Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 1469090, 11 pages
https://doi.org/10.1155/2020/1469090

Research Article

Upper Bound on the Bit Error Probability of Systematic Binary Linear Codes via Their Weight Spectra

Jia Liu,[1] **Mingyu Zhang**,[1] **Chaoyong Wang**,[1] **Rongjun Chen**,[2] **Xiaofeng An**,[1] **and Yufei Wang**[1]

[1]*School of Information Engineering, Jilin Engineering Normal University, Changchun 130052, China*
[2]*School of Computer Science, Guangdong Polytechnic Normal University, Guangzhou 510665, China*

Correspondence should be addressed to Rongjun Chen; crj321@163.com

Received 18 November 2019; Accepted 11 December 2019; Published 29 January 2020

Guest Editor: jinchang ren

Copyright © 2020 Jia Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, upper bound on the probability of maximum *a posteriori* (MAP) decoding error for systematic binary linear codes over additive white Gaussian noise (AWGN) channels is proposed. The proposed bound on the bit error probability is derived with the framework of Gallager's first bounding technique (GFBT), where the Gallager region is defined to be an irregular high-dimensional geometry by using a list decoding algorithm. The proposed bound on the bit error probability requires only the knowledge of weight spectra, which is helpful when the input-output weight enumerating function (IOWEF) is not available. Numerical results show that the proposed bound on the bit error probability matches well with the maximum-likelihood (ML) decoding simulation approach especially in the high signal-to-noise ratio (SNR) region, which is better than the recently proposed Ma bound.

1. Introduction

Upper bounds on the maximum *a posteriori* (MAP) decoding error probability, as a key technique for evaluating the performance of the binary linear codes over additive white Gaussian noise (AWGN) channels, bring a profound impact on the reliable transmission of the next-generation mobile communication systems since they can be used to not only predict the performance without resorting to computer simulations but also guide the design of coding [1]. In order to improve the looseness of the union bound in the low signal-to-noise ratio (SNR) region, many improved upper bounds, on the bit error probability [2–5] and on the frame error probability [2–4, 6–15], are proposed. As surveyed in [1], the improved upper bounds on the bit error probability [2–4] are based on Gallager's first bounding technique (GFBT):

$$\Pr\{E_b\} = \Pr\{E_b, \underline{y} \in \mathscr{R}\} + \Pr\{E_b, \underline{y} \notin \mathscr{R}\}, \tag{1}$$

$$\leq \Pr\{E_b, \underline{y} \in \mathscr{R}\} + \Pr\{\underline{y} \notin \mathscr{R}\}, \tag{2}$$

in which E_b denotes the event that represents an error in one of the information bits of the decoded codeword, $\underline{y}$ denotes the received signal vector, and $\mathscr{R}$ denotes an arbitrary region around the transmitted signal vector (called the Gallager region). Divsalar [2] chose the region $\mathscr{R}$ to be an Euclidean sphere centered at the point along the line connecting the origin to the transmitted signal vector. Sason and Shamai [3] chose the region $\mathscr{R}$ to be a circular cone whose central line passes through the origin and the transmitted signal vector. The upper bounds [2, 3] on the bit error probability based on equation (2) can be considered to be simply replaced by the weight spectra $\{A_d, 1 \leq d \leq n\}$ in the upper bound on the frame error probability by

$$A_d^* \triangleq \sum_{i=0}^{k} \frac{i}{k} A_{i,d}, \quad 1 \leq d \leq n, \tag{3}$$

where $A_{i,d}$ denotes the number of code words of Hamming weight d encoded by information bits of Hamming weight i

and k denotes the dimension of the linear code. However, as noted by Zangl and Herzog [4], computing the expression $\Pr\{\underline{y} \notin \mathscr{R}\}$ by the factor 1.0 in (2) means that the worst case of k bit errors is assumed if $\underline{y}$ falls outside the good region $\mathscr{R}$, and then Zangl and Herzog [4] improved the tangential-sphere bound (TSB) on the bit error probability [3] by computing this probability in a more accurate way. The upper bounds on the bit error probability [2–4] require the whole input-output weight enumerating function (IOWEF), which can be applied to both systematic codes and non-systematic codes. The upper bound on the bit error probability by Ma et al. [16] can be evaluated by calculating partial IOWEF with truncated information weight $(\{A_{i,d}, 0 \le i \le T, 0 \le d \le n-k+T\}$, where $T \ge 0$ is a positive integer), which holds only for systematic codes. However, for most codes, the IOWEF is usually not computable. In contrast, it is reasonable to assume that the weight spectra $\{A_d, 0 \le d \le n\}$ of codes are available, such as the BCH code [17]. In this paper, different from most of the existing bounds, we derive a tighter upper bound on the bit error probability of systematic binary linear codes via their weight spectra.

The main results as well as the structure of this paper are summarized as follows:

(1) In Section 2, we present the preliminaries and necessary notation. The conventional union bound and four reported upper bounds based on GFBT are also reviewed in Section 2.

(2) In Section 3, in a detailed way, we rederive the recently proposed bound on the bit error probability by Liu [5], in which the union bound is firstly truncated and then amended for the systematic linear codes over AWGN channels. In this paper, the proposed upper bound on the bit error probability is derived in a much more detailed way by considering more information of the Gallager region $\mathscr{R}$ and the truncated IOWEF of the code. Finally, with the framework of GFBT, we derive the upper bound on the bit error probability which requires only the knowledge of weight spectra of the code.

(3) In Section 4, numerical examples are given to show that the proposed bound is helpful to evaluate the performance of the systematic binary linear codes which can predict the performance of the code in the high-SNR region, avoiding the time-consuming computer simulations.

(4) Section 5 concludes this paper.

2. Preliminaries

Let $\mathbb{F}_2 = \{0, 1\}$ be the binary field. Let $\mathscr{C}[n, k]$ be a systematic binary linear block code of dimension k and length n with a generator matrix $G = [I_k, P]$, where I_k is the $k \times k$ identity matrix. Let $\underline{u} \in \mathbb{F}_2^k$ be the information vector and $\underline{c} \in \mathbb{F}_2^n$ be the associated codeword. We have the encoding function as follows:

$$\underline{u} \longrightarrow \underline{c} = \underline{u}G. \tag{4}$$

Considering the binary phase shift keying (BPSK) mapping, we have $\underline{c} \longrightarrow \underline{s}$ by $s_t = 1 - 2c_t$ for $0 \le t \le n-1$. Suppose that $\underline{s}$ is transmitted over an AWGN channel. Let $\underline{y} = \underline{s} + \underline{z}$ be the received vector, where $\underline{z}$ is a vector of independent Gaussian random variables with zero mean and variance σ^2. We have the decoding function as follows:

$$\underline{y} \longrightarrow \hat{\underline{u}}. \tag{5}$$

Without loss of generality, assume that the all-zero codeword $\underline{c}^{(0)}$ is transmitted. The conventional union bound and four reported upper bounds based on GFBT are also reviewed in the following sections.

