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Research Article

Global Dynamics of Some Exponential Type Systems

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We explore the boundedness and persistence, existence of an invariant rectangle, local dynamical properties about the unique positive fixed point, global dynamics by the discrete-time Lyapunov function, and the rate of convergence of some $(2,3)$-type exponential systems of difference equations. Finally, theoretical results are numerically verified.

1. Introduction

Recently, global dynamical properties of $(2,2)$ as well as $(2,3)$-type exponential difference equations or systems of exponential difference equations have widely been explored. In this regard, Ozturk et al. [1] have explored the global dynamical properties of the following $(2,2)$-type exponential difference equation:

$$x_{n+1} = \frac{\alpha_{10} + \alpha_{11} e^{-x_n}}{\alpha_{12} + x_{n-1}}, \tag{1}$$

where $\alpha_s\,(s = 10, 11, 12)$ and $x_s\,(s = -1, 0)$ are the positive real numbers. Equation (1) may be viewed as a model in mathematical biology where α_{10} is the immigration rate, α_{11} is the population growth rate, and α_{12} is the carrying capacity. Bozkurt [2] has explored the global dynamical properties of the following $(2,3)$-type exponential difference equation:

$$x_{n+1} = \frac{\alpha_{10} e^{-x_n} + \alpha_{11} e^{-x_{n-1}}}{\alpha_{12} + \alpha_{10} x_n + \alpha_{11} x_{n-1}}, \tag{2}$$

where $\alpha_s\,(s = 10, 11, 12)$ and $x_s\,(s = -1, 0)$ are the positive real numbers. More precisely, .Bozkurt [2] has explored the local asymptotic stability of the equilibrium point by linearized stability theorem, gasymptotic stability behavior by Lyapunov function, and semicycle analysis of positive solutions of the exponential difference equation, which is depicted in (2). Finally, theoretical results are verified numerically. Motivated from the aforementioned studies, here our purpose is to explore the global dynamical properties of the following $(2,3)$-type exponential systems, that is the extension of the work of Bozkurt [2]:

$$\begin{aligned}
x_{n+1} &= \frac{\alpha_{10} e^{-y_n} + \alpha_{11} e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10} y_n + \alpha_{11} y_{n-1}},\\
y_{n+1} &= \frac{\alpha_{13} e^{-z_n} + \alpha_{14} e^{-z_{n-1}}}{\alpha_{15} + \alpha_{13} z_n + \alpha_{14} z_{n-1}},\\
z_{n+1} &= \frac{\alpha_{16} e^{-x_n} + \alpha_{17} e^{-x_{n-1}}}{\alpha_{18} + \alpha_{16} x_n + \alpha_{17} x_{n-1}},
\end{aligned} \tag{3}$$

$$x_{n+1}=\frac{\alpha_{19}e^{-z_n}+\alpha_{20}e^{-z_{n-1}}}{\alpha_{21}+\alpha_{19}z_n+\alpha_{20}z_{n-1}},\quad y_{n+1}=\frac{\alpha_{22}e^{-x_n}+\alpha_{23}e^{-x_{n-1}}}{\alpha_{24}+\alpha_{22}x_n+\alpha_{23}x_{n-1}},\quad z_{n+1}=\frac{\alpha_{25}e^{-y_n}+\alpha_{26}e^{-y_{n-1}}}{\alpha_{27}+\alpha_{25}y_n+\alpha_{26}y_{n-1}},\tag{4}$$

$$x_{n+1}=\frac{\alpha_{28}e^{-x_n}+\alpha_{29}e^{-x_{n-1}}}{\alpha_{30}+\alpha_{28}x_n+\alpha_{29}x_{n-1}},\quad y_{n+1}=\frac{\alpha_{31}e^{-y_n}+\alpha_{32}e^{-y_{n-1}}}{\alpha_{33}+\alpha_{31}z_n+\alpha_{32}z_{n-1}},\quad z_{n+1}=\frac{\alpha_{34}e^{-z_n}+\alpha_{35}e^{-z_{n-1}}}{\alpha_{36}+\alpha_{34}y_n+\alpha_{35}y_{n-1}},\tag{5}$$

$$x_{n+1}=\frac{\alpha_{37}e^{-x_n}+\alpha_{38}e^{-x_{n-1}}}{\alpha_{39}+\alpha_{37}y_n+\alpha_{38}y_{n-1}},\quad y_{n+1}=\frac{\alpha_{40}e^{-y_n}+\alpha_{41}e^{-y_{n-1}}}{\alpha_{42}+\alpha_{40}x_n+\alpha_{41}x_{n-1}},\quad z_{n+1}=\frac{\alpha_{43}e^{-z_n}+\alpha_{44}e^{-z_{n-1}}}{\alpha_{45}+\alpha_{43}z_n+\alpha_{44}z_{n-1}},\tag{6}$$

$$x_{n+1}=\frac{\alpha_{46}e^{-x_n}+\alpha_{47}e^{-x_{n-1}}}{\alpha_{48}+\alpha_{46}z_n+\alpha_{47}z_{n-1}},\quad y_{n+1}=\frac{\alpha_{49}e^{-y_n}+\alpha_{50}e^{-y_{n-1}}}{\alpha_{51}+\alpha_{49}y_n+\alpha_{50}y_{n-1}},\quad z_{n+1}=\frac{\alpha_{52}e^{-z_n}+\alpha_{53}e^{-z_{n-1}}}{\alpha_{54}+\alpha_{52}x_n+\alpha_{53}x_{n-1}},\tag{7}$$

$$x_{n+1}=\frac{\alpha_{55}e^{-z_n}+\alpha_{56}e^{-z_{n-1}}}{\alpha_{57}+\alpha_{55}x_n+\alpha_{56}x_{n-1}},\quad y_{n+1}=\frac{\alpha_{58}e^{-x_n}+\alpha_{59}e^{-x_{n-1}}}{\alpha_{60}+\alpha_{58}y_n+\alpha_{59}y_{n-1}},\quad z_{n+1}=\frac{\alpha_{61}e^{-y_n}+\alpha_{62}e^{-y_{n-1}}}{\alpha_{63}+\alpha_{61}z_n+\alpha_{62}z_{n-1}},\tag{8}$$

where $\alpha_s\,(s=10,\ldots,63)$ and $x_s, y_s, z_s\,(s=-1,0)$ are the positive real numbers.

The rest of the paper is organized as follows: in Section 2, we explore that every positive solution of systems (3)–(8) is bounded and persists, whereas construction of an invariant rectangle is explored in Section 3. In Section 4, we explore the existence as well as uniqueness of the positive equilibrium point of systems (3)–(8). In Section 5, we explore the local dynamical properties about the unique positive equilibrium point of systems (3)–(8). In Section 6, we explore global dynamics about the positive equilibrium by the discrete-time Lyapunov function. We study the rate of convergence in Section 7, whereas discussion along with numerical simulations is presented in Section 8.

2. Boundedness and Persistence of Systems (3)–(8)

Theorem 1. *Every positive solution $\{\Omega_n\}_{n=-1}^{\infty}$ of systems (3)–(8) is bounded and persists.*

Proof. (i) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (3), then

$$x_n\le\frac{\alpha_{10}+\alpha_{11}}{\alpha_{12}}=U_1,\quad y_n\le\frac{\alpha_{13}+\alpha_{14}}{\alpha_{15}}=U_2,\quad z_n\le\frac{\alpha_{16}+\alpha_{17}}{\alpha_{18}}=U_3,\quad n=1,2,\ldots.\tag{9}$$

From (3) and (9), one gets

$$x_n\ge\frac{(\alpha_{10}+\alpha_{11})e^{-((\alpha_{13}+\alpha_{14})/\alpha_{15})}}{\alpha_{10}+(\alpha_{11}+\alpha_{12})((\alpha_{13}+\alpha_{14})/\alpha_{15})}=L_1,\quad y_n\ge\frac{(\alpha_{13}+\alpha_{14})e^{-((\alpha_{16}+\alpha_{17})/\alpha_{18})}}{\alpha_{15}+(\alpha_{13}+\alpha_{14})((\alpha_{16}+\alpha_{17})/\alpha_{18})}=L_2,\quad z_n\ge\frac{(\alpha_{16}+\alpha_{17})e^{-((\alpha_{10}+\alpha_{11})/\alpha_{12})}}{\alpha_{18}+(\alpha_{16}+\alpha_{17})((\alpha_{10}+\alpha_{11})/\alpha_{12})}=L_3,\quad n=2,3,\ldots.\tag{10}$$

So, from (9) and (10), one has

$$L_1\le x_n\le U_1,\quad L_2\le y_n\le U_2,\quad L_3\le z_n\le U_3,\quad n=3,4,\ldots.\tag{11}$$

(ii) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (4), then

$$x_n\le\frac{\alpha_{19}+\alpha_{20}}{\alpha_{21}}=U_4,\quad y_n\le\frac{\alpha_{22}+\alpha_{23}}{\alpha_{24}}=U_5,\quad z_n\le\frac{\alpha_{25}+\alpha_{26}}{\alpha_{27}}=U_6,\quad n=1,2,\ldots.\tag{12}$$

From (4) and (12), one gets

$$x_n \geq \frac{(\alpha_{19}+\alpha_{20})e^{-((\alpha_{25}+\alpha_{26})/\alpha_{27})}}{\alpha_{21}+(\alpha_{19}+\alpha_{20})((\alpha_{25}+\alpha_{26})/\alpha_{27})} = L_4,$$
$$y_n \geq \frac{(\alpha_{22}+\alpha_{23})e^{-((\alpha_{19}+\alpha_{20})/\alpha_{21})}}{\alpha_{24}+(\alpha_{22}+\alpha_{23})((\alpha_{19}+\alpha_{20})/\alpha_{21})} = L_5, \quad (13)$$
$$z_n \geq \frac{(\alpha_{25}+\alpha_{26})e^{-((\alpha_{22}+\alpha_{23})/\alpha_{24})}}{\alpha_{27}+(\alpha_{25}+\alpha_{26})((\alpha_{22}+\alpha_{23})/\alpha_{24})} = L_6,$$
$$n = 2, 3, \ldots.$$

So, from (12) and (13), one gets

$$L_4 \leq x_n \leq U_4,\quad L_5 \leq y_n \leq U_5,\quad L_6 \leq z_n \leq U_6,\quad n = 3, 4, \ldots. \quad (14)$$

(iii) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (5), then

$$x_n \leq \frac{\alpha_{28}+\alpha_{29}}{\alpha_{30}} = U_7,\quad y_n \leq \frac{\alpha_{31}+\alpha_{32}}{\alpha_{33}} = U_8,\quad z_n \leq \frac{\alpha_{34}+\alpha_{35}}{\alpha_{36}} = U_9,\quad n = 1, 2, \ldots. \quad (15)$$

From (5) and (15), one gets

$$x_n \geq \frac{(\alpha_{28}+\alpha_{29})e^{-((\alpha_{28}+\alpha_{29})/\alpha_{30})}}{\alpha_{30}+(\alpha_{28}+\alpha_{29})((\alpha_{28}+\alpha_{29})/\alpha_{30})} = L_7,$$
$$y_n \geq \frac{(\alpha_{31}+\alpha_{32})e^{-((\alpha_{31}+\alpha_{32})/\alpha_{33})}}{\alpha_{33}+(\alpha_{31}+\alpha_{32})((\alpha_{34}+\alpha_{35})/\alpha_{36})} = L_8, \quad (16)$$
$$z_n \geq \frac{(\alpha_{34}+\alpha_{35})e^{-((\alpha_{34}+\alpha_{35})/\alpha_{35})}}{\alpha_{36}+(\alpha_{34}+\alpha_{35})((\alpha_{31}+\alpha_{32})/\alpha_{33})} = L_9,$$
$$n = 2, 3, \ldots.$$

So, from (15) and (16), one gets

$$L_7 \leq x_n \leq U_7,\quad L_8 \leq y_n \leq U_8,\quad L_9 \leq z_n \leq U_9,\quad n = 3, 4, \ldots. \quad (17)$$

(iv) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (6), then

$$x_n \leq \frac{\alpha_{37}+\alpha_{38}}{\alpha_{39}} = U_{10},\quad y_n \leq \frac{\alpha_{40}+\alpha_{41}}{\alpha_{42}} = U_{11},\quad z_n \leq \frac{\alpha_{43}+\alpha_{44}}{\alpha_{45}} = U_{12},\quad n = 1, 2, \ldots. \quad (18)$$

From (6) and (18), one gets

$$x_n \geq \frac{(\alpha_{37}+\alpha_{38})e^{-((\alpha_{37}+\alpha_{38})/\alpha_{39})}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})((\alpha_{40}+\alpha_{41})/\alpha_{42})} = L_{10},$$
$$y_n \geq \frac{(\alpha_{40}+\alpha_{41})e^{-((\alpha_{40}+\alpha_{41})/\alpha_{42})}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})((\alpha_{37}+\alpha_{38})/\alpha_{39})} = L_{11}, \quad (19)$$
$$z_n \geq \frac{(\alpha_{43}+\alpha_{44})e^{-((\alpha_{43}+\alpha_{44})/\alpha_{45})}}{\alpha_{45}+(\alpha_{43}+\alpha_{44})((\alpha_{43}+\alpha_{44})/\alpha_{45})} = L_{12},$$
$$n = 2, 3, \ldots.$$

So, from (18) and (19), one gets

$$L_{10} \leq x_n \leq U_{10},\quad L_{11} \leq y_n \leq U_{11},\quad L_{12} \leq z_n \leq U_{12},\quad n = 3, 4, \ldots. \quad (20)$$

(v) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (7), then

$$x_n \leq \frac{\alpha_{46}+\alpha_{47}}{\alpha_{48}} = U_{13},\quad y_n \leq \frac{\alpha_{49}+\alpha_{50}}{\alpha_{51}} = U_{14},\quad z_n \leq \frac{\alpha_{52}+\alpha_{53}}{\alpha_{54}} = U_{15},\quad n = 1, 2, \ldots. \quad (21)$$

From (7) and (21), one gets

$$x_n \geq \frac{(\alpha_{46}+\alpha_{47})e^{-((\alpha_{46}+\alpha_{47})/\alpha_{48})}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})((\alpha_{52}+\alpha_{53})/\alpha_{54})} = L_{13},$$
$$y_n \geq \frac{(\alpha_{49}+\alpha_{50})e^{-((\alpha_{49}+\alpha_{50})/\alpha_{51})}}{\alpha_{51}+(\alpha_{49}+\alpha_{50})((\alpha_{49}+\alpha_{50})/\alpha_{51})} = L_{14}, \quad (22)$$
$$z_n \geq \frac{(\alpha_{52}+\alpha_{53})e^{-((\alpha_{52}+\alpha_{53})/\alpha_{54})}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})((\alpha_{46}+\alpha_{47})/\alpha_{48})} = L_{15},$$
$$n = 2, 3, \ldots.$$

So, from (21) and (22), one gets

$$L_{13} \leq x_n \leq U_{13}, \quad L_{14} \leq y_n \leq U_{14}, \quad L_{15} \leq z_n \leq U_{15}, \quad n = 3, 4, \ldots. \tag{23}$$

(vi) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (8), then

$$x_n \leq \frac{\alpha_{55} + \alpha_{56}}{\alpha_{57}} = U_{16}, \quad y_n \leq \frac{\alpha_{58} + \alpha_{59}}{\alpha_{60}} = U_{17}, \quad z_n \leq \frac{\alpha_{61} + \alpha_{62}}{\alpha_{63}} = U_{18}, \quad n = 1, 2, \ldots. \tag{24}$$

From (8) and (24), one gets

$$x_n \geq \frac{(\alpha_{55} + \alpha_{56})e^{-((\alpha_{61}+\alpha_{62})/\alpha_{63})}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})((\alpha_{55} + \alpha_{56})/\alpha_{57})} = L_{16},$$
$$y_n \geq \frac{(\alpha_{58} + \alpha_{59})e^{-((\alpha_{55}+\alpha_{56})/\alpha_{57})}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})((\alpha_{58} + \alpha_{59})/\alpha_{60})} = L_{17},$$
$$z_n \geq \frac{(\alpha_{61} + \alpha_{62})e^{-((\alpha_{58}+\alpha_{59})/\alpha_{60})}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})((\alpha_{61} + \alpha_{62})/\alpha_{63})} = L_{18}, \quad n = 2, 3, \ldots. \tag{25}$$

So, from (24) and (25), one gets

$$L_{16} \leq x_n \leq U_{16}, \quad L_{17} \leq y_n \leq U_{17}, \quad L_{18} \leq z_n \leq U_{18}, \quad n = 3, 4, \ldots. \tag{26}$$

□

3. Existence of Invariant Rectangle of Systems (3)–(8)

Theorem 2. *If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of systems (3)–(8), then their corresponding invariant rectangles, respectively, are $[L_1, U_1] \times [L_2, U_2] \times [L_3, U_3]$, $[L_4, U_4] \times [L_5, U_5] \times [L_6, U_6]$, $[L_7, U_7] \times [L_8, U_8] \times [L_9, U_9]$, $[L_{10}, U_{10}] \times [L_{11}, U_{11}] \times [L_{12}, U_{12}]$, $[L_{13}, U_{13}] \times [L_{14}, U_{14}] \times [L_{15}, U_{15}]$, and $[L_{16}, U_{16}] \times [L_{17}, U_{17}] \times [L_{18}, U_{18}]$.*

Proof. If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution with $x_0, x_{-1} \in [L_1, U_1]$, $y_0, y_{-1} \in [L_2, U_2]$, and $z_0, z_{-1} \in [L_3, U_3]$, then from (3), one has

$$x_1 = \frac{\alpha_{10}e^{-y_0} + \alpha_{11}e^{-y_{-1}}}{\alpha_{12} + \alpha_{10}y_0 + \alpha_{11}y_{-1}} \leq \frac{\alpha_{10} + \alpha_{11}}{\alpha_{12}},$$
$$x_1 = \frac{\alpha_{10}e^{-y_0} + \alpha_{11}e^{-y_{-1}}}{\alpha_{12} + \alpha_{10}y_0 + \alpha_{11}y_{-1}} \geq \frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13}+\alpha_{14})/\alpha_{15})}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})((\alpha_{13} + \alpha_{14})/\alpha_{15})},$$
$$y_1 = \frac{\alpha_{13}e^{-z_0} + \alpha_{14}e^{-z_{-1}}}{\alpha_{15} + \alpha_{13}z_0 + \alpha_{14}z_{-1}} \leq \frac{\alpha_{13} + \alpha_{14}}{\alpha_{15}},$$
$$y_1 = \frac{\alpha_{13}e^{-z_0} + \alpha_{14}e^{-z_{-1}}}{\alpha_{15} + \alpha_{13}z_0 + \alpha_{14}z_{-1}} \geq \frac{(\alpha_{13} + \alpha_{14})e^{-((\alpha_{16}+\alpha_{17})/\alpha_{18})}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})((\alpha_{16} + \alpha_{17})/\alpha_{18})},$$
$$z_1 = \frac{\alpha_{16}e^{-x_0} + \alpha_{17}e^{-x_{-1}}}{\alpha_{18} + \alpha_{16}x_0 + \alpha_{17}x_{-1}} \leq \frac{\alpha_{16} + \alpha_{17}}{\alpha_{18}},$$
$$z_1 = \frac{\alpha_{16}e^{-x_0} + \alpha_{17}e^{-x_{-1}}}{\alpha_{18} + \alpha_{16}x_0 + \alpha_{17}x_{-1}} \geq \frac{(\alpha_{16} + \alpha_{17})e^{-((\alpha_{10}+\alpha_{11})/\alpha_{12})}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})((\alpha_{10} + \alpha_{11})/\alpha_{12})}. \tag{27}$$

Hence (27) then implies that $x_1 \in [L_1, U_1]$, $y_1 \in [L_2, U_2]$, and $z_1 \in [L_3, U_3]$. Finally from (3), it is easy to establish that $x_{k+1} \in [L_1, U_1]$ (resp., $y_{k+1} \in [L_2, U_2]$ and $z_{k+1} \in [L_3, U_3]$) if $x_k \in [L_1, U_1]$ (resp., $y_k \in [L_2, U_2]$ and $z_k \in [L_3, U_3]$). □

Remark 1. In a similar way, one can prove the invariant rectangle for systems (4)–(8).

4. Existence as well as Uniqueness of Positive Fixed Point of Systems (3)–(8)

Existence as well as uniqueness of a positive fixed point of systems (3)–(8) is explored in this section, as follows.

Theorem 3.

(i) System (3) has a unique positive fixed point: $\Upsilon_1 = (\overline{x}, \overline{y}, \overline{z}) \in [L_1, U_1] \times [L_2, U_2] \times [L_3, U_3]$ if

$$\frac{(\alpha_{13} + \alpha_{14})e^{-L_1}((U_1 + 1)(\alpha_{13} + \alpha_{14}) + \alpha_{15})(\alpha_{10} + \alpha_{11})e^{-(((\alpha_{13}+\alpha_{14})e^{-U_1})/(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1))}}{(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)^2(\alpha_{18} + (\alpha_{16} + \alpha_{17}))^2}$$
$$\times \frac{((\alpha_{10} + \alpha_{11})((((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)) + 1) + \alpha_{12})}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)))^2}(\alpha_{16} + \alpha_{17})$$
$$\cdot e^{-((\alpha_{10}+\alpha_{11})e^{-(((\alpha_{13}+\alpha_{14})e^{-L_1})/(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1))}/(\alpha_{12}+(\alpha_{10}+\alpha_{11})(((\alpha_{13}+\alpha_{14})e^{-L_1})/(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1))))}$$
$$\times((\alpha_{16} + \alpha_{17})(\Lambda_1 + 1) + \alpha_{18}) < \Lambda_1^2, \tag{28}$$

where

$$\Lambda_1 = \frac{(\alpha_{10} + \alpha_{11})e^{-(((\alpha_{13}+\alpha_{14})e^{-U_3})/(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3))}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_3})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3))}; \tag{29}$$

(ii) System (4) has a unique positive fixed point: $\Upsilon_2 = (\overline{x}, \overline{y}, \overline{z}) \in [L_4, U_4] \times [L_5, U_5] \times [L_6, U_6]$ if

$$\begin{aligned} &\frac{(\alpha_{19} + \alpha_{20})e^{-L_6}((U_6 + 1)(\alpha_{19} + \alpha_{20}) + \alpha_{21})(\alpha_{22} + \alpha_{23})e^{-(((\alpha_{19}+\alpha_{20})e^{-U_6})/(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6))}}{(\alpha_{21} + (\alpha_{19} + \alpha_{20})U_6)^2} \\ &\quad\times \frac{((\alpha_{22} + \alpha_{23})((((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)) + 1) + \alpha_{24})(\alpha_{25} + \alpha_{26})e^{-(((\alpha_{22}+\alpha_{23})e^{-(((\alpha_{19}+\alpha_{20})e^{-L_6})/(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6))})/((\alpha_{24}+(\alpha_{22}+\alpha_{23}))(((\alpha_{19}+\alpha_{20})e^{-L_6})/(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6))))}}{(\alpha_{24} + (\alpha_{22} + \alpha_{23})(((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)))^2} \\ &\quad\times \frac{((\alpha_{25} + \alpha_{26})(\Lambda_2 + 1) + \alpha_{27})}{(\alpha_{27} + (\alpha_{25} + \alpha_{26}))^2} < \Lambda_2^2, \end{aligned} \tag{30}$$

where

$$\Lambda_2 = \frac{(\alpha_{22} + \alpha_{23})e^{-(((\alpha_{19}+\alpha_{20})e^{-U_6})/(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6))}}{\alpha_{24} + (\alpha_{22} + \alpha_{23})(((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6))}; \tag{31}$$

(iii) System (5) has a unique positive fixed point: $\Upsilon_3 = (\overline{x}, \overline{y}, \overline{z}) \in [L_7, U_7] \times [L_8, U_8] \times [L_9, U_9]$ if

$$\begin{aligned} &\frac{e^{-((e^{-U_8}/L_8)-(\alpha_{33}/(\alpha_{31}+\alpha_{32})))}e^{-L_8}(U_8 + 1)((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})) + 1)}{((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))^2 U_8^2} \\ &\quad\cdot e^{-((e^{-((e^{-L_8/U_8})-(\alpha_{33}/(\alpha_{31}+\alpha_{32})))}/(e^{-L_8}/L_8)-(\alpha_{33}/(\alpha_{31}+\alpha_{32})))-(\alpha_{36}/(\alpha_{34}+\alpha_{35})))} < \Lambda_3^2, \end{aligned} \tag{32}$$

where

$$\Lambda_3 = \frac{e^{-((e^{-U_8}/L_8)-(\alpha_{33}/(\alpha_{31}+\alpha_{32})))}}{((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))} - \frac{\alpha_{36}}{\alpha_{34} + \alpha_{35}}; \tag{33}$$

(iv) System (6) has a unique positive fixed point: $\Upsilon_4 = (\overline{x}, \overline{y}, \overline{z}) \in [L_{10}, U_{10}] \times [L_{11}, U_{11}] \times [L_{12}, U_{12}]$ if

$$\begin{aligned} &\frac{e^{-((e^{-U_{10}}/L_{10})-(\alpha_{39}/(\alpha_{37}+\alpha_{38})))}e^{-L_{10}}(U_{10} + 1)((e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})) + 1)}{((e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})))^2 U_{10}^2} \\ &\quad\cdot e^{-((e^{-((e^{-L_{10}/U_{10}})-(\alpha_{39}/(\alpha_{37}+\alpha_{38})))}/(e^{-L_{10}}/L_{10})-(\alpha_{39}/(\alpha_{37}+\alpha_{38})))-(\alpha_{42}/(\alpha_{40}+\alpha_{41})))} < \Lambda_4^2, \end{aligned} \tag{34}$$

where

$$\Lambda_4 = \frac{e^{-\left(\left(e^{-U_{10}}/L_{10}\right)-\left(\alpha_{39}/\left(\alpha_{37}+\alpha_{38}\right)\right)\right)}}{\left(\left(e^{-L_{10}}/L_{10}\right)-\left(\alpha_{39}/\left(\alpha_{37}+\alpha_{38}\right)\right)\right)} - \frac{\alpha_{42}}{\alpha_{40}+\alpha_{41}}; \tag{35}$$

(v) System (7) has a unique positive fixed point: $\Upsilon_5 = (\overline{x}, \overline{y}, \overline{z}) \in [L_{13}, U_{13}] \times [L_{14}, U_{14}] \times [L_{15}, U_{15}]$ *if*

$$\begin{aligned}&\frac{e^{-\left(\left(e^{-U_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)}e^{-L_{13}}\left(U_{13}+1\right)\left(\left(e^{-L_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)+1\right)}{\left(\left(e^{-L_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)^2 U_{13}^2}\\ &\quad \cdot e^{-\left(\left(e^{-\left(\left(e^{-L_{13}}/U_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)}/\left(e^{-L_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)-\left(\alpha_{54}/\left(\alpha_{52}+\alpha_{53}\right)\right)\right)} < \Lambda_5^2,\end{aligned} \tag{36}$$

where

$$\Lambda_5 = \frac{e^{-\left(\left(e^{-U_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)}}{\left(\left(\left(e^{-L_{13}}/L_{13}\right)-\left(\alpha_{48}/\left(\alpha_{46}+\alpha_{47}\right)\right)\right)\right)} - \frac{\alpha_{54}}{\alpha_{52}+\alpha_{53}}; \tag{37}$$

(vi) System (8) has a unique positive fixed point: $\Upsilon_6 = (\overline{x}, \overline{y}, \overline{z}) \in [L_{16}, U_{16}] \times [L_{17}, U_{17}] \times [L_{18}, U_{18}]$ *if*

$$\begin{aligned}&\frac{\left(\alpha_{57}+2U_{16}\left(\alpha_{55}+\alpha_{56}\right)\right)\left(\alpha_{63}+2U_{16}\left(\alpha_{61}+\alpha_{62}\right)\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)\right)}{\left(\alpha_{63}\ln\left(\alpha_{55}+\alpha_{56}\right)/L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)+\left(\alpha_{61}+\alpha_{62}\right)\left(\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)\right)^2\right)}\\ &\quad\times\left(\frac{\alpha_{57}+2U_{16}\left(\alpha_{55}+\alpha_{56}\right)\left(\alpha_{63}+2\left(\alpha_{61}+\alpha_{62}\right)\right)\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)}{\alpha_{63}\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)}+\Lambda_6\right)\\ &\quad\times\frac{\alpha_{60}\left(\alpha_{58}+\alpha_{59}\right)}{L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\left(\alpha_{60}+\left(\alpha_{58}+\alpha_{59}\right)\right)} < \Lambda_6,\end{aligned} \tag{38}$$

where

$$\Lambda_6 = \ln\left(\frac{\alpha_{61}+\alpha_{62}}{\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)\left(\alpha_{63}+\left(\alpha_{61}+\alpha_{62}\right)\ln\left(\left(\alpha_{55}+\alpha_{56}\right)/\left(L_{16}\left(\alpha_{57}+\left(\alpha_{55}+\alpha_{56}\right)L_{16}\right)\right)\right)\right)}\right). \tag{39}$$

Proof. (i) From (3), one has

$$\begin{aligned}x &= \frac{\left(\alpha_{10}+\alpha_{11}\right)e^{-y}}{\alpha_{12}+\left(\alpha_{10}+\alpha_{11}\right)y},\\ y &= \frac{\left(\alpha_{13}+\alpha_{14}\right)e^{-z}}{\alpha_{15}+\left(\alpha_{13}+\alpha_{14}\right)z},\\ z &= \frac{\left(\alpha_{16}+\alpha_{17}\right)e^{-x}}{\alpha_{18}+\left(\alpha_{16}+\alpha_{17}\right)x}.\end{aligned} \tag{40}$$

From (40), one gets

$$\begin{aligned}z &= \frac{\left(\alpha_{16}+\alpha_{17}\right)e^{-x}}{\alpha_{18}+\left(\alpha_{16}+\alpha_{17}\right)x},\\ x &= \frac{\left(\alpha_{10}+\alpha_{11}\right)e^{-y}}{\alpha_{12}+\left(\alpha_{10}+\alpha_{11}\right)y},\\ y &= \frac{\left(\alpha_{13}+\alpha_{14}\right)e^{-z}}{\alpha_{15}+\left(\alpha_{13}+\alpha_{14}\right)z}.\end{aligned} \tag{41}$$

From (41), setting

$$\begin{aligned}y = g(z) &= \frac{\left(\alpha_{13}+\alpha_{14}\right)e^{-z}}{\alpha_{15}+\left(\alpha_{13}+\alpha_{14}\right)z},\\ x = f(y) &= \frac{\left(\alpha_{10}+\alpha_{11}\right)e^{-y}}{\alpha_{12}+\left(\alpha_{10}+\alpha_{11}\right)y}.\end{aligned} \tag{42}$$

Denote

$$F(z) := \frac{(\alpha_{16}+\alpha_{17})e^{-k(z)}}{\alpha_{18}+(\alpha_{16}+\alpha_{17})k(z)} - z, \tag{43}$$

where

$$\begin{aligned} k(z) &:= x = f(g(z)), \\ &= f\left(\frac{(\alpha_{13}+\alpha_{14})e^{-z}}{\alpha_{15}+(\alpha_{13}+\alpha_{14})z}\right), \\ &= \frac{(\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}, \end{aligned} \tag{44}$$

and $z \in [L_3, U_3]$. Here, our finding is that $F(z) = 0$ has a unique solution, where $z \in [L_3, U_3]$. From (43) and (44), one gets

$$F'(z) = -\frac{k'(z)(\alpha_{16}+\alpha_{17})e^{-k(z)}\left((k(z)+1)(\alpha_{16}+\alpha_{17})+\alpha_{18}\right)}{\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})k(z)\right)^2} - 1, \tag{45}$$

where

$$\begin{aligned} k'(z) = &-\frac{(\alpha_{13}+\alpha_{14})e^{-z}\left((z+1)(\alpha_{13}+\alpha_{14})+\alpha_{15}\right)(\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}}{\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)^2\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)\right)^2} \\ &\times\left((\alpha_{10}+\alpha_{11})\left(\frac{(\alpha_{13}+\alpha_{14})e^{-z}}{\alpha_{15}+(\alpha_{13}+\alpha_{14})z}+1\right)+\alpha_{12}\right). \end{aligned} \tag{46}$$

Using (44) and (46) in (45), we obtain

$$\begin{aligned} F'(z) = &\frac{(\alpha_{13}+\alpha_{14})e^{-z}\left((z+1)(\alpha_{13}+\alpha_{14})+\alpha_{15}\right)(\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}}{\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)^2\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})\left(\left((\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}\right)/\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)\right)\right)\right)^2}\times \\ &\frac{\left((\alpha_{10}+\alpha_{11})\left(\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)+1\right)+\alpha_{12}\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)\right)^2}(\alpha_{16}+\alpha_{17})e^{-\left(\left((\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}\right)/\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)\right)\right)}\times \\ &\cdot\left((\alpha_{16}+\alpha_{17})\left(\frac{(\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-z}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})z\right)\right)}+1\right)+\alpha_{18}\right)-1 \\ \le &\frac{(\alpha_{13}+\alpha_{14})e^{-L_1}\left((U_1+1)(\alpha_{13}+\alpha_{14})+\alpha_{15}\right)(\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-U_1}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)\right)}}{\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)^2\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})\Lambda_1\right)^2}\times \\ &\frac{\left((\alpha_{10}+\alpha_{11})\left(\left(\left((\alpha_{13}+\alpha_{14})e^{-L_1}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)\right)+1\right)+\alpha_{12}\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-L_1}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)\right)\right)^2}(\alpha_{16}+\alpha_{17})e^{-\left((\alpha_{10}+\alpha_{11})e^{-\left(\left((\alpha_{13}+\alpha_{14})e^{-L_1}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)\right)}/\alpha_{12}+(\alpha_{10}+\alpha_{11})\left(\left((\alpha_{13}+\alpha_{14})e^{-L_1}\right)/\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_1\right)\right)\right)}\times \\ &\cdot\left((\alpha_{16}+\alpha_{17})(\Lambda_1+1)+\alpha_{18}\right)-1, \end{aligned} \tag{47}$$

where Λ_1 is depicted in (29). Now, assuming that if (28) along with (29) holds then from (47), one gets $F'(z) < 0$. □

Remark 2. The proof of (ii)–(vi) is same as the proof of (i). So, it is omitted.

5. Local Dynamics about Unique Positive Fixed Point of Systems (3)–(8)

The local dynamics about Υ_i $(i = 1, \ldots, 6)$, respectively, of systems (3) to (8) is explored in this section, as follows.

Theorem 4. *For Υ_1 of (3), the following holds:*

(i) Υ_1 is a sink if

$$\begin{aligned}&\frac{\alpha_{10}\alpha_{13}(\alpha_{14}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)}\\&+\frac{\alpha_{10}\alpha_{14}(\alpha_{13}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)}\\&+\frac{(\alpha_{11}\alpha_{13}+\alpha_{10}\alpha_{14})(\alpha_{16}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)}<1;\end{aligned} \tag{48}$$

(ii) Υ_1 is a source if

$$\begin{aligned}&\frac{\alpha_{10}\alpha_{13}(\alpha_{14}+\alpha_{17})(e^{-U_1}+U_3)(e^{-U_2}+U_1)(e^{-U_3}+U_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1)}\\&+\frac{\alpha_{10}\alpha_{14}(\alpha_{13}+\alpha_{17})(e^{-U_1}+U_3)(e^{-U_2}+U_1)(e^{-U_3}+U_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1)}\\&+\frac{(\alpha_{11}\alpha_{13}+\alpha_{10}\alpha_{14})(\alpha_{16}+\alpha_{17})(e^{-U_1}+U_3)(e^{-U_2}+U_1)(e^{-U_3}+U_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1)}>1.\end{aligned} \tag{49}$$

Proof. (i) If Υ_1 is a fixed point of (3), then

$$\begin{aligned}\overline{x}&=\frac{(\alpha_{10}+\alpha_{11})e^{-\overline{y}}}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y}},\\ \overline{y}&=\frac{(\alpha_{13}+\alpha_{14})e^{-\overline{z}}}{\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z}},\\ \overline{z}&=\frac{(\alpha_{16}+\alpha_{17})e^{-\overline{x}}}{\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x}}.\end{aligned} \tag{50}$$

Moreover, we have the following map for constructing the corresponding linearized form of (3):

$$(x_{n+1},x_n,y_{n+1},y_n,z_{n+1},z_n)\longrightarrow(f,f_1,g,g_1,h,h_1), \tag{51}$$

where

$$\begin{aligned}f&=\frac{\alpha_{10}e^{-y_n}+\alpha_{11}e^{-y_{n-1}}}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}},\\ f_1&=x_n,\\ g&=\frac{\alpha_{13}e^{-z_n}+\alpha_{14}e^{-z_{n-1}}}{\alpha_{15}+\alpha_{13}z_n+\alpha_{14}z_{n-1}},\\ g_1&=y_n,\\ h&=\frac{\alpha_{16}e^{-x_n}+\alpha_{17}e^{-x_{n-1}}}{\alpha_{18}+\alpha_{16}x_n+\alpha_{17}x_{n-1}},\\ h_1&=z_n.\end{aligned} \tag{52}$$

The $J|_{\Upsilon_1}$ about Υ_1 subject to the map (51) is

$$J|_{\Upsilon_1}=\begin{pmatrix}0&0&b_{13}&b_{14}&0&0\\1&0&0&0&0&0\\0&0&0&0&b_{35}&b_{36}\\0&0&1&0&0&0\\b_{51}&b_{52}&0&0&0&0\\0&0&0&0&1&0\end{pmatrix}, \tag{53}$$

where

$$\begin{aligned}b_{13}&=-\frac{\alpha_{10}(e^{-\overline{y}}+\overline{x})}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y}},\\ b_{14}&=-\frac{\alpha_{11}(e^{-\overline{y}}+\overline{x})}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y}},\\ b_{35}&=-\frac{\alpha_{13}(e^{-\overline{z}}+\overline{y})}{\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z}},\\ b_{36}&=-\frac{\alpha_{14}(e^{-\overline{z}}+\overline{y})}{\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z}},\\ b_{51}&=-\frac{\alpha_{16}(e^{-\overline{x}}+\overline{z})}{\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x}},\\ b_{52}&=-\frac{\alpha_{17}(e^{-\overline{x}}+\overline{z})}{\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x}}.\end{aligned} \tag{54}$$

The auxiliary equation of $J|_{\Upsilon_1}$ about Υ_1 is

$$P(\lambda) = \lambda^6 + L_1\lambda^3 + L_2\lambda^2 + L_3\lambda + L_4 = 0, \quad (55)$$

where

$$\begin{aligned} L_1 &= -b_{13}b_{35}b_{51}, \\ L_2 &= -b_{13}b_{35}b_{52} - b_{13}b_{36}b_{51} - b_{14}b_{35}b_{51}, \\ L_3 &= -b_{13}b_{36}b_{52} - b_{14}b_{35}b_{52} - b_{14}b_{36}b_{51}, \\ L_4 &= -b_{14}b_{36}b_{52}. \end{aligned} \quad (56)$$

Now,

$$\begin{aligned}
\sum_{i=1}^{4} |L_i| &= \frac{\alpha_{10}\alpha_{13}(\alpha_{14}+\alpha_{17})\left(e^{-\overline{x}}+\overline{z}\right)\left(e^{-\overline{y}}+\overline{x}\right)\left(e^{-\overline{z}}+\overline{y}\right)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y})(\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z})(\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x})} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13}+\alpha_{17})\left(e^{-\overline{x}}+\overline{z}\right)\left(e^{-\overline{y}}+\overline{x}\right)\left(e^{-\overline{z}}+\overline{y}\right)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y})(\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z})(\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x})} \\
&+ \frac{\alpha_{11}\alpha_{13}(\alpha_{16}+\alpha_{17})\left(e^{-\overline{x}}+\overline{z}\right)\left(e^{-\overline{y}}+\overline{x}\right)\left(e^{-\overline{z}}+\overline{y}\right)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y})(\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z})(\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x})} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{16}+\alpha_{17})\left(e^{-\overline{x}}+\overline{z}\right)\left(e^{-\overline{y}}+\overline{x}\right)\left(e^{-\overline{z}}+\overline{y}\right)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y})(\alpha_{15}+(\alpha_{13}+\alpha_{14})\overline{z})(\alpha_{18}+(\alpha_{16}+\alpha_{17})\overline{x})} \\
&\leq \frac{\alpha_{10}\alpha_{13}(\alpha_{14}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&+ \frac{\alpha_{11}\alpha_{13}(\alpha_{16}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{16}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&= \frac{\alpha_{10}\alpha_{13}(\alpha_{14}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)} \\
&+ \frac{(\alpha_{11}\alpha_{13}+\alpha_{10}\alpha_{14})(\alpha_{16}+\alpha_{17})(e^{-L_1}+L_3)(e^{-L_2}+L_1)(e^{-L_3}+L_2)}{(\alpha_{12}+(\alpha_{10}+\alpha_{11})L_2)(\alpha_{15}+(\alpha_{13}+\alpha_{14})L_3)(\alpha_{18}+(\alpha_{16}+\alpha_{17})L_1)}.
\end{aligned} \quad (57)$$

Assuming that (48) holds, and then from (57), one gets $\sum_{i=1}^{4}|L_i| < 1$. Hence, Υ_1 of (3) is a sink.

(ii) Using similar manipulation as in the proof of (i), one has

$$\sum_{i=1}^{4}\left|L_i\right| \geq \frac{\alpha_{10}\alpha_{13}\left(\alpha_{14}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}$$
$$+\frac{\alpha_{10}\alpha_{14}\left(\alpha_{13}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}$$
$$+\frac{\alpha_{11}\alpha_{13}\left(\alpha_{16}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}$$
$$+\frac{\alpha_{10}\alpha_{14}\left(\alpha_{16}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)} \tag{58}$$
$$=\frac{\alpha_{10}\alpha_{13}\left(\alpha_{14}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}$$
$$+\frac{\alpha_{10}\alpha_{14}\left(\alpha_{13}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}$$
$$+\frac{\left(\alpha_{11}\alpha_{13}+\alpha_{10}\alpha_{14}\right)\left(\alpha_{16}+\alpha_{17}\right)\left(e^{-U_1}+U_3\right)\left(e^{-U_2}+U_1\right)\left(e^{-U_3}+U_2\right)}{\left(\alpha_{12}+(\alpha_{10}+\alpha_{11})U_2\right)\left(\alpha_{15}+(\alpha_{13}+\alpha_{14})U_3\right)\left(\alpha_{18}+(\alpha_{16}+\alpha_{17})U_1\right)}.$$

Assuming that (49) holds, and then from (58), one gets $\sum_{i=1}^{4}|L_i|>1$. Hence, Υ_1 of (3) is a source. □

In similar manner, one can explore the local dynamics about $Y_i\,(i=2,\ldots,6)$, respectively, of systems (4)–(8), as follows.

Theorem 5.

(i) For Υ_2 of (4), the following holds:

(i.1) Υ_2 is a sink if

$$\frac{\alpha_{19}\alpha_{25}\left(\alpha_{22}+\alpha_{23}\right)\left(e^{-L_4}+L_5\right)\left(e^{-L_6}+L_4\right)\left(e^{-L_4}+L_6\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})L_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})L_4\right)}$$
$$+\frac{\alpha_{19}\alpha_{26}\left(\alpha_{22}+\alpha_{23}\right)\left(e^{-L_4}+L_5\right)\left(e^{-L_5}+L_6\right)\left(e^{-L_6}+L_4\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})L_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})L_4\right)}$$
$$+\frac{\left(\alpha_{20}\alpha_{22}+\alpha_{20}\alpha_{23}\right)\left(\alpha_{26}+\alpha_{25}\right)\left(e^{-L_4}+L_5\right)\left(e^{-L_5}+L_6\right)\left(e^{-L_6}+L_4\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})L_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})L_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})L_4\right)}<1. \tag{59}$$

(i.2) Υ_2 is a source if

$$\frac{\alpha_{19}\alpha_{25}\left(\alpha_{22}+\alpha_{23}\right)\left(e^{-U_4}+U_5\right)\left(e^{-U_6}+U_4\right)\left(e^{-U_4}+U_6\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})U_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})U_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})U_4\right)}$$
$$+\frac{\alpha_{19}\alpha_{26}\left(\alpha_{22}+\alpha_{23}\right)\left(e^{-U_4}+U_5\right)\left(e^{-U_5}+U_6\right)\left(e^{-U_6}+U_4\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})U_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})U_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})U_4\right)}$$
$$+\frac{\left(\alpha_{20}\alpha_{22}+\alpha_{20}\alpha_{23}\right)\left(\alpha_{26}+\alpha_{25}\right)\left(e^{-U_4}+U_5\right)\left(e^{-U_5}+U_6\right)\left(e^{-U_6}+U_4\right)}{\left(\alpha_{27}+(\alpha_{25}+\alpha_{26})U_5\right)\left(\alpha_{21}+(\alpha_{19}+\alpha_{20})U_6\right)\left(\alpha_{24}+(\alpha_{22}+\alpha_{23})U_4\right)}>1, \tag{60}$$

with

$$J|_{\Upsilon_2}=\begin{pmatrix} 0 & 0 & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{61}$$

where

$$\begin{aligned} b_{15} &= -\frac{\alpha_{19}\left(e^{-\overline{x}}+\overline{y}\right)}{\alpha_{21}+(\alpha_{19}+\alpha_{20})\overline{x}}, \\ b_{16} &= -\frac{\alpha_{20}\left(e^{-\overline{x}}+\overline{y}\right)}{\alpha_{21}+(\alpha_{19}+\alpha_{20})\overline{x}}, \\ b_{31} &= -\frac{\alpha_{22}\left(e^{-\overline{z}}+\overline{x}\right)}{\alpha_{24}+(\alpha_{22}+\alpha_{23})\overline{z}}, \\ b_{32} &= -\frac{\alpha_{23}\left(e^{-\overline{z}}+\overline{x}\right)}{\alpha_{24}+(\alpha_{22}+\alpha_{23})\overline{z}}, \\ b_{53} &= -\frac{\alpha_{25}\left(e^{-\overline{y}}+\overline{z}\right)}{\alpha_{27}+(\alpha_{25}+\alpha_{26})\overline{y}}, \\ b_{54} &= -\frac{\alpha_{26}\left(e^{-\overline{y}}+\overline{z}\right)}{\alpha_{27}+(\alpha_{25}+\alpha_{26})\overline{y}}; \end{aligned} \tag{62}$$

(ii) For Υ_3 of (5), the following holds:

(ii.1) Υ_3 is a sink if

$$\begin{aligned}
&\frac{(\alpha_{28}+\alpha_{29})(e^{-L_7}+L_7)}{\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7}+\frac{(\alpha_{31}+\alpha_{32})e^{-L_8}}{\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9}+\frac{(\alpha_{34}+\alpha_{35})e^{-L_9}}{\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8}\\
&+\frac{(\alpha_{34}\alpha_{28}+\alpha_{29}\alpha_{34}+\alpha_{35}\alpha_{28}+\alpha_{29}\alpha_{35}+\alpha_{29}\alpha_{32})e^{-L_7-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{31}\alpha_{34}+\alpha_{35}\alpha_{31}+\alpha_{32}\alpha_{35})e^{-L_8-L_9}}{(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{28}\alpha_{34}+\alpha_{29}\alpha_{34}+\alpha_{28}\alpha_{35}+\alpha_{29}\alpha_{32}+\alpha_{29}\alpha_{35})L_7e^{-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{31}\alpha_{34}+\alpha_{32}\alpha_{34}+\alpha_{31}\alpha_{35}+\alpha_{32}\alpha_{35})L_7L_8}{(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})e^{-L_7-L_8-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})e^{-L_7-L_8-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})L_7L_8L_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})L_7L_8L_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})e^{-L_7}L_8L_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})e^{-L_7}L_8L_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})L_7e^{-L_8-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})L_7e^{-L_8-L_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})L_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})L_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})L_8)}<1.
\end{aligned}\tag{63}$$

(ii.2) Υ_3 *is a source if*

$$
\begin{aligned}
&\frac{(\alpha_{28}+\alpha_{29})(e^{-U_7}+U_7)}{\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7}+\frac{(\alpha_{31}+\alpha_{32})e^{-U_8}}{\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9}+\frac{(\alpha_{34}+\alpha_{35})e^{-U_9}}{\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8}\\
&+\frac{(\alpha_{34}\alpha_{28}+\alpha_{29}\alpha_{34}+\alpha_{35}\alpha_{28}+\alpha_{29}\alpha_{35}+\alpha_{29}\alpha_{32})e^{-U_7-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{31}\alpha_{34}+\alpha_{35}\alpha_{31}+\alpha_{32}\alpha_{35})e^{-U_8-U_9}}{(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{28}\alpha_{34}+\alpha_{29}\alpha_{34}+\alpha_{28}\alpha_{35}+\alpha_{29}\alpha_{32}+\alpha_{29}\alpha_{35})U_7e^{-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{31}\alpha_{34}+\alpha_{32}\alpha_{34}+\alpha_{31}\alpha_{35}+\alpha_{32}\alpha_{35})U_7U_8}{(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})e^{-U_7-U_8-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})e^{-U_7-U_8-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})U_7U_8U_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})U_7U_8U_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})e^{-U_7}U_8U_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})e^{-U_7}U_8U_9}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{28}\alpha_{31}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{34}+\alpha_{28}\alpha_{32}\alpha_{34}+\alpha_{28}\alpha_{35}\alpha_{31})U_7e^{-U_8-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}\\
&+\frac{(\alpha_{29}\alpha_{32}\alpha_{34}+\alpha_{29}\alpha_{31}\alpha_{35}+\alpha_{28}\alpha_{32}\alpha_{35}+\alpha_{29}\alpha_{32}\alpha_{35})U_7e^{-U_8-U_9}}{(\alpha_{30}+(\alpha_{28}+\alpha_{29})U_7)(\alpha_{33}+(\alpha_{31}+\alpha_{32})U_9)(\alpha_{36}+(\alpha_{34}+\alpha_{35})U_8)}>1,
\end{aligned}
\tag{64}
$$

with

$$J|_{\Upsilon_3} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & b_{35} & b_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (65)$$

where

$$\begin{aligned} b_{11} &= -\frac{\alpha_{28}\left(e^{-\overline{x}} + \overline{x}\right)}{\alpha_{30} + \left(\alpha_{28} + \alpha_{29}\right)\overline{x}}, \\ b_{12} &= -\frac{\alpha_{29}\left(e^{-\overline{x}} + \overline{x}\right)}{\alpha_{30} + \left(\alpha_{28} + \alpha_{29}\right)\overline{x}}, \\ b_{33} &= -\frac{\alpha_{31}e^{-\overline{y}}}{\alpha_{33} + \left(\alpha_{31} + \alpha_{32}\right)\overline{z}}, \\ b_{34} &= -\frac{\alpha_{32}e^{-\overline{y}}}{\alpha_{33} + \left(\alpha_{31} + \alpha_{32}\right)\overline{z}}, \\ b_{35} &= -\frac{\alpha_{31}\overline{y}}{\alpha_{33} + \left(\alpha_{31} + \alpha_{32}\right)\overline{z}}, \\ b_{36} &= -\frac{\alpha_{32}\overline{y}}{\alpha_{33} + \left(\alpha_{31} + \alpha_{32}\right)\overline{z}}, \\ b_{53} &= -\frac{\alpha_{34}\overline{z}}{\alpha_{36} + \left(\alpha_{34} + \alpha_{35}\right)\overline{y}}, \\ b_{54} &= -\frac{\alpha_{35}\overline{z}}{\alpha_{36} + \left(\alpha_{34} + \alpha_{35}\right)\overline{y}}, \\ b_{55} &= -\frac{\alpha_{34}e^{-\overline{z}}}{\alpha_{36} + \left(\alpha_{34} + \alpha_{35}\right)\overline{y}}, \\ b_{56} &= -\frac{\alpha_{35}e^{-\overline{z}}}{\alpha_{36} + \left(\alpha_{34} + \alpha_{35}\right)\overline{y}}; \end{aligned} \quad (66)$$

(iii) For Υ_4 of (6), the following holds:

(iii.1) Υ_4 is a sink if

$$\begin{aligned} &\frac{\left(\alpha_{37} + \alpha_{38}\right)e^{-L_{10}}}{\alpha_{39} + \left(\alpha_{37} + \alpha_{38}\right)L_{11}} + \frac{\left(\alpha_{40} + \alpha_{41}\right)e^{-L_{11}}}{\alpha_{42} + \left(\alpha_{40} + \alpha_{41}\right)L_{10}} \\ &+ \frac{\left(\alpha_{43} + \alpha_{44}\right)\left(e^{-L_{12}} + L_{12}\right)}{\alpha_{45} + \left(\alpha_{43} + \alpha_{44}\right)L_{12}} \\ &+ \frac{\left(\alpha_{37}\alpha_{40} + \alpha_{41}\alpha_{37} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41}\right)e^{-L_{10}-L_{11}}}{\left(\alpha_{39} + \left(\alpha_{37} + \alpha_{38}\right)L_{11}\right)\left(\alpha_{42} + \left(\alpha_{40} + \alpha_{41}\right)L_{10}\right)} \\ &+ \frac{\left(\alpha_{37}\alpha_{40} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41}\right)L_{10}L_{11}}{\left(\alpha_{39} + \left(\alpha_{37} + \alpha_{38}\right)L_{11}\right)\left(\alpha_{42} + \left(\alpha_{40} + \alpha_{41}\right)L_{10}\right)} \\ &+ \frac{\left(\alpha_{40}\alpha_{43} + \alpha_{41}\alpha_{44} + \alpha_{41}\alpha_{43} + \alpha_{4}\alpha_{44}\right)e^{-L_{11}}\left(e^{-L_{12}} + L_{12}\right)}{\left(\alpha_{42} + \left(\alpha_{40} + \alpha_{41}\right)L_{10}\right)\left(\alpha_{45} + \left(\alpha_{43} + \alpha_{44}\right)L_{12}\right)} \\ &+ \frac{\left(\alpha_{37}\alpha_{43} + \alpha_{38}\alpha_{43} + \alpha_{38}\alpha_{44} + \alpha_{37}\alpha_{44}\right)e^{-L_{10}}\left(e^{-L_{12}} + L_{10}\right)}{\left(\alpha_{39} + \left(\alpha_{37} + \alpha_{38}\right)L_{11}\right)\left(\alpha_{45} + \left(\alpha_{43} + \alpha_{44}\right)L_{12}\right)} \\ &+ \big[\left(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44}\right. \\ &\left.+\alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44}\right)e^{-L_{10}-L_{11}-L_{12}} \\ &+ \left(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44}\right. \\ &\left.+\alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44}\right)L_{10}L_{11}L_{12} \\ &+ \left(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44}\right. \\ &\left.+\alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44}\right)e^{-L_{12}}L_{10}L_{11} \\ &+ \left(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{43}\right. \\ &\left.+\alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44}\right)L_{12}e^{-L_{10}-L_{11}}\big] \\ &\frac{1}{\left(\alpha_{39} + \left(\alpha_{37} + \alpha_{38}\right)L_{11}\right)\left(\alpha_{42} + \left(\alpha_{40} + \alpha_{41}\right)L_{10}\right)\left(\alpha_{45} + \left(\alpha_{43} + \alpha_{44}\right)L_{12}\right)} \\ &< 1. \end{aligned} \quad (67)$$

(iii.2) Υ_4 is a source if

$$\frac{(\alpha_{37}+\alpha_{38})e^{-U_{10}}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})U_{11}}+\frac{(\alpha_{40}+\alpha_{41})e^{-U_{11}}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})U_{10}}$$
$$+\frac{(\alpha_{43}+\alpha_{44})\left(e^{-U_{12}}+L_{12}\right)}{\alpha_{45}+(\alpha_{43}+\alpha_{44})U_{12}}$$
$$+\frac{(\alpha_{37}\alpha_{40}+\alpha_{41}\alpha_{37}+\alpha_{38}\alpha_{40}+\alpha_{37}\alpha_{41}+\alpha_{38}\alpha_{41})e^{-U_{10}-U_{11}}}{\left(\alpha_{39}+(\alpha_{37}+\alpha_{38})U_{11}\right)\left(\alpha_{42}+(\alpha_{40}+\alpha_{41})U_{10}\right)}$$
$$+\frac{(\alpha_{37}\alpha_{40}+\alpha_{38}\alpha_{40}+\alpha_{37}\alpha_{41}+\alpha_{38}\alpha_{41})U_{10}U_{11}}{\left(\alpha_{39}+(\alpha_{37}+\alpha_{38})U_{11}\right)\left(\alpha_{42}+(\alpha_{40}+\alpha_{41})U_{10}\right)}$$
$$+\frac{(\alpha_{40}\alpha_{43}+\alpha_{41}\alpha_{44}+\alpha_{41}\alpha_{43}+\alpha_{4}\alpha_{44})e^{-U_{11}}\left(e^{-U_{12}}+U_{12}\right)}{\left(\alpha_{42}+(\alpha_{40}+\alpha_{41})U_{10}\right)\left(\alpha_{45}+(\alpha_{43}+\alpha_{44})U_{12}\right)}$$
$$+\frac{(\alpha_{37}\alpha_{43}+\alpha_{38}\alpha_{43}+\alpha_{38}\alpha_{44}+\alpha_{37}\alpha_{44})e^{-U_{10}}\left(e^{-U_{12}}+U_{10}\right)}{\left(\alpha_{39}+(\alpha_{37}+\alpha_{38})U_{11}\right)\left(\alpha_{45}+(\alpha_{43}+\alpha_{44})U_{12}\right)}$$
$$+\big[(\alpha_{37}\alpha_{40}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{43}+\alpha_{37}\alpha_{41}\alpha_{43}+\alpha_{37}\alpha_{40}\alpha_{44}+$$
$$\alpha_{38}\alpha_{41}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{44}+\alpha_{37}\alpha_{41}\alpha_{44}+\alpha_{38}\alpha_{41}\alpha_{44})e^{-U_{10}-U_{11}-U_{12}}$$
$$+(\alpha_{37}\alpha_{40}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{43}+\alpha_{37}\alpha_{41}\alpha_{43}+\alpha_{37}\alpha_{40}\alpha_{44}$$
$$+\alpha_{38}\alpha_{41}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{44}+\alpha_{37}\alpha_{41}\alpha_{44}+\alpha_{38}\alpha_{41}\alpha_{44})U_{10}U_{11}U_{12}$$
$$+(\alpha_{37}\alpha_{40}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{43}+\alpha_{37}\alpha_{41}\alpha_{43}+\alpha_{37}\alpha_{40}\alpha_{44}$$
$$+\alpha_{38}\alpha_{41}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{44}+\alpha_{37}\alpha_{41}\alpha_{44}+\alpha_{38}\alpha_{41}\alpha_{44})e^{-U_{12}}U_{10}U_{11}$$
$$+(\alpha_{37}\alpha_{40}\alpha_{43}+\alpha_{38}\alpha_{40}\alpha_{43}+\alpha_{37}\alpha_{41}\alpha_{43}+\alpha_{37}\alpha_{40}\alpha_{44}+\alpha_{38}\alpha_{41}\alpha_{43}$$
$$+\alpha_{38}\alpha_{40}\alpha_{44}+\alpha_{37}\alpha_{41}\alpha_{44}+\alpha_{38}\alpha_{41}\alpha_{44})U_{12}e^{-U_{10}-U_{11}}\big]$$
$$\times\frac{1}{\left(\alpha_{39}+(\alpha_{37}+\alpha_{38})U_{11}\right)\left(\alpha_{42}+(\alpha_{40}+\alpha_{41})U_{10}\right)\left(\alpha_{45}+(\alpha_{43}+\alpha_{44})U_{12}\right)}$$
$$>1, \tag{68}$$

with

$$J|_{\Upsilon_4}=\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & b_{55} & b_{56}\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{69}$$

where

$$\begin{aligned}
b_{11} &= -\frac{\alpha_{37}e^{-\overline{x}}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})\overline{y}},\\
b_{12} &= -\frac{\alpha_{38}e^{-\overline{x}}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})\overline{y}},\\
b_{13} &= -\frac{\alpha_{37}\overline{x}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})\overline{y}},\\
b_{14} &= -\frac{\alpha_{38}\overline{x}}{\alpha_{39}+(\alpha_{37}+\alpha_{38})\overline{y}},\\
b_{31} &= -\frac{\alpha_{40}\overline{y}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})\overline{x}},\\
b_{32} &= -\frac{\alpha_{41}\overline{y}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})\overline{x}},\\
b_{33} &= -\frac{\alpha_{40}e^{-\overline{y}}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})\overline{x}},\\
b_{34} &= -\frac{\alpha_{41}e^{-\overline{y}}}{\alpha_{42}+(\alpha_{40}+\alpha_{41})\overline{x}},\\
b_{55} &= -\frac{\alpha_{43}\left(e^{-\overline{z}}+\overline{z}\right)}{\alpha_{45}+(\alpha_{43}+\alpha_{44})\overline{z}},\\
b_{56} &= -\frac{\alpha_{44}\left(e^{-\overline{z}}+\overline{z}\right)}{\alpha_{45}+(\alpha_{43}+\alpha_{44})\overline{z}};
\end{aligned} \tag{70}$$

(iv) For Υ_5 of (7), the following holds:

(iv.1) Υ_5 is a sink if

$$
\begin{aligned}
&\frac{(\alpha_{46}+\alpha_{47})e^{-L_{13}}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})L_{15}}+\frac{(\alpha_{52}+\alpha_{53})e^{-L_{15}}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})L_{13}}+\frac{(\alpha_{49}+\alpha_{50})(e^{-L_{14}}+L_{14})}{\alpha_{51}+(\alpha_{49}+\alpha_{50})L_{14}}\\
&+\frac{(\alpha_{46}\alpha_{52}+\alpha_{47}\alpha_{52}+\alpha_{46}\alpha_{53}+\alpha_{47}\alpha_{53})e^{-L_{13}-L_{15}}}{(\alpha_{48}+(\alpha_{46}+\alpha_{47})L_{15})(\alpha_{54}+(\alpha_{52}+\alpha_{53})L_{13})}\\
&+\frac{(\alpha_{46}\alpha_{52}+\alpha_{47}\alpha_{52}+\alpha_{46}\alpha_{53}+\alpha_{47}\alpha_{53})L_{13}L_{15}}{(\alpha_{48}+(\alpha_{46}+\alpha_{47})L_{15})(\alpha_{54}+(\alpha_{52}+\alpha_{53})L_{13})}\\
&+\frac{(\alpha_{\alpha_{46}}\alpha_{49}+\alpha_{47}\alpha_{47}+\alpha_{46}\alpha_{50}+\alpha_{47}\alpha_{50})e^{-L_{13}}(e^{-L_{14}}+L_{14})}{(\alpha_{48}+(\alpha_{46}+\alpha_{47})L_{15})(\alpha_{51}+(\alpha_{49}+\alpha_{50})L_{14})}\\
&+\frac{(\alpha_{49}\alpha_{52}+\alpha_{50}\alpha_{52}+\alpha_{49}\alpha_{53}+\alpha_{50}\alpha_{53})e^{-L_{15}}(e^{-L_{14}}+L_{14})}{(\alpha_{54}+(\alpha_{52}+\alpha_{53})L_{13})(\alpha_{51}+(\alpha_{49}+\alpha_{50})L_{14})}\\
&+\big[(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}\\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})e^{-L_{13}-L_{14}-L_{15}}\\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}\\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})L_{13}L_{14}L_{15}\\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}\\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})e^{-L_{14}}L_{13}L_{15}\\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}\\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})L_{14}e^{-L_{13}-L_{15}}\big]\\
&\times\frac{1}{(\alpha_{48}+(\alpha_{46}+\alpha_{47})L_{15})(\alpha_{51}+(\alpha_{49}+\alpha_{50})L_{14})(\alpha_{54}+(\alpha_{52}+\alpha_{53})L_{13})}<1.
\end{aligned}
\tag{71}
$$

(iv.2) Υ_5 is a source if

$$\begin{aligned}
&\frac{(\alpha_{46}+\alpha_{47})e^{-U_{13}}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})U_{15}}+\frac{(\alpha_{52}+\alpha_{53})e^{-U_{15}}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})U_{13}} \\
&+\frac{(\alpha_{49}+\alpha_{50})\left(e^{-U_{14}}+U_{14}\right)}{\alpha_{51}+(\alpha_{49}+\alpha_{50})U_{14}}+ \\
&\frac{(\alpha_{46}\alpha_{52}+\alpha_{47}\alpha_{52}+\alpha_{46}\alpha_{53}+\alpha_{47}\alpha_{53})e^{-U_{13}-U_{15}}}{\left(\alpha_{54}+(\alpha_{52}+\alpha_{53})U_{13}\right)\left(\alpha_{48}+(\alpha_{46}+\alpha_{47})U_{15}\right)} \\
&+\frac{(\alpha_{46}\alpha_{52}+\alpha_{47}\alpha_{52}+\alpha_{46}\alpha_{53}+\alpha_{47}\alpha_{53})U_{13}U_{15}}{\left(\alpha_{54}+(\alpha_{52}+\alpha_{53})U_{13}\right)\left(\alpha_{48}+(\alpha_{46}+\alpha_{47})U_{15}\right)} \\
&+\frac{\left(\alpha_{\alpha_{46}}\alpha_{49}+\alpha_{47}\alpha_{47}+\alpha_{46}\alpha_{50}+\alpha_{47}\alpha_{50}\right)e^{-U_{13}}\left(e^{-U_{14}}+U_{14}\right)}{\left(\alpha_{54}+(\alpha_{52}+\alpha_{53})U_{15}\right)\left(\alpha_{51}+(\alpha_{49}+\alpha_{48})U_{14}\right)} \\
&+\frac{(\alpha_{49}\alpha_{52}+\alpha_{50}\alpha_{52}+\alpha_{49}\alpha_{53}+\alpha_{50}\alpha_{53})e^{-U_{15}}\left(e^{-U_{14}}+U_{14}\right)}{\left(\alpha_{54}+(\alpha_{52}+\alpha_{53})U_{15}\right)\left(\alpha_{51}+(\alpha_{49}+\alpha_{48})U_{14}\right)} \\
&+\big[(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}+ \\
&\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})e^{-U_{13}-U_{14}-U_{15}} \\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53} \\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})U_{13}U_{14}U_{15} \\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53} \\
&+\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})e^{-U_{14}}U_{13}U_{15} \\
&+(\alpha_{46}\alpha_{49}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{52}+\alpha_{46}\alpha_{50}\alpha_{52}+\alpha_{46}\alpha_{49}\alpha_{53}+ \\
&\alpha_{47}\alpha_{50}\alpha_{52}+\alpha_{47}\alpha_{49}\alpha_{53}+\alpha_{46}\alpha_{50}\alpha_{53}+\alpha_{47}\alpha_{50}\alpha_{53})U_{14}e^{-U_{13}-U_{15}}\big] \\
&\times\frac{1}{\left(\alpha_{48}+(\alpha_{46}+\alpha_{47})U_{15}\right)\left(\alpha_{51}+(\alpha_{49}+\alpha_{50})U_{14}\right)\left(\alpha_{54}(\alpha_{52}+\alpha_{53})U_{13}\right)} \\
&>1,
\end{aligned} \tag{72}$$

with

$$J|_{\Upsilon_5}=\begin{pmatrix} b_{11} & b_{12} & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{51} & b_{52} & 0 & 0 & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{73}$$

where

$$\begin{aligned}
b_{11}&=-\frac{\alpha_{46}e^{-\overline{x}}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})\overline{z}}, \\
b_{12}&=-\frac{\alpha_{47}e^{-\overline{x}}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})\overline{z}}, \\
b_{15}&=-\frac{\alpha_{46}\overline{x}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})\overline{z}}, \\
b_{16}&=-\frac{\alpha_{47}\overline{x}}{\alpha_{48}+(\alpha_{46}+\alpha_{47})\overline{z}}, \\
b_{33}&=-\frac{\alpha_{49}\left(e^{-\overline{y}}+\overline{y}\right)}{\alpha_{51}+(\alpha_{49}+\alpha_{50})\overline{y}}, \\
b_{34}&=-\frac{\alpha_{50}\left(e^{-\overline{y}}+\overline{y}\right)}{\alpha_{51}+(\alpha_{49}+\alpha_{50})\overline{y}}, \\
b_{51}&=-\frac{\alpha_{52}\overline{z}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})\overline{x}}, \\
b_{52}&=-\frac{\alpha_{53}\overline{z}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})\overline{x}}, \\
b_{55}&=-\frac{\alpha_{52}e^{-\overline{z}}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})\overline{x}}, \\
b_{56}&=-\frac{\alpha_{53}e^{-\overline{z}}}{\alpha_{54}+(\alpha_{52}+\alpha_{53})\overline{x}};
\end{aligned} \tag{74}$$

(v) For Υ_6 of (8), the following holds:

(v.1) Υ_6 is a sink if

$$\begin{aligned}
&\frac{(\alpha_{55}\alpha_{58}+\alpha_{56}\alpha_{58}+\alpha_{55}\alpha_{59}+\alpha_{56}\alpha_{59})L_{16}L_{17}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})L_{16})(\alpha_{60}+(\alpha_{58}+\alpha_{59})L_{17})}+\frac{(\alpha_{55}+\alpha_{56})L_{16}}{\alpha_{57}+(\alpha_{55}+\alpha_{56})L_{16}}\\
&+\frac{(\alpha_{55}\alpha_{61}+\alpha_{55}\alpha_{62}+\alpha_{56}\alpha_{61}+\alpha_{56}\alpha_{62})L_{16}L_{18}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})L_{16})(\alpha_{63}+(\alpha_{61}+\alpha_{62})L_{18})}+\frac{(\alpha_{58}+\alpha_{59})L_{17}}{\alpha_{60}+(\alpha_{58}+\alpha_{59})L_{17}}\\
&+\frac{(\alpha_{58}\alpha_{62}+\alpha_{59}\alpha_{61}+\alpha_{59}\alpha_{62}+\alpha_{58}\alpha_{61})L_{16}L_{18}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})L_{16})(\alpha_{63}+(\alpha_{61}+\alpha_{62})L_{18})}+\frac{(\alpha_{61}+\alpha_{62})L_{18}}{\alpha_{63}+(\alpha_{61}+\alpha_{62})L_{18}}\\
&+\left[(\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{62}\right.\\
&+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{56}\alpha_{58}\alpha_{62}+\alpha_{55}\alpha_{58}\alpha_{62}+\alpha_{56}\alpha_{58}\alpha_{62})e^{-L_{16}-L_{17}-L_{18}}\\
&+(\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{62}+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{59}\alpha_{61}\\
&\left.+\alpha_{56}\alpha_{58}\alpha_{62}+\alpha_{55}\alpha_{59}\alpha_{62}+\alpha_{56}\alpha_{59}\alpha_{61}+\alpha_{56}\alpha_{59}\alpha_{62})L_{16}L_{17}L_{18}\right]\\
&\times\frac{1}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})L_{16})(\alpha_{60}+(\alpha_{58}+\alpha_{59})L_{17})(\alpha_{63}+(\alpha_{61}+\alpha_{62})L_{18})}<1.
\end{aligned}\tag{75}$$

(v.2) Υ_6 is a source if

$$\begin{aligned}
&\frac{(\alpha_{55}\alpha_{58}+\alpha_{56}\alpha_{58}+\alpha_{55}\alpha_{59}+\alpha_{56}\alpha_{59})U_{16}U_{17}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})U_{16})(\alpha_{60}+(\alpha_{58}+\alpha_{59})U_{17})}+\frac{(\alpha_{55}+\alpha_{56})U_{16}}{\alpha_{57}+(\alpha_{55}+\alpha_{56})U_{16}}\\
&+\frac{(\alpha_{55}\alpha_{61}+\alpha_{55}\alpha_{62}+\alpha_{56}\alpha_{61}+\alpha_{56}\alpha_{62})U_{16}U_{18}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})U_{16})(\alpha_{63}+(\alpha_{61}+\alpha_{62})U_{18})}+\frac{(\alpha_{58}+\alpha_{59})U_{17}}{\alpha_{60}+(\alpha_{58}+\alpha_{59})U_{17}}\\
&+\frac{(\alpha_{58}\alpha_{62}+\alpha_{59}\alpha_{61}+\alpha_{59}\alpha_{62}+\alpha_{58}\alpha_{61})U_{16}U_{18}}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})U_{16})(\alpha_{63}+(\alpha_{61}+\alpha_{62})U_{18})}+\frac{(\alpha_{61}+\alpha_{62})U_{18}}{\alpha_{63}+(\alpha_{61}+\alpha_{62})U_{18}}\\
&+\left[(\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{62}\right.\\
&+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{56}\alpha_{58}\alpha_{62}+\alpha_{55}\alpha_{58}\alpha_{62}+\alpha_{56}\alpha_{58}\alpha_{62})e^{-U_{16}-U_{17}-U_{18}}\\
&+(\alpha_{55}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{58}\alpha_{62}+\alpha_{56}\alpha_{58}\alpha_{61}+\alpha_{55}\alpha_{59}\alpha_{61}\\
&\left.+\alpha_{56}\alpha_{58}\alpha_{62}+\alpha_{55}\alpha_{59}\alpha_{62}+\alpha_{56}\alpha_{59}\alpha_{61}+\alpha_{56}\alpha_{59}\alpha_{62})U_{16}U_{17}U_{18}\right]\\
&\times\frac{1}{(\alpha_{57}+(\alpha_{55}+\alpha_{56})U_{16})(\alpha_{60}+(\alpha_{58}+\alpha_{59})U_{17})(\alpha_{63}+(\alpha_{61}+\alpha_{62})U_{18})}>1,
\end{aligned}\tag{76}$$

with

$$J|_{\Upsilon_6} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (77)$$

where

$$\begin{aligned} b_{11} &= -\frac{\alpha_{55}\overline{x}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\overline{x}}, \\ b_{12} &= -\frac{\alpha_{56}\overline{x}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\overline{x}}, \\ b_{15} &= -\frac{\alpha_{55}e^{-\overline{z}}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\overline{x}}, \\ b_{16} &= -\frac{\alpha_{56}e^{-\overline{z}}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\overline{x}}, \\ b_{31} &= -\frac{\alpha_{58}e^{-\overline{x}}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\overline{y}}, \\ b_{32} &= -\frac{\alpha_{59}e^{-\overline{x}}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\overline{y}}, \\ b_{33} &= -\frac{\alpha_{58}\overline{y}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\overline{y}}, \\ b_{34} &= -\frac{\alpha_{59}\overline{y}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\overline{y}}, \\ b_{53} &= -\frac{\alpha_{61}e^{-\overline{y}}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\overline{z}}, \\ b_{54} &= -\frac{\alpha_{62}e^{-\overline{y}}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\overline{z}}, \\ b_{55} &= -\frac{\alpha_{61}\overline{z}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\overline{z}}, \\ b_{56} &= -\frac{\alpha_{62}\overline{z}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\overline{z}}. \end{aligned} \quad (78)$$

Proof. It is similar to the proof of Theorem 4. So its proof is omitted. □

Hereafter, by constructing a Lyapunov function with discrete time motivated from the work of [2], global dynamics about $\Upsilon_i (i = 1, \ldots, 6)$, respectively, of systems (3)–(8) is explored. □

6. Global Dynamics of Systems (3)–(8)

Theorem 6. *For global dynamics about $\Upsilon_i (i = 1, \ldots, 6)$, respectively, of systems (3)–(8), following statements hold:*

(i) Υ_1 of (3) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{10} + \alpha_{11})e^{-L_2} < (2\overline{x} - U_1)(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2), \\ &(\alpha_{13} + \alpha_{14})e^{-L_3} < (2\overline{y} - U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3), \\ &(\alpha_{16} + \alpha_{17})e^{-L_1} < (2\overline{z} - U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1); \end{aligned} \quad (79)$$

(ii) Υ_2 of (4) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{19} + \alpha_{20})e^{-L_6} < (2\overline{x} - U_4)(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6), \\ &(\alpha_{22} + \alpha_{23})e^{-L_4} < (2\overline{y} - U_5)(\alpha_{24} + (\alpha_{22} + \alpha_{23})L_4), \\ &(\alpha_{25} + \alpha_{26})e^{-L_5} < (2\overline{z} - U_6)(\alpha_{27} + (\alpha_{25} + \alpha_{26})L_5); \end{aligned} \quad (80)$$

(iii) Υ_3 of (5) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{28} + \alpha_{29})e^{-L_7} < (2\overline{x} - U_7)(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7), \\ &(\alpha_{31} + \alpha_{32})e^{-L_8} < (2\overline{y} - U_8)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9), \\ &(\alpha_{34} + \alpha_{35})e^{-L_9} < (2\overline{z} - U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8); \end{aligned} \quad (81)$$

(iv) Υ_4 of (6) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{37} + \alpha_{38})e^{-L_{10}} < (2\overline{x} - U_{10})(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11}), \\ &(\alpha_{40} + \alpha_{41})e^{-L_{11}} < (2\overline{y} - U_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10}), \\ &(\alpha_{43} + \alpha_{44})e^{-L_{12}} < (2\overline{z} - U_{12})(\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12}); \end{aligned} \quad (82)$$

(v) Υ_5 of (7) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{46} + \alpha_{47})e^{-L_{13}} < (2\overline{x} - U_{13})(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15}), \\ &(\alpha_{49} + \alpha_{50})e^{-L_{14}} < (2\overline{y} - U_{14})(\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14}), \\ &(\alpha_{52} + \alpha_{53})e^{-L_{15}} < (2\overline{z} - U_{15})(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13}); \end{aligned} \quad (83)$$

(vi) Υ_6 of (8) is global asymptotically stable if

$$\begin{aligned} &(\alpha_{55} + \alpha_{56})e^{-L_{18}} < (2\overline{x} - U_{16})(\alpha_{57} + (\alpha_{56} + \alpha_{57})L_{16}), \\ &(\alpha_{58} + \alpha_{59})e^{-L_{16}} < (2\overline{y} - U_{17})(\alpha_{60} + (\alpha_{58} + \alpha_{59})L_{17}), \\ &(\alpha_{61} + \alpha_{62})e^{-L_{17}} < (2\overline{z} - U_{18})(\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18}). \end{aligned} \quad (84)$$

Proof. (i) Consider the following discrete-time Lyapunov function:

$$\Gamma_n = \left(x_n - \overline{x}\right)^2 + \left(y_n - \overline{y}\right)^2 + \left(z_n - \overline{z}\right)^2. \tag{85}$$

Now

$$\begin{aligned}
\Delta\Gamma_n &= \Gamma_{n+1} - \Gamma_n \\
&= \left(x_{n+1} - x_n\right)\left(x_{n+1} + x_n - 2\overline{x}\right) + \left(y_{n+1} - y_n\right)\left(y_{n+1} + y_n - 2\overline{y}\right) + \left(z_{n+1} - z_n\right)\left(z_{n+1} + z_n - 2\overline{z}\right), \\
&= \left(x_{n+1} - x_n\right)\left(\frac{\alpha_{10}e^{-y_n} + \alpha_{11}e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} + x_n - 2\overline{x}\right) \\
&\quad + \left(y_{n+1} - y_n\right)\left(\frac{\alpha_{13}e^{-z_n} + \alpha_{14}e^{-z_{n-1}}}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}} + y_n - 2\overline{y}\right) \\
&\quad + \left(z_{n+1} - z_n\right)\left(\frac{\alpha_{16}e^{-x_n} + \alpha_{17}e^{-x_{n-1}}}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}} + z_n - 2\overline{z}\right) \\
&\leq \left(U_1 - L_1\right)\left(\frac{(\alpha_{10} + \alpha_{11})e^{-L_2}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2} + U_1 - 2\overline{x}\right) \\
&\quad + \left(U_2 - L_2\right)\left(\frac{(\alpha_{13} + \alpha_{14})e^{-L_3}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3} + U_2 - 2\overline{y}\right) \\
&\quad + \left(U_3 - L_3\right)\left(\frac{(\alpha_{16} + \alpha_{17})e^{-L_1}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1} + U_3 - 2\overline{z}\right) \\
&= \left(U_1 - L_1\right)\left(\frac{(\alpha_{10} + \alpha_{11})e^{-L_2} - \left(2\overline{x} - U_1\right)\left(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2\right)}{\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2}\right) \\
&\quad + \left(U_2 - L_2\right)\left(\frac{(\alpha_{13} + \alpha_{14})e^{-L_3} - \left(2\overline{y} - U_2\right)\left(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3\right)}{\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3}\right) \\
&\quad + \left(U_3 - L_3\right)\left(\frac{(\alpha_{16} + \alpha_{17})e^{-L_1} - \left(2\overline{z} - U_3\right)\left(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1\right)}{\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1}\right).
\end{aligned} \tag{86}$$

From (80) and (87), one gets $\Delta\Gamma_n < 0\, \forall\, n \geq 0$. Hence, we obtain that $\lim_{n\longrightarrow\infty}(\Gamma_{n+1} - \Gamma_n) = 0$, and thus Υ_1 of (3) is global asymptotically stable. □ □

Remark 3. The proof of (ii)–(vi) is same as the proof of (i).

7. Rate of Convergence

We will explore the convergence result about the equilibrium point of systems (3)–(8) motivated from the existing literature [3–5], in this section.

Theorem 7. *If the positive solution of (3) is* $\{\Omega_n\}_{n=-1}^{\infty}$, *s.t.*

$$\begin{aligned}
\lim_{n\longrightarrow\infty} x_n &= \overline{x}, \\
\lim_{n\longrightarrow\infty} y_n &= \overline{y}, \\
\lim_{n\longrightarrow\infty} z_n &= \overline{z},
\end{aligned} \tag{87}$$

where

$$\begin{aligned}
\overline{x} &\in \left[L_1, U_1\right], \\
\overline{y} &\in \left[L_2, U_2\right], \\
\overline{z} &\in \left[L_3, U_3\right],
\end{aligned} \tag{88}$$

then the error vector, i.e.,

$$\theta_n = \begin{pmatrix} \theta_n^1 \\ \theta_{n-1}^1 \\ \theta_n^2 \\ \theta_{n-1}^2 \\ \theta_n^3 \\ \theta_{n-1}^3 \end{pmatrix}, \tag{89}$$

satisfies the following relation:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=|\lambda_{1,\ldots,6}J_{|\Upsilon_1|}|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=|\lambda_{1,\ldots,6}J_{|\Upsilon_1|}|, \tag{90}$$

where $|\lambda_{1,\ldots,6}J|_{\Upsilon_1|}$ *are the roots of* $J|_{\Upsilon_1}$.

Proof. If the positive solution of (3) is $\{\Omega_n\}_{n=-1}^{\infty}$, s.t. (88) along with (89) holds. To find the error terms, one has

$$\begin{aligned}
x_{n+1}-\overline{x}&=\frac{\alpha_{10}e^{-y_n}+\alpha_{11}e^{-y_{n-1}}}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}-\frac{(\alpha_{10}+\alpha_{11})e^{-\overline{y}}}{\alpha_{12}+(\alpha_{10}+\alpha_{11})\overline{y}}\\
&=-\frac{\alpha_{10}e^{-y_n}\left(e^{y_n-\overline{y}}-1\right)}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}-\frac{\alpha_{11}e^{-y_{n-1}}\left(e^{y_{n-1}-\overline{y}}-1\right)}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}-\frac{\alpha_{10}\overline{x}(y_n-\overline{y})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}\\
&\quad-\frac{\alpha_{11}\overline{x}(y_{n-1}-\overline{y})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}\\
&=-\frac{\alpha_{10}\left(e^{-y_n}\left(y_n-\overline{y}+O_1\left((y_n-\overline{y})^2\right)\right)+\overline{x}(y_n-\overline{y})\right)}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_n-\overline{y})\\
&\quad-\frac{\alpha_{11}\left(e^{-y_{n-1}}\left(y_{n-1}-\overline{y}+O_2\left((y_{n-1}-\overline{y})^2\right)+\overline{x}(y_{n-1}-\overline{y})\right)\right)}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_{n-1}-\overline{y}).
\end{aligned}\tag{91}$$

So,

$$\begin{aligned}
x_{n+1}-\overline{x}&=-\frac{\alpha_{10}(e^{-y_n}+\overline{x})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_n-\overline{y})-\frac{\alpha_{11}(e^{-y_{n-1}}+\overline{x})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_{n-1}-\overline{y})\\
&\quad+O_1\left((y_n-\overline{y})^2\right)+O_2\left((y_{n-1}-\overline{y})^2\right).
\end{aligned}\tag{92}$$

Similarly,

$$\begin{aligned}
y_{n+1}-\overline{y}&=-\frac{\alpha_{13}(e^{-z_n}+\overline{y})}{\alpha_{15}+\alpha_{13}z_n+\alpha_{14}z_{n-1}}(z_n-\overline{z})-\frac{\alpha_{14}(e^{-z_{n-1}}+\overline{y})}{\alpha_{15}+\alpha_{13}z_n+\alpha_{14}z_{n-1}}(z_{n-1}-\overline{z})\\
&\quad+O_3\left((z_n-\overline{z})^2\right)+O_4\left((z_{n-1}-\overline{z})^2\right),\\
z_{n+1}-\overline{z}&=-\frac{\alpha_{16}(e^{-x_n}+\overline{z})}{\alpha_{18}+\alpha_{16}x_n+\alpha_{17}x_{n-1}}(x_n-\overline{x})-\frac{\alpha_{17}(e^{-x_{n-1}}+\overline{z})}{\alpha_{18}+\alpha_{16}x_n+\alpha_{17}x_{n-1}}(x_{n-1}-\overline{x})+\\
&\quad O_5\left((x_n-\overline{x})^2\right)+O_6\left((x_{n-1}-\overline{x})^2\right).
\end{aligned}\tag{93}$$

From (92) and (93), one gets

$$\begin{aligned}
x_{n+1}-\overline{x}&\approx-\frac{\alpha_{10}(e^{-y_n}+\overline{x})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_n-\overline{y})-\frac{\alpha_{11}(e^{-y_{n-1}}+\overline{x})}{\alpha_{12}+\alpha_{10}y_n+\alpha_{11}y_{n-1}}(y_{n-1}-\overline{y}),\\
y_{n+1}-\overline{y}&\approx-\frac{\alpha_{13}(e^{-z_n}+\overline{y})}{\alpha_{15}+\alpha_{13}z_n+\alpha_{14}z_{n-1}}(z_n-\overline{z})-\frac{\alpha_{14}(e^{-z_{n-1}}+\overline{y})}{\alpha_{15}+\alpha_{13}z_n+\alpha_{14}z_{n-1}}(z_{n-1}-\overline{z}),\\
z_{n+1}-\overline{z}&\approx-\frac{\alpha_{16}(e^{-x_n}+\overline{z})}{\alpha_{18}+\alpha_{16}x_n+\alpha_{17}x_{n-1}}(x_n-\overline{x})-\frac{\alpha_{17}(e^{-x_{n-1}}+\overline{z})}{\alpha_{18}+\alpha_{16}x_n+\alpha_{17}x_{n-1}}(x_{n-1}-\overline{x}).
\end{aligned}\tag{94}$$

Denote

$$\begin{aligned}\theta_n^1 &= x_n - \overline{x},\\ \theta_n^2 &= y_n - \overline{y},\\ \theta_n^3 &= z_n - \overline{z}.\end{aligned} \tag{95}$$

In view of (95), from (94), one gets

$$\begin{aligned}\theta_{n+1}^1 &= \omega_{1n}\theta_n^2 + \omega_{2n}\theta_{n-1}^2,\\ \theta_{n+1}^2 &= \omega_{3n}\theta_n^3 + \omega_{4n}\theta_{n-1}^3,\\ \theta_{n+1}^3 &= \omega_{5n}\theta_n^1 + \omega_{6n}\theta_{n-1}^1,\end{aligned} \tag{96}$$

where

$$\begin{aligned}\omega_{1n} &= -\frac{\alpha_{10}\left(e^{-y_n} + \overline{x}\right)}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}},\\ \omega_{2n} &= -\frac{\alpha_{11}\left(e^{-y_{n-1}} + \overline{x}\right)}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}},\\ \omega_{3n} &= -\frac{\alpha_{13}\left(e^{-z_n} + \overline{y}\right)}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}},\\ \omega_{4n} &= -\frac{\alpha_{14}\left(e^{-z_{n-1}} + \overline{y}\right)}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}},\\ \omega_{5n} &= -\frac{\alpha_{16}\left(e^{-x_n} + \overline{z}\right)}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}},\\ \omega_{6n} &= -\frac{\alpha_{17}\left(e^{-x_{n-1}} + \overline{z}\right)}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}.\end{aligned} \tag{97}$$

From (97), one gets

$$\begin{aligned}\lim_{n\longrightarrow\infty}\omega_{1n} &= -\frac{\alpha_{10}\left(e^{-\overline{y}} + \overline{x}\right)}{\alpha_{12} + \left(\alpha_{10} + \alpha_{11}\right)\overline{y}},\\ \lim_{n\longrightarrow\infty}\omega_{2n} &= -\frac{\alpha_{11}\left(e^{-\overline{y}} + \overline{x}\right)}{\alpha_{12} + \left(\alpha_{10} + \alpha_{11}\right)\overline{y}},\\ \lim_{n\longrightarrow\infty}\omega_{3n} &= -\frac{\alpha_{13}\left(e^{-\overline{z}} + \overline{y}\right)}{\alpha_{15} + \left(\alpha_{13} + \alpha_{14}\right)\overline{z}},\\ \lim_{n\longrightarrow\infty}\omega_{4n} &= -\frac{\alpha_{14}\left(e^{-\overline{z}} + \overline{y}\right)}{\alpha_{15} + \left(\alpha_{13} + \alpha_{14}\right)\overline{z}},\\ \lim_{n\longrightarrow\infty}\omega_{5n} &= -\frac{\alpha_{16}\left(e^{-\overline{x}} + \overline{z}\right)}{\alpha_{18} + \left(\alpha_{16} + \alpha_{17}\right)\overline{x}},\\ \lim_{n\longrightarrow\infty}\omega_{6n} &= -\frac{\alpha_{17}\left(e^{-\overline{x}} + \overline{z}\right)}{\alpha_{18} + \left(\alpha_{16} + \alpha_{17}\right)\overline{x}},\end{aligned} \tag{98}$$

that is,

$$\begin{aligned}\omega_{1n} &= -\frac{\alpha_{10}\left(e^{-\overline{y}} + \overline{x}\right)}{\alpha_{12} + \left(\alpha_{10} + \alpha_{11}\right)\overline{y}} + b1_n,\\ \omega_{2n} &= -\frac{\alpha_{11}\left(e^{-\overline{y}} + \overline{x}\right)}{\alpha_{12} + \left(\alpha_{10} + \alpha_{11}\right)\overline{y}} + b1_{n-1},\\ \omega_{3n} &= -\frac{\alpha_{13}\left(e^{-\overline{z}} + \overline{y}\right)}{\alpha_{15} + \left(\alpha_{13} + \alpha_{14}\right)\overline{z}} + b2_n,\\ \omega_{4n} &= -\frac{\alpha_{14}\left(e^{-\overline{z}} + \overline{y}\right)}{\alpha_{15} + \left(\alpha_{13} + \alpha_{14}\right)\overline{z}} + b2_{n-1},\\ \omega_{5n} &= -\frac{\alpha_{16}\left(e^{-\overline{x}} + \overline{z}\right)}{\alpha_{18} + \left(\alpha_{16} + \alpha_{17}\right)\overline{x}} + b3_n,\\ \omega_{6n} &= -\frac{\alpha_{17}\left(e^{-\overline{x}} + \overline{z}\right)}{\alpha_{18} + \left(\alpha_{16} + \alpha_{17}\right)\overline{x}} + b3_{n-1},\end{aligned} \tag{99}$$

where $b1_n, b1_{n-1}, b2_n, b2_{n-1}, b3_n$, and $b3_{n-1} \longrightarrow 0$ as $n \longrightarrow \infty$. Now, we have system 1.10 of [6], where $A = J|_{\Upsilon_1}$ and

$$B(n) = \begin{pmatrix} 0 & 0 & b1_n & b1_{n-1} & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & b2_n & b2_{n-1}\\ 0 & 0 & 1 & 0 & 0 & 0\\ b3_n & b3_{n-1} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}, \tag{100}$$

such that $\|B(n)\| \longrightarrow \infty, n \longrightarrow \infty$. So, about Υ_1 of (3) the limiting system becomes

$$\begin{pmatrix}\theta_{n+1}^1\\ \theta_n^1\\ \theta_{n+1}^2\\ \theta_n^2\\ \theta_{n+1}^3\\ \theta_n^3\end{pmatrix} = J|_{\Upsilon_1}\begin{pmatrix}\theta_n^1\\ \theta_{n-1}^1\\ \theta_n^2\\ \theta_{n-1}^2\\ \theta_n^3\\ \theta_{n-1}^3\end{pmatrix}, \tag{101}$$

which is as $J|_{\Upsilon_1}$ about Υ_1. □

In the following theorem, we will summarize the convergence results for systems (4) to (8).

Theorem 8. *(i) If the positive solution of (4) is $\{\Omega_n\}_{n=-1}^{\infty}$, s.t. (87) along with the following relation holds:*

$$\begin{aligned}\overline{x} &\in \left[L_4, U_4\right],\\ \overline{y} &\in \left[L_5, U_5\right],\\ \overline{z} &\in \left[L_6, U_6\right],\end{aligned} \tag{102}$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_2\right|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_2\right|, \quad (103)$$

where $|\lambda_{1,\ldots,6}J|\Upsilon_2|$ *are the roots of* $J|\Upsilon_2$.

(ii) If the positive solution of (5) is $\{\Omega_n\}_{n=-1}^{\infty}$, *s.t. (87) along with the following relation holds:*

$$\overline{x}\in[L_7,U_7],$$
$$\overline{y}\in[L_8,U_8], \quad (104)$$
$$\overline{z}\in[L_9,U_9],$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_3\right|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_3\right|, \quad (105)$$

where $|\lambda_{1,\ldots,6}J|\Upsilon_3|$ *are the roots of* $J|\Upsilon_3$.

(iii) If the positive solution of (6) is $\{\Omega_n\}_{n=-1}^{\infty}$, *s.t. (87) along with the following relation holds:*

$$\overline{x}\in[L_{10},U_{10}],$$
$$\overline{y}\in[L_{11},U_{11}], \quad (106)$$
$$\overline{z}\in[L_{12},U_{12}],$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_4\right|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_4\right|, \quad (107)$$

where $|\lambda_{1,\ldots,6}J|\Upsilon_4|$ *are the roots of* $J|\Upsilon_4$.

(iv) If the positive solution of (7) is $\{\Omega_n\}_{n=-1}^{\infty}$, *s.t. (87) along with the following relation holds:*

$$\overline{x}\in[L_{13},U_{13}],$$
$$\overline{y}\in[L_{14},U_{14}], \quad (108)$$
$$\overline{z}\in[L_{15},U_{15}],$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_5\right|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_5\right|, \quad (109)$$

where $|\lambda_{1,\ldots,6}J|\Upsilon_5|$ *are the roots of* $J|\Upsilon_5$.

(v) If the positive solution of (8) is $\{\Omega_n\}_{n=-1}^{\infty}$, *s.t. (87) along with the following relation holds:*

$$\overline{x}\in[L_{16},U_{16}],$$
$$\overline{y}\in[L_{17},U_{17}], \quad (110)$$
$$\overline{z}\in[L_{18},U_{18}],$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\left(\|\theta_n\|\right)^{1/n}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_6\right|,$$
$$\lim_{n\longrightarrow\infty}\frac{\|\theta_{n+1}\|}{\|\theta_n\|}=\left|\lambda_{1,\ldots,6}J\right|\Upsilon_6\right|, \quad (111)$$

where $|\lambda_{1,\ldots,6}J|\Upsilon_6|$ *are the roots of* $J|\Upsilon_6$.

Proof. It is similar to Theorem 7, and hence its proof is omitted. □

8. Discussion and Simulations

In the reported work, we have explored the global dynamics of $(2,3)$-type exponential systems of difference equations. We have investigated that $\{\Omega_n\}_{n=-1}^{\infty}$ of systems (3) to (8) is bounded and persists, and the corresponding invariant rectangles, respectively, are $[L_1,U_1]\times[L_2,U_2]\times[L_3,U_3]$, $[L_4,U_4]\times[L_5,U_5]\times[L_6,U_6]$, $[L_7,U_7]\times[L_8,U_8]\times[L_9,U_9]$, $[L_{10},U_{10}]\times[L_{11},U_{11}]\times[L_{112},U_{12}]$, $[L_{13},U_{13}]\times[L_{14},U_{14}]\times[L_{15},U_{15}]$, and $[L_{16},U_{16}]\times[L_{17},U_{17}]\times[L_{18},U_{18}]$. Further, we have explored the existence and uniqueness of the positive equilibrium and the global and local dynamics of systems (3)–(8). We have also investigated the rate of convergence of the positive solution of systems (3)–(8). Finally, some numerical examples are provided to support the theoretical results. For instance, if $\alpha_i\,(i=10,\ldots,18)$, respectively, are 13, 24, 319, 12, 0.1, 0.2, 1.5, 0.4, and 0.002, then from Figures 1(a)–1(c), the positive fixed point (0.06304125281075143, 0.567941569702224, 0.4220224516533974) of (3) is stable and its corresponding attractor is shown in Figure 1(s). Now, if $\alpha_i\,(i=19,\ldots,27)$, respectively, are 19, 14, 9, 112, 0.1, 0.2, 15, 14, and 2, then from Figures 1(d)–1(f), the positive equilibrium point (0.6122355979161732, 0.996719892698477, 0.8425637558539959) of (4) is stable and its corresponding attractor is shown in Figure 1(t). For (5), if $\alpha_i\,(i=28,\ldots,36)$, respectively, are 9, 0.4, 9, 12, 0.1, 0.2, 15, 2, and 15, then from Figures 1(g)–1(i), its unique positive equilibrium point (0.277465951, 0.924713, 0.88573) is stable and its corresponding attractor is shown in Figure 1(u). For (6), if $\alpha_i\,(i=37,\ldots,45)$,

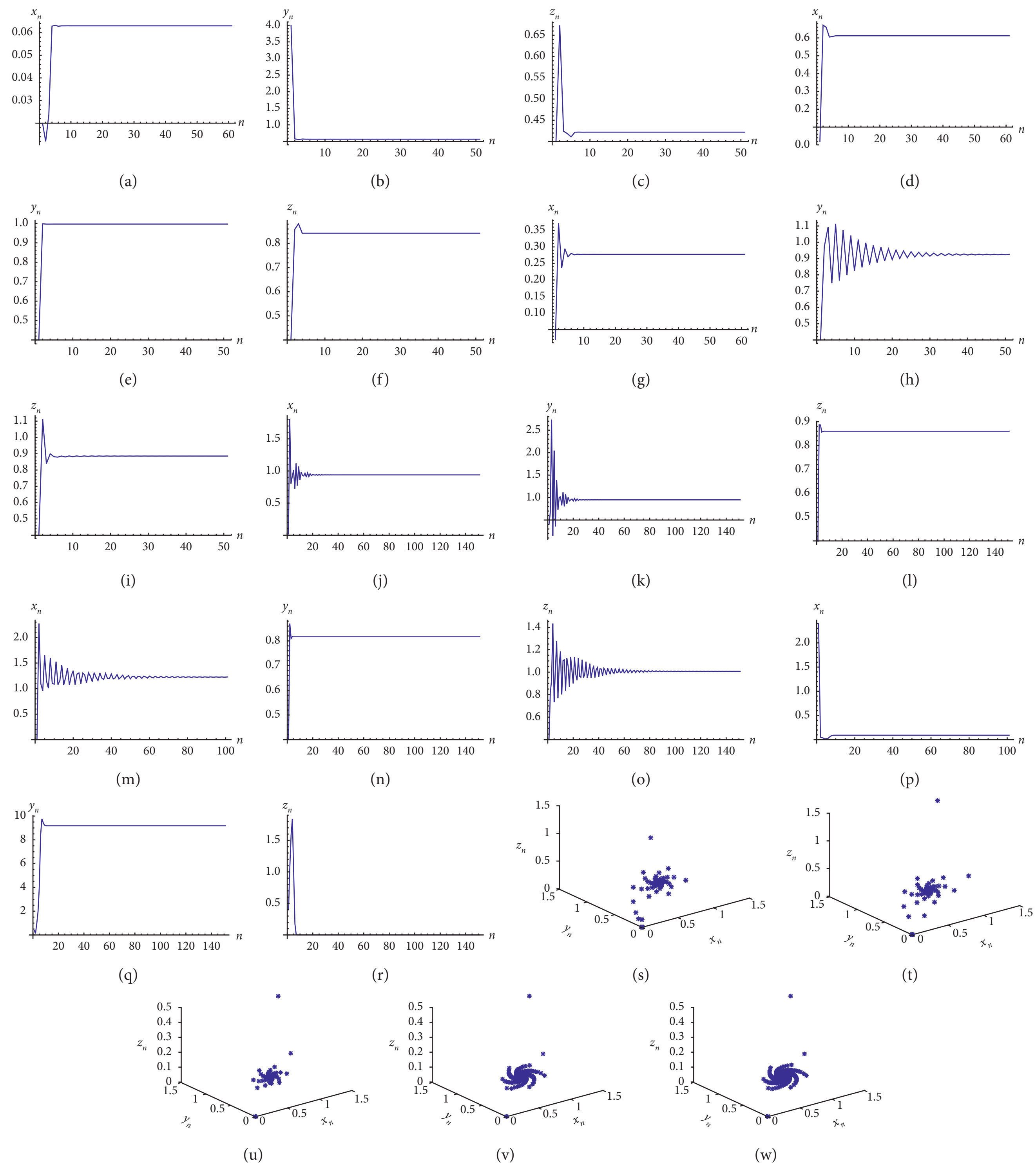

Figure 1: Trajectories for (3) to (8) with initial values $x_s, y_s, z_s\,(s = -1, 0)$, respectively, being $0.7, 2.4, 0.9, 0.4, 1.9$, and 0.4.

respectively, