u}_t^\delta\|_{H^{m+1}}\leq(C/\delta)$.

Similar to a previous report [4], for any $(d/2)<s<m-1$, we have

$$\left\|\overline{u}^\delta-\overline{u}\right\|_{L^\infty(H^m)}\leq C\left(\left\|u_0^\delta-u_0\right\|_{H^m}+\delta^{m-s-1}T\right). \tag{45}$$

For equation (30), we have an energy estimate for $w^{\nu,\alpha,\delta}=u^{\nu,\alpha}-\overline{u}^\delta$, and we obtain the following (dropping ν,α and δ):

$$\begin{aligned}&\frac{d}{dt}\left(\|w\|_{H^m}^2+\alpha^2\|\nabla u\|_{H^m}^2-\alpha^2\left(\nabla u,\nabla\overline{u}^\delta\right)_{H^m}\right)\\ &\quad+\nu\|\nabla u\|_{H^m}^2+\alpha^2\left(\nabla u,\nabla\overline{u}_t^\delta\right)_{H^m}\\ &\leq\nu\left(\nabla u,\nabla\overline{u}^\delta\right)_{H^m}+C\left(\|w\|_{L^\infty}\left\|\overline{u}^\delta\right\|_{H^{m+1}}\|w\|_{H^s}\right.\\ &\quad+\left(\left\|\overline{u}^\delta\right\|_{H^m}+\|u\|_{H^m}\right)\|w\|_{H^m}^2\\ &\quad\left.+C\alpha^2\|u\|_{H^m}\|\nabla u\|_{H^m}\left(\left\|\nabla\overline{u}^\delta\right\|_{H^{m+2}}+\left\|\overline{u}^\delta\right\|_{H^{m+2}}\right)\right),\end{aligned} \tag{46}$$

since

$$\begin{aligned}&\|w\|_{L^\infty}\left\|\overline{u}^\delta\right\|_{H^{m+1}}\|w\|_{H^m}\\ &\leq\left\|u^{\alpha,\nu}-\overline{u}\right\|_{L^\infty}\left\|\overline{u}^\delta\right\|_{H^{m+1}}\|w\|_{H^m}+\left\|\overline{u}-t\overline{u}^\delta\right\|_{L^\infty}\left\|\overline{u}^\delta\right\|_{H^{m+1}}\|w\|_{H^m}.\end{aligned} \tag{47}$$

From Step 2, $u^{\alpha,\nu}$ converges to $\overline{u}$ in H^{m-1}, and we deduce the following:

$$\begin{aligned}\left\|u^{\alpha,\nu}-\overline{u}\right\|_{L^\infty} &\le C\left\|u^{\alpha,\nu}-\overline{u}\right\|_{L^\infty(H^{m+1})}\\ &\le C\alpha^{(m-1/m)}t^{(m-1/2m)}+C(\nu t)^{(m-1/2m)}+g^{(1/2)}(0),\end{aligned} \tag{48}$$

where $g(0)=\|u_0^{\alpha,\nu}-u_0\|_m^2+\alpha\|\nabla u_0^{\alpha,\nu}\|_m^2$. On contrary, from (45), we know that

$$\left\|\overline{u}-t\overline{u}^{\delta}\right\|_{L^\infty}\le C\left(\left\|u_0^{\delta}-u_0\right\|_{H^m}+\delta^{m-s\prime}T\right):=\widetilde{f}(\delta), \tag{49}$$

for some $s_0<s\prime<m-1$. Combining estimates (48) and (49) and using (46) and the Gronwall inequality, we obtain

$$\begin{aligned}\left(\|w\|_{H^m}^2+\alpha^2\|\nabla u\|_{H^m}^2\right) &\le \alpha^2\left\|\overline{u}^{\delta}\right\|_{H^3}+\nu\int_0^T\left\|\nabla\overline{u}^{\delta}\right\|_{H^m}^2\\ &+\alpha^2\int_0^T\|\nabla u\|_{H^m}+\alpha^2\int_0^T\|\nabla\overline{u}\|_{H^m}\mathrm{d}s\\ &+\int_0^T\frac{f(\alpha,\nu,u_0,\nu_0)}{\delta}\|w\|_{H^m}\\ &+\int_0^T C\alpha^2\|\nabla u\|_{H^m}\|\overline{u}\|_{H^{m+3}}\\ &+\int_0^T\widetilde{f}(\delta)\|w\|_{H^m}+\int_0^T C\|w\|_{H^m}^2,\end{aligned} \tag{50}$$

where $f(\alpha,\nu,u_0,\nu_0)$ is defined by (37). It follows that

$$\begin{aligned}&\|u-\overline{u}\|_{L^\infty(0,T_0;H^m)}^2+\alpha^2\|\nabla u\|_{L^\infty(0;T_0;H^m)}^2\\ &\le CT_0\left(\frac{\alpha^2}{\delta}+\frac{\nu}{\delta^2}\right)+\frac{f(\alpha,\nu,u_0,\nu_0)}{\delta}+\frac{\alpha^2}{\delta^3}+\left\|u_0^{\delta}-u_0\right\|_{H^m}+f^2(\delta),\end{aligned} \tag{51}$$

where δ is chosen such that (α^2/δ^3), (ν/δ^2), and $(\|u_0^{\alpha,\nu}-u_0\|_{H^m}/\delta)$ are sufficiently small. Letting $\alpha,\nu\longrightarrow 0$ and $\delta\longrightarrow 0$, the following can be shown:

$$\lim_{\alpha,\nu\longrightarrow 0}\left(\left\|u^{\alpha,\beta}-\overline{u}\right\|_{H^m}^2+\alpha^2\|\nabla u\|_{H^m}^2\right)=0. \tag{52}$$

Using the continuous method, we ensure convergence on any time interval $[0,T_0]$ for $T_0<T^*$, where T^* is the maximum existence time of the Euler equations. This completes the proof. □

Data Availability

Previously reported (regular article) data were used to support this study and are available at (DOI: 10.1142/S0218202518500458). These prior studies are cited at relevant places within the text as reference [7].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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