2.1. Union Bound. The simplest upper bound on the bit error probability is the union bound:

$$\begin{aligned}\Pr\{E_b\} &\le \sum_{i=1}^{k} \frac{i}{k} \sum_{d=1}^{n} \Pr\{E_b^{i,d}\} \\ &\le \sum_{i=1}^{k} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right),\end{aligned} \tag{6}$$

where $E_b^{i,d}$ is the event that there exists at least one codeword of Hamming weight d encoded by information bits of Hamming weight i that is nearer than $\underline{c}^{(0)}$ to $\underline{y}$, and $Q(\sqrt{d}/\sigma)$ is the pairwise error probability with

$$Q(x) \triangleq \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)} \mathrm{d}z. \tag{7}$$

However, the above conventional union bound is loose and even diverges (≥ 1) in the low-SNR region. Then, the improved upper bounds on the bit error probability based on GFBT were proposed, such as the Divsalar bound [2], the tangential-sphere bound (TSB) [3], the improved tangential-sphere bound (ITSB) [4], and the Ma bound [16].

2.2. The Divsalar Bound. In 1999, Divsalar derived a simple upper bound [2] on the bit error probability based on GFBT, where the region $\mathscr{R}$ is chosen to be an n-dimensional sphere centered at a scaled transmitted signal vector. Both the radius and the center of the sphere can be optimized. Let $d_{\min}$ denote the minimum Hamming weight. Taking into account the definition of A_d^* in (3), we have the Divsalar bound on the bit error probability:

$$P_b \le \sum_{d=d_{\min}}^{n-k+1} \min\left\{e^{-nE_b(\delta,\beta,\gamma)}, A_d^* Q\left(\sqrt{2d\gamma}\right)\right\}, \tag{8}$$

where

$$E_b(\delta,\beta,\gamma) = -r_b(\delta) + \frac{1}{2}\ln\left(\beta + (1-\beta)e^{2r_b(\delta)}\right) + \frac{\beta\gamma\delta}{1-(1-\beta)\delta},$$

$$\gamma = \frac{1}{2\sigma^2},$$

$$\delta = \frac{d}{n},$$

$$r_b(\delta) = \frac{\ln A_d^*}{n},$$

$$\beta = \sqrt{\frac{\gamma(1-\delta)}{\delta}\frac{2}{1-e^{-2r_b(\delta)}} + \left(\frac{1-\delta}{\delta}\right)^2\left[(1+\gamma)^2-1\right]} - \frac{1-\delta}{\delta}(1+\gamma). \tag{9}$$

2.3. The Tangential-Sphere Bound. In 2000, Sason and Shamai [3] derived the tangential-sphere bound on the bit error probability based on GFBT, where the region $\mathscr{R}$ is chosen to be an n-dimensional circular cone whose central line passes through the origin O and the transmitted signal. Let

$$\overline{\gamma}(a,x) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}\mathrm{d}t, \quad a>0,\ x\geq 0, \tag{10}$$

denote the normalized incomplete gamma function. Taking into account the definition of A_d^* in (3), we have the TSB with a parameter r on the bit error probability:

$$P_b = \int_{-\infty}^{+\infty} P_b(z_1)\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(z_1^2/2\sigma^2\right)}\mathrm{d}z_1, \tag{11}$$

where

$$P_b(z_1) \leq \sum_{d:\,(\delta_d/2)<\alpha_d}\left\{A_d^*\int_{\beta_d(z_1)}^{r_{z_1}}\frac{e^{-\left(z_2^2/2\sigma^2\right)}}{\sqrt{2\pi}\sigma}\overline{\gamma}\left(\frac{n-2}{2},\frac{r_{z_1}^2-z_2^2}{2\sigma^2}\right)\mathrm{d}z_2\right\} + 1 - \overline{\gamma}\left(\frac{n-1}{2},\frac{r_{z_1}^2}{2\sigma^2}\right),$$

$$\delta_d = 2\sqrt{d},$$

$$\alpha_d = r\sqrt{1-\frac{\delta_d^2}{4n}},$$

$$r_{z_1} = r\left(1-\frac{z_1}{\sqrt{n}}\right),$$

$$\beta_d(z_1) = \frac{r_{z_1}\sqrt{d}}{r\sqrt{1-(d/n)}}. \tag{12}$$

The parameter r in the TSB can be optimized by a numerical solution of the following equation:

$$\sum_{d:\,(\delta_d/2)<\alpha_d} A_d^*\int_0^{\theta_d}\sin^{n-3}\phi\,\mathrm{d}\phi = \frac{\sqrt{\pi}\Gamma(n-2/2)}{\Gamma(n-1/2)}, \tag{13}$$

where

$$\theta_d = \arccos\left(\frac{\delta_d}{2r\sqrt{1-\left(\delta_d^2/4n\right)}}\right). \tag{14}$$

2.4. The Improved Tangential-Sphere Bound. In 2001, Zangl and Herzog [4] derived the improved tangential-sphere bound on the bit error probability based on GFBT by computing the expression $\Pr\{\underline{y}\notin\mathscr{R}\}$ in a more accurate way. We have the ITSB with a parameter r_Ψ on the bit error probability:

$$P_b \leq \int_{-\infty}^{+\infty}\left\{\sum_{d:(\delta_d/2)<\alpha_d}\left\{A_d\frac{\overline{g}_d}{k}\overline{\gamma}\left(\frac{n-2}{2},\frac{r_{z_1}^2-\beta_d^2(z_1)}{2\sigma^2}\right)\cdot\left(Q\left(\frac{\beta_d(z_1)}{\sigma}\right)-Q\left(\frac{r_{z_1}}{\sigma}\right)\right)\right\} + \frac{\max\left(\overline{g}_0,\ldots,\overline{g}_{\mathrm{opt}}\right)}{k}\cdot\overline{\gamma}\left(\frac{n-1}{2},\frac{\beta_{d_{\mathrm{opt}}+1}^2(z_1)}{2\sigma^2}\right)\right.$$
$$\left.\cdot\left(\sum_{d=d_{\mathrm{opt}}+1}^{n}\frac{\max(\overline{g}_0,\ldots,\overline{g}_{d-1})-\max(\overline{g}_0,\ldots,\overline{g}_d)}{k}\cdot\overline{\gamma}\left(\frac{n-1}{2},\frac{r_{z_1}^2}{2\sigma^2}\right)\right)+\frac{\max(\overline{g}_0,\ldots,\overline{g}_n)}{k}\right\}\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\left(z_1^2/2\sigma^2\right)}\mathrm{d}z_1. \tag{15}$$

The parameter r_Ψ in the ITSB can be optimized by a numerical solution of the following equation:

$$\sum_{d:\ (\delta_d/2)<\alpha_d} A_d \frac{\overline{g}_d}{k} \int_0^{\theta_d} \sin^{n-3}\phi \, d\phi = \frac{\max\left(\overline{g}_0, \ldots, \overline{g}_{\text{opt}}\right)}{k} \cdot \frac{\sqrt{\pi}\Gamma(n-2/2)}{\Gamma(n-1/2)}, \tag{16}$$

where

$$\theta_d = \arccos\left(\frac{\delta_d}{2r_\Psi\sqrt{1-\left(\delta_d^2/4n\right)}}\right). \tag{17}$$

2.5. The Ma Bound. In 2018, Ma et al. [16] derived the upper bound on the bit error probability under maximum *a posteriori* (MAP) decoding. The Ma bound can be evaluated by calculating partial IOWEF with truncated information weight. We have the Ma bound with a parameter r^* on the bit error probability

$$\text{BER}_{\text{MAP}} \le \min_{0\le r^*\le T/2}\left\{\sum_{i\le 2r^*}\frac{i}{k}\left(\sum_d A_{i,d} Q\frac{\sqrt{d}}{\sigma}\right) + \sum_{i=r^*+1}^{k}\frac{\min\{i+r^*,k\}}{k}\binom{k}{i}p_b^i(1-p_b)^{k-i}\right\}, \tag{18}$$

where

$$p_b = Q\left(\frac{1}{\sigma}\right). \tag{19}$$

3. Upper Bound on the Bit Error Probability Based on GFBT

3.1. The Gallager Region $\mathscr{R}$. We define the region $\mathscr{R}$ by the Hamming distance based on a list decoding algorithm which is shown in Figure 1, resulting in an irregular high-dimensional geometry (Algorithm 1).

The list decoding algorithm is similar to but different from the algorithm presented in [14]. The *list region* in [14] is an n-dimensional Hamming sphere with center at the hard decision of the whole received sequence, while the list region here is a k-dimensional Hamming sphere with center at the hard decision of the information part of the received sequence.

The Gallager region $\mathscr{R}$ can be defined by

$$\mathscr{R} \triangleq \left\{\underline{y} \,\middle|\, \overset{(0)}{\underline{u}} \in \mathscr{L}_{\underline{y}}\right\} = \left\{\underline{y} \,\middle|\, W_H\left(\hat{\underline{y}}_0^{k-1}\right) \le r^*\right\}. \tag{20}$$

3.2. Upper Bound on the Bit Error Probability via IOWEF. We assume that $A_{i,d} \ge 1$ and denote all the code words of Hamming weight d encoded by information bits of Hamming weight i by $\underline{c}^{(\ell)}$, $1 \le \ell \le A_{i,d}$. Let $E_{0\longrightarrow\ell}$ be the event that $\underline{c}^{(\ell)}$ is nearer than $\underline{c}^{(0)}$ to $\underline{y}$.

With the framework of GFBT, we have

$$\Pr\{E_b\} = \Pr\{E_b, \underline{y} \in \mathscr{R}\} + \Pr\{E_b, \underline{y} \notin \mathscr{R}\}. \tag{21}$$

As shown in Figure 1(b), we have

$$\Pr\{E_b, \underline{y} \in \mathscr{R}\} \le \sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}\Pr\{E_b^{i,d}, \underline{y} \in \mathscr{R}\}, \tag{22}$$

$$\le \sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}A_{i,d}\Pr\{E_{0\longrightarrow 1}, \underline{y} \in \mathscr{R}\}. \tag{23}$$

As shown in Figure 1(a), we have

$$\Pr\{E_b, \underline{y} \notin \mathscr{R}\} \le \sum_{i=r^*+1}^{k}\frac{\min\{i+r^*,k\}}{k}\binom{k}{i}p_b^i(1-p_b)^{k-i}, \tag{24}$$

which means that the decoder outputs at most $i + r^*$ erroneous bits.

Assuming a binary vector of total length N_t passes through a BSC with cross error probability p, we denote $B(p, N_t, N_\ell, N_u)$ to be the probability that the number of bit errors occurring ranges from N_ℓ to N_u, that is,

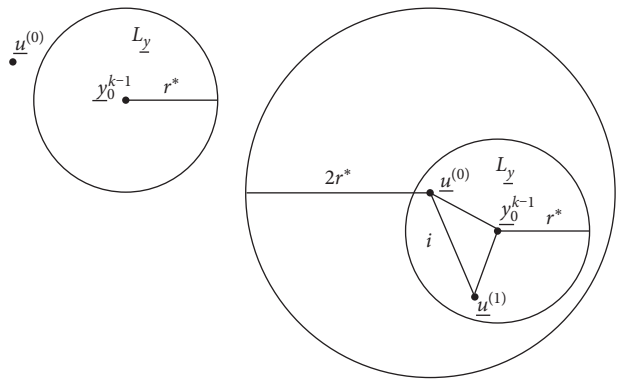
$$B(p, N_t, N_\ell, N_u) \triangleq \sum_{m=N_\ell}^{N_u}\binom{N_t}{m}p^m(1-p)^{N_t-m}. \tag{25}$$

Then, we define

$$C(r, p, N_t, N_\ell, N_u) \triangleq \sum_{m=N_\ell}^{N_u}\frac{\min\{m+r, N_t\}}{N_t}\binom{N_t}{m}p^m(1-p)^{N_t-m}. \tag{26}$$

Theorem 1. *We have the upper bound on the bit error probability of systematic binary linear codes under MAP decoding*

$$\Pr\{E_b\} \le \min_{0\le r^*\le k}\left\{\sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b, k-i, 0, \lfloor r-\frac{i}{2}\rfloor\right) + C\left(r^*, p_b, k, r^*+1, k\right)\right\}. \tag{27}$$



(a) (b)

Figure 1: Graphical illustrations of the decoding error events. (a) The error event that the all-zero sequence $\underline{u}^{(0)}$ is not in the list. (b) The error event that the all-zero sequence $\underline{u}^{(0)}$ is in the list but not the closest one.

(1) We denote
$$\underline{y} \triangleq (\underline{y}_0^{k-1}\,\underline{y}_k^{n-1})$$
in which
$$\underline{y}_0^{k-1} = \underbrace{(y_0 \cdots y_{k-1})}_{k}$$
and
$$\underline{y}_k^{n-1} = \underbrace{(y_k \cdots y_{n-1})}_{n-k}$$
Make hard decisions on the information part $\underline{y}_0^{k-1} = \underbrace{(y_0 \cdots y_{k-1})}_{k}$ of the received vector $\underline{y}$, resulting in a vector $\hat{\underline{y}}_0^{k-1} = \underbrace{(\hat{y}_0 \cdots \hat{y}_{k-1})}_{k}$ of length k. Then, the channel becomes a memoryless binary symmetric channel (BSC) with cross probability
$$p_b \triangleq Q(1/\sigma)$$
(2) List all sequences of length k within the Hamming sphere with center at $\hat{\underline{y}}_0^{k-1}$ of radius r^*, where r^* is a positive integer. The resulting list is denoted as $\mathscr{L}_{\underline{y}}$.
(3) Encode each sequence in $\mathscr{L}_{\underline{y}}$ by the encoding algorithm of the systematic code, resulting in a list of code words, denoted as $\mathscr{L}_{\underline{c}}$.
(4) Find the codeword $\underline{c}^* \in \mathscr{L}_{\underline{c}}$ that is closest to $\underline{y}$. Output the information part $\hat{\underline{u}}$ of $\underline{c}^*$ as the decoding result.

Algorithm 1: A list decoding algorithm.

Proof. Without loss of generality, we denote

$$\begin{aligned} \underline{c}^{(1)} &\triangleq \left(\underbrace{1\cdots1}_{i}\underbrace{0\cdots0}_{k-i}\underbrace{1\cdots1}_{d-i}\underbrace{0\cdots0}_{n-k-d+i}\right), \\ \hat{\underline{y}}_0^{k-1} &\triangleq \left(\underbrace{1\cdots1}_{i_1}\underbrace{0\cdots0}_{i-i_1}\underbrace{1\cdots1}_{i_2}\underbrace{0\cdots0}_{k-i-i_2}\right). \end{aligned} \tag{28}$$

Notice that a necessary condition for the event $E_{0\longrightarrow1}$ is that the corresponding input information sequence of the codeword $\underline{c}^{(1)}$ is in the list $\mathscr{L}_{\underline{y}}$. Hence,

$$W_H\left(\underline{c}_0^{(1)k-1} - \hat{\underline{y}}_0^{k-1}\right) \le r^*. \tag{29}$$

We have

$$i - i_1 + i_2 \le r^*. \tag{30}$$

Also notice that, for $\underline{y} \in \mathscr{R}$,

$$W_H\left(\hat{\underline{y}}_0^{k-1}\right) \le r^*. \tag{31}$$

We have

$$i_1 + i_2 \le r^*. \tag{32}$$

By combining (30) and (32), we can verify that

$$i_2 \le \lfloor r^* - \frac{i}{2} \rfloor. \tag{33}$$

By the union bounds, we have

$$\begin{aligned} &\Pr\left\{E_{0\longrightarrow1}, \underline{y} \in \mathscr{R}\right\} \\ &\quad = \Pr\left\{E_{0\longrightarrow1}, W_H\left(\hat{\underline{y}}_0^{k-1}\right) \le r^*\right\} \\ &\quad \le \sum_{i_2=1}^{r^*} \Pr\left\{E_{0\longrightarrow1}, W_H\left(\hat{\underline{y}}_0^{i-1}\right) \le r^* - i_2, W_H\left(\hat{\underline{y}}_i^{k-1}\right) = i_2\right\} \\ &\quad \le \sum_{i_2=1}^{r^*} \Pr\left\{E_{0\longrightarrow1}, W_H\left(\hat{\underline{y}}_i^{k-1}\right) = i_2\right\} \\ &\quad = \sum_{i_2=1}^{\lfloor r^* - (i/2) \rfloor} \Pr\left\{E_{0\longrightarrow1}, W_H\left(\hat{\underline{y}}_i^{k-1}\right) = i_2\right\}, \end{aligned} \tag{34}$$

for $i_2 \le r^* - (i/2)$ from (33).

Since the event $\Pr\{E_{0\longrightarrow1}\}$ is independent of $\hat{\underline{y}}_i^{k-1}$ and $\Pr\{E_{0\longrightarrow1}\} = Q(\sqrt{d}/\sigma)$, we have

$$\sum_{i_2=1}^{(r^*-i/2)} \Pr\left\{E_{0\longrightarrow 1}, W_H\left(\hat{y}_i^{k-1}\right) = i_2\right\}, \tag{35}$$

$$= \sum_{i_2=1}^{(r^*-i/2)} \Pr\{E_{0\longrightarrow 1}\}\Pr\left\{W_H\left(\hat{y}_i^{k-1}\right) = i_2\right\}, \tag{36}$$

$$= \Pr\{E_{0\longrightarrow 1}\} \sum_{i_2=1}^{r^*-(i/2)} \Pr\left\{W_H\left(\hat{y}_i^{k-1}\right) = i_2\right\}, \tag{37}$$

$$= Q\left(\frac{\sqrt{d}}{\sigma}\right) B\left(p_b, k-i, 0, \lfloor r^* - \frac{i}{2}\rfloor\right). \tag{38}$$

Then, by substituting (38) in (23), we have

$$\Pr\left\{E_b, \underline{y} \in \mathscr{R}\right\} \le \sum_{i=1}^{2r^*} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right) B\left(p_b, k-i, 0, \lfloor r^* - \frac{i}{2}\rfloor\right). \tag{39}$$

Therefore, it can be verified by substituting (24) and (39) in (1) to complete the proof. □

Corollary 1. *The proposed upper bound on the bit error probability (Theorem 1) can improve the conventional union bound on the bit error probability.*

Proof. As to the proposed bound (Theorem 1), note that

$$C\left(r^*, p_b, k, r^* + 1, k\right) = 0, \tag{40}$$

by setting

$$r^* = k. \tag{41}$$

we have

$$\begin{aligned}\Pr\{E_b\} &\le \sum_{i=1}^{k} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right) B\left(p_b, k-i, 0, \lfloor r^* - \frac{i}{2}\rfloor\right) \\ &\le \sum_{i=1}^{k} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right).\end{aligned} \tag{42}$$

Since $B(p_b, k-i, 0, \lfloor r^* - (i/2)\rfloor) \le 1$, the proof is completed. □

Corollary 2. *The proposed upper bound on the bit error probability (Theorem 1) can improve the Ma bound on the bit error probability (18).*

Proof. Assuming that we know only partial IOWEF with truncated information weight $\left\{A_{i,d}, 0 \le i \le T, 0 \le d \le n - k + T\right\}$, the parameter r^* in the proposed bound (Theorem 1) is optimized in the interval $[0, \lfloor T/2\rfloor]$. Theorem 1 can be written as

$$\Pr\{E_b\} \le \min_{0\le r^* \le \lfloor T/2\rfloor} \left\{\sum_{i=1}^{2r^*} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right) B\left(p_b, k-i, 0, \lfloor r - \frac{i}{2}\rfloor\right) + C\left(r^*, p_b, k, r^* + 1, k\right)\right\}. \tag{43}$$

Since $B(p_b, k-i, 0, \lfloor r - (i/2)\rfloor)$ is the probability that the number of bit errors occurring in a binary vector of total length $k-i$, when passing through a BSC with cross error probability p_b, it ranges from 0 to $r - (i/2)$. Then, it can be verified by

$$B\left(p_b, k-i, 0, \lfloor r^* - \frac{i}{2}\rfloor\right) \le 1, \tag{44}$$

to complete the proof. □

The objective of this paper is to derive the upper bound on bit error probability with only knowing of the weight spectrum.

3.3. Upper Bound on the Bit Error Probability via Weight Spectra. In this section, we focus on how to derive the upper bound on the bit error probability via weight spectra. The IOWEF is usually not computable, but the weight spectra $\{A_d, 0 \le d \le n\}$ of the code are usually available, such as the BCH code [17]. Let $T \ge 0$ be a positive integer that is relatively small. Assuming that we know only the truncated IOWEF $\{A_{i,d}, 0 \le i \le T, 0 \le d \le n - k + T\}$ which can be obtained by using a brute-force method and the weight spectrum $\{A_d, 0 \le d \le n\}$.

Define

$$A'_d = A_d - \sum_{i=1}^{T} A_{i,d}, \tag{45}$$

for $0 \le d \le n$.

Then, we focus on how to obtain the upper bound on the bit error probability by using the IOWEF $\{A_{i,d}, 0 \le i \le T, 0 \le d \le n - k + T\}$ and the weight spectrum $\{A_d, 0 \le d \le n\}$. We derive the upper bound in the two following cases.

Case 1: if the radius of the Hamming sphere $r^* \in [0, \lfloor T/2\rfloor]$ in Figure 1, we can get the IOWEF $\{A_{i,d}, 0 \le i \le T, 0 \le d \le n - k + T\}$ by using a brute-force method, and we have

$$\Pr\{E_b\} \le \min_{0\le r^* \le \lfloor T/2\rfloor} \left\{\sum_{i=1}^{2r^*} \frac{i}{k} \sum_{d=1}^{n} A_{i,d} Q\left(\frac{\sqrt{d}}{\sigma}\right) B \cdot \left(p_b, k-i, 0, \lfloor r - \frac{i}{2}\rfloor\right) + C\left(r^*, p_b, k, r^* + 1, k\right)\right\}, \tag{46}$$

which can be verified by Theorem 1.

Case 2: if the radius of the Hamming sphere $r^* \in [\lfloor T/2\rfloor + 1, k]$ in Figure 1, we can derive the upper bound on the bit error probability by employing both the IOWEF $\{A_{i,d}, 0 \le i \le T, 0 \le d \le n - k + T\}$ and the weight spectrum $\{A_d, 0 \le d \le n\}$.

From Theorem 1, we have

$$\Pr\{E_b\} \le \min_{\lfloor T/2\rfloor+1\le r^*\le k}\left\{\sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r-\frac{i}{2}\rfloor\right) + C\left(r^*,p_b,k,r^*+1,k\right)\right\}, \tag{47}$$

in which

$$\begin{aligned}\sum_{i=1}^{2r^*}\frac{i}{k}\left(\sum_{d=1}^{n}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)\right)&\\ =\sum_{i=1}^{T}\sum_{d=1}^{n}\frac{i}{k}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)&\\ +\sum_{i=T+1}^{2r^*}\sum_{d=1}^{n}\frac{i}{k}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right).&\end{aligned} \tag{48}$$

Note that

$$\sum_{i=T+1}^{2r^*}\sum_{d=1}^{n}A_{i,d}=\sum_{i=T+1}^{2r^*}\sum_{d=i}^{2r^*}A_{i,d}+\sum_{i=T+1}^{2r^*}\sum_{d=2r^*+1}^{2r^*+n-k}A_{i,d}. \tag{49}$$

Firstly, it is easy to verify that the first term in the right-hand side (RHS) of (49)

$$\sum_{i=T+1}^{2r^*}\sum_{d=i}^{2r^*}A_{i,d}=\sum_{d=T+1}^{2r^*}\sum_{i=T+1}^{d}A_{i,d}. \tag{50}$$

Since

$$\sum_{i=T+1}^{d}A_{i,d}=A'_d, \tag{51}$$

for $d\in[T+1,2r^*]$ and i is obviously not greater than $\min\{d,k\}$, we have

$$\sum_{i=T+1}^{d}\frac{i}{k}A_{i,d}\le\frac{\min\{d,k\}}{k}A'_d. \tag{52}$$

Therefore,

$$\begin{aligned}\sum_{i=T+1}^{2r^*}\sum_{d=i}^{2r^*}\frac{i}{k}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)&\\ \le\sum_{d=T+1}^{2r^*}\frac{\min\{d,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right),&\end{aligned} \tag{53}$$

for

$$B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)\le 1. \tag{54}$$

Secondly, it is easy to verify that the second term in the RHS of (49)

$$\sum_{i=T+1}^{2r^*}\sum_{d=2r^*+1}^{2r^*+N-K}A_{i,d}\le\sum_{d=2r^*+1}^{2r^*+N-K}A'_d. \tag{55}$$

Since

$$\sum_{i=T+1}^{2r^*}A_{i,d}\le A'_d, \tag{56}$$

for $d\in[T+1,2r^*+n-k]$ and i is obviously not greater than $\min\{2r^*,k\}$, we have

$$\sum_{i=T+1}^{2r^*}\frac{i}{k}A_{i,d}\le\frac{\min\{2r^*,k\}}{k}A'_d. \tag{57}$$

Therefore,

$$\begin{aligned}\sum_{i=T+1}^{2r^*}\sum_{d=2r^*+1}^{2r^*+n-k}\frac{i}{k}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)&\\ \le\sum_{d=2r^*+1}^{2r^*+n-k}\frac{\min\{2r^*,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right),&\end{aligned} \tag{58}$$

for $B(p_b,k-i,0,\lfloor r^*-(i/2)\rfloor)\le 1$.

Then, we have

$$\begin{aligned}\sum_{i=T+1}^{2r^*}\sum_{d=1}^{n}\frac{i}{k}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r^*-\frac{i}{2}\rfloor\right)&\\ \le\sum_{d=T+1}^{2r^*}\frac{\min\{d,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right)+\sum_{d=2r^*+1}^{2r^*+n-k}\frac{\min\{2r^*,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right),&\end{aligned} \tag{59}$$

by combining (53) and (58) with (59).

Finally, we have

$$\begin{aligned}\Pr\{E_b\}\le\min_{\lfloor T/2\rfloor+1\le r^*\le k}\Bigg\{&\sum_{i=1}^{T}\frac{i}{k}\sum_{d=1}^{n}h(i,d,r^*)\\ &+\sum_{d=T+1}^{2r^*}\frac{\min\{d,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right)\\ &+\sum_{d=2r^*+1}^{2r^*+n-k}\frac{\min\{2r^*,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right)\\ &+C\left(r^*,p_b,k,r^*+1,k\right)\Bigg\},\end{aligned} \tag{60}$$

by combining (48) and (59) with (47).

Then, we have the following theorem.

Theorem 2. *We have the upper bound on the bit error probability of systematic binary linear codes via their weight spectra*

$$\Pr\{E_b\}\le\min\left\{\Pr\left\{E_{b_1}\right\},\Pr\left\{E_{b_2}\right\}\right\}, \tag{61}$$

where

$$\Pr\{E_{b_1}\} = \min_{0\le r^*\le\lfloor T/2\rfloor}\left\{\sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r-\frac{i}{2}\rfloor\right)+C(r^*,p_b,k,r^*+1,k)\right\},$$

$$\Pr\{E_{b_2}\} = \min_{\lfloor T/2\rfloor+1\le r^*\le k}\left\{\sum_{i=1}^{T}\frac{i}{k}\sum_{d=1}^{n}h(i,d,r^*)+\sum_{d=T+1}^{2r^*}\frac{\min\{d,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right)+\sum_{d=2r^*+1}^{2r^*+n-k}\frac{\min\{2r^*,k\}}{k}A'_dQ\left(\frac{\sqrt{d}}{\sigma}\right)+C(r^*,p_b,k,r^*+1,k)\right\}. \tag{62}$$

Proof. We can complete the theorem by combining (46) and (60). □

Corollary 3. *The proposed upper bound on the bit error probability (Theorem 2) can improve the proposed upper bound on the bit error probability (Theorem 1).*

Assuming that we know only the truncated IOWEF $\{A_{i,d}, 0\le i\le T, 0\le d\le n-k+T\}$ and the proof weight spectrum $\{A_d, 0\le d\le n\}$, Theorem 1 can be written as

$$\Pr\{E_b\} \le \min_{0\le r^*\le\lfloor T/2\rfloor}\left\{\sum_{i=1}^{2r^*}\frac{i}{k}\sum_{d=1}^{n}A_{i,d}Q\left(\frac{\sqrt{d}}{\sigma}\right)B\left(p_b,k-i,0,\lfloor r-\frac{i}{2}\rfloor\right)+C(r^*,p_b,k,r^*+1,k)\right\}, \tag{63}$$

implying that $\Pr\{E_b\}\le\Pr\{E_{b_1}\}$. Theorem 2 needs a minimization over $0\le r^*\le k$ if the optimal parameter r^* is in the interval $[0,\lfloor T/2\rfloor]$, and Theorem 2 is exactly Theorem 1; if r^* is in the interval $[\lfloor T/2\rfloor+1,k]$, Theorem 2 is tighter than Theorem 1. Therefore, we claim that Theorem 2 can improve Theorem 1 to complete the proof.

Corollary 4. *The proposed upper bound on the bit error probability (Theorem 2) can improve the Ma bound on the bit error probability (18).*

Proof. It can be verified by combining Corollaries 2 and 3. □

Remark. The proposed bound (Theorem 2) has a little higher computational loads than the conventional union bound. Firstly, the overhead is caused by recursively computing $B(\cdot,\cdot,\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot,\cdot,\cdot)$. The probability $B(\cdot,\cdot,\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot,\cdot,\cdot)$ is the summation with at most k summands, which are independent of the IOWEF and hence can be calculated and stored for use. Secondly, the overhead is caused by minimizing over r^* $(0\le r^*\le k)$. A brute-force method can be implemented by computing the bound for each r^*, which can be done recursively.

4. Numerical Examples

In this section, we need to point out that the weight spectra of the compared BCH codes can be found in [17]. For all the upper bounds on the bit error probability except the Ma bound, we need the whole IOWEF. Then, in this paper, the compared bounds are the Ma bound (18) and the proposed bound (61) in Theorem 2 on the bit error probability, which are also compared with the simulation results under the maximum-likelihood (ML) decoding.

Figures 2 and 3 show the comparisons between the upper bounds of BCH codes [127, 106] and [127, 113], respectively, which are also compared with the simulation results under ML decoding. A partial IOWEF $\{A_{i,d}, 0\le i\le T, 0\le d\le n-k+T\}$ with $T=8$ of the BCH codes [127, 106] and [127, 113] can be obtained by using a brute-force method, respectively. The computed IOWEF $\{A_{i,d}, 0\le i\le 8, 0\le d\le 29\}$ of the BCH code [127, 106] is given in Table 1. The Ma bound is obtained by this truncated IOWEF according to (18). The proposed upper bound is obtained by this truncated IOWEF and the weight spectrum according to (61) in Theorem 2 (note that (61) is different from [5], (23)) since Theorem 2 here is derived in a much more detailed way when the Hamming weight $d\in(T,2r^*]$). As pointed out in [18], multidimensional signal processing plays a very important role in effective data analytics and interpretation. In this paper, we tighten the upper bound by analysing the k-dimensional vector. We can see that the proposed bound is tighter than the Ma bound. We can also see that, for the same code length n, the higher the code rate is, the tighter the Ma bound is. The proposed bound is always tighter whatever the code rate is.

Figure 4 shows the comparisons between the upper bounds on the bit error probability of the BCH code [255, 239], which are also compared with the simulation results under ML decoding. A partial IOWEF $\{A_{i,d}, 0\le i\le T, 0\le d\le n-k+T\}$ with $T=5$ of the BCH code [255, 239] can be obtained by using a brute-force method. The Ma bound is obtained by this truncated IOWEF according to (18). The proposed upper bound is obtained by this truncated IOWEF and the weight spectrum according to (61). We can see that the proposed bound is tighter than the Ma bound. We can also see that, the proposed bound coincides nicely with the ML decoding results in the high-SNR region when we only

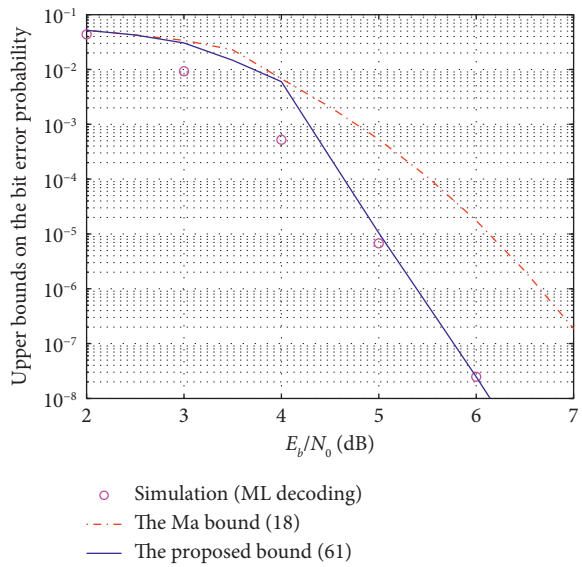


Figure 2: comparison between the proposed bound (Theorem 2) and the Ma bound on the bit error probability of the BCH code [127, 106], which is also compared with the simulation results under ML decoding.

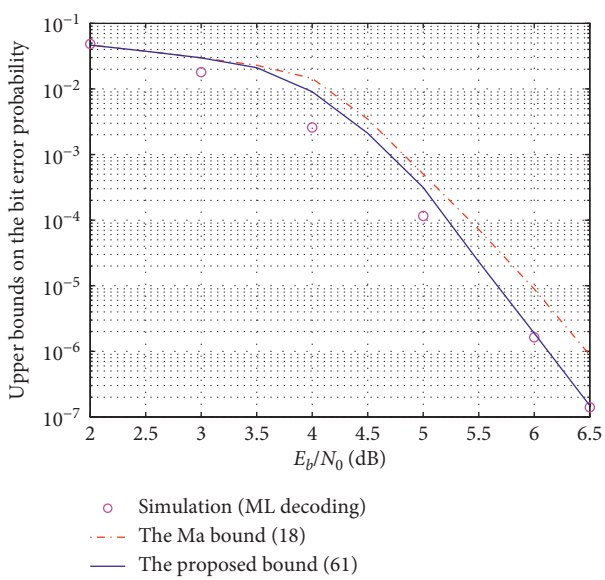


Figure 3: comparison between the proposed bound (Theorem 2) and the Ma bound on the bit error probability of the BCH code [127, 113], which is also compared with the simulation results under ML decoding.

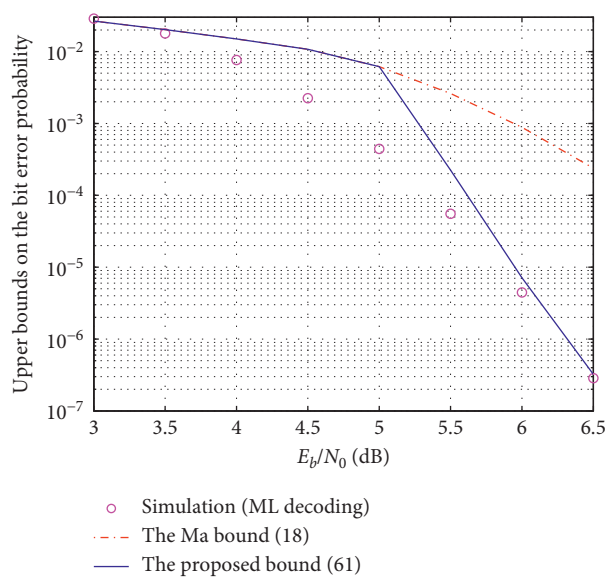


Figure 4: comparison between the proposed bound (Theorem 2) and the Ma bound on the bit error probability of the BCH code [255, 239], which is also compared with the simulation results under ML decoding.

know less IOWEF $\{A_{i,d}, 0 \le i \le 5, 0 \le d \le 21\}$ of the BCH code [255, 239].

Figure 5 shows the comparisons between the upper bounds on the bit error probability of the BCH code [31, 21], which are also compared with the simulation results under ML decoding. The whole IOWEF $A_{i,d}$, $0 \le i \le 21$, $0 \le d \le 31\}$ of the BCH code [31, 21] can be obtained by using a brute-force method. The Ma bound is obtained by the whole IOWEF according to (18), where $T = 21$. The proposed upper bound is obtained by the whole IOWEF and the weight spectrum according to (61). We can see that the proposed bound is tighter than the Ma bound in the low-SNR region when we know the whole IOWEF. We can also see that, the proposed bound and the Ma bound coincide nicely with the ML decoding results in the high-SNR region.

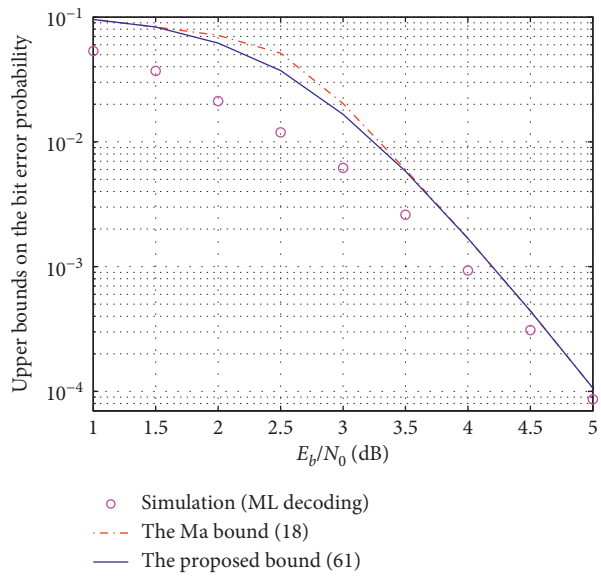


Figure 5: comparison between the proposed bound (Theorem 2) and the Ma bound on the bit error probability of the BCH code [31, 21], which is also compared with the simulation results under ML decoding.

Table 1: The partial IOWEF $\{A_{i,d},\ 0 \le i \le 8,\ 0 \le d \le 29\}$ with $T = 8$ of the BCH code [127, 106].

1	7	17	3	19	1847	5	20	2627123	7	15	$2.364896e+009$
1	8	8	3	20	584	5	21	984117	7	16	$3.415877e+009$
1	9	2	3	21	127	5	22	288687	7	17	$4.098554e+009$
1	10	4	3	22	21	5	23	64158	7	18	$4.098554e+009$
1	11	23	4	7	3470	5	24	10105	7	19	$3.415535e+009$
1	12	27	4	8	15952	5	25	1049	7	20	$2.364660e+009$
1	13	13	4	9	46995	5	26	59	7	21	1351255504
1	14	3	4	10	125279	6	7	19462	7	22	630548101
1	15	8	4	11	275791	6	8	194386	7	23	236462455
1	16	1	4	12	484126	6	9	1059704	7	24	69546072
2	7	71	4	13	697948	6	10	4752113	7	25	15457425
2	8	188	4	14	834510	6	11	16623036	7	26	2439979
2	9	305	4	15	834260	6	12	44331232	7	27	243848
2	10	472	4	16	696623	6	13	94527372	7	28	11614
2	11	711	4	17	482965	6	14	165420289	8	8	163532
2	12	1007	4	18	275130	6	15	239125721	8	9	2946230
2	13	964	4	19	128772	6	16	286915986	8	10	29469636
2	14	794	4	20	47734	6	17	286915986	8	11	192045523
2	15	511	4	21	14385	6	18	239077900	8	12	864339336
2	16	340	4	22	3172	6	19	165519571	8	13	$2.924072e+009$
2	17	135	4	23	513	6	20	94593564	8	14	$7.797383e+009$
2	18	48	4	24	37	6	21	44141017	8	15	$1.672354e+010$
2	19	12	4	25	2	6	22	16553366	8	16	$2.926719e+010$
2	20	7	5	7	11611	6	23	4864138	8	17	$4.226740e+010$
3	7	614	5	8	73116	6	24	1082409	8	18	$5.072263e+010$
3	8	2097	5	9	281110	6	25	170775	8	19	$5.072258e+010$
3	9	4816	5	10	960971	6	26	17090	8	20	$4.226969e+010$
3	10	10482	5	11	2633699	6	27	823	8	21	$2.926271e+010$
3	11	18637	5	12	5642851	7	7	13142	8	22	$1.672131e+010$
3	12	27236	5	13	9825582	7	8	276537	8	23	$7.803514e+009$
3	13	32706	5	14	14197834	7	9	2380926	8	24	$2.926377e+009$
3	14	32212	5	15	17043850	7	10	15092372	8	25	860707256
3	15	27027	5	16	17047259	7	11	69866721	8	26	191259366
3	16	18689	5	17	14202094	7	12	237457686	8	27	30195788
3	17	10826	5	18	9829514	7	13	630106871	8	28	3021987
3	18	5000	5	19	5614707	7	14	1350174932	8	29	143670

5. Conclusions

In this paper, upper bound on the bit error probability of systematic binary linear codes under MAP decoding is derived. The proposed bound just requires the weight spectra of the code, which is helpful when the whole IOWEF of the code is not available. The proposed bound (Theorem 2) is proved to be tighter than the recently proposed Ma bound. The numerical results show that the proposed bound on the bit error probability via weight spectra coincides nicely with the ML decoding results in the high-SNR region, which can predict the BER performance without resorting to computer simulations since the simulation is time-consuming in high-SNR region.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Scientific Research Planning Project of Education Department of Jilin Province (JJKH20200180KJ), the Doctoral Initiated Research Foundation Project (BSKJ201820, BSKJ201821, and BSKJ201822), the Science and Technology Planning Project of Jilin Province in 2020 named Research on Key Technologies of Wireless Wearable Health Monitoring Equipment, the Distinctive Innovation of Ordinary Universities of Guangdong Province (2018KTSCX120), and the PhD Start-up Fund of Natural Science Foundation of Guangdong Province (2016A030310335).

References

[1] I. Sason and S. Shamai, "Performance analysis of linear codes under maximum-likelihood decoding: a tutorial," in *Foundations and Trends in Communications and Information Theory*, vol. 3, pp. 1–225, , no. 1-2, NOW, Delft, Netherlands, 2006.

[2] D. Divsalar, "A simple tight bound on error probability of block codes with application to turbo codes," TMO Progress Report 42–139, pp. 1–35, JPL, Pasadena, CA, USA, 1999.

[3] I. Sason and S. Shamai, "Improved upper bounds on the ML decoding error probability of parallel and serial concatenated turbo codes via their ensemble distance spectrum," *IEEE Transactions on Information Theory*, vol. 46, no. 1, pp. 24–47, 2000.

[4] J. Zangl and R. Herzog, "Improved tangential sphere bound on the bit error probability of concatenated codes," *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 5, pp. 825–830, 2001.

[5] J. Liu, "Performance analysis of systematic linear codes over AWGN channels," in *Proceedings of the 2016 IEEE International Conference on RFID Technology and Applications*, pp. 133–137, Foshan, China, September 2016.

[6] E. R. Berlekamp, "The technology of error-correcting codes," *Proceedings of the IEEE*, vol. 68, no. 5, pp. 564–593, 1980.

[7] T. Kasami, T. Fujiwara, T. Takata, K. Tomita, and S. Lin, "Evaluation of the block error probability of block modulation codes by the maximum-likelihood decoding for an AWGN channel," in *Proceedings of the 15th Symposium on Information Theory and its Applications*, Minakami, Japan, September 1992.

[8] T. Kasami, T. Fujiwara, T. Takata, and S. Lin, "Evaluation of the block error probability of block modulation codes by the maximum-likelihood decoding for an AWGN channel," in *Proceedings of the 1993 IEEE International Symposium on Information Theory*, p. 68, San Antonio, TX, USA, January 1993.

[9] H. Herzberg and G. Poltyrev, "Techniques of bounding the probability of decoding error for block coded modulation structures," *IEEE Transactions on Information Theory*, vol. 40, no. 3, pp. 903–911, 1994.

[10] G. Poltyrev, "Bounds on the decoding error probability of binary linear codes via their spectra," *IEEE Transactions on Information Theory*, vol. 40, no. 4, pp. 1284–1292, 1994.

[11] D. Divsalar and E. Biglieri, "Upper bounds to error probabilities of coded systems beyond the cutoff rate," *IEEE Transactions on Communications*, vol. 51, no. 12, pp. 2011–2018, 2003.

[12] S. Yousefi and A. K. Khandani, "Generalized tangential sphere bound on the ML decoding error probability of linear binary block codes in AWGN interference," *IEEE Transactions on Information Theory*, vol. 50, no. 11, pp. 2810–2815, 2004.

[13] A. Mehrabian and S. Yousefi, "Improved tangential sphere bound on the ML decoding error probability of linear binary block codes in AWGN and block fading channels," *IEE Proceedings-Communications*, vol. 153, no. 6, pp. 885–893, 2006.

[14] X. Ma, J. Liu, and B. Bai, "New techniques for upper-bounding the ML decoding performance of binary linear codes," *IEEE Transactions on Communications*, vol. 61, no. 3, pp. 842–851, 2013.

[15] J. Liu, "The parameterized Gallager's first bounds based on conditional triplet-wise error probability," *Mathematics and Computers in Simulation*, vol. 163, pp. 32–46, 2019.

[16] X. Ma, K. Huang, and B. Bai, "Systematic block Markov superposition transmission of repetition codes,," *IEEE Transactions on Information Theory*, vol. 64, no. 3, pp. 1604–1620, 2018.

[17] The OEIS Foundation Inc., *List of Weight Distributions*, The OEIS Foundation Inc., NJ, USA, 2013, http://oeis.org/wiki/List_of_weigth_distributions.

[18] J. Ren, J. Han, and M. Dalla Mura, "Special issue on multimodal data fusion for multidimensional signal processing," *Multidimensional Systems and Signal Processing*, vol. 27, no. 4, pp. 801–805, 2016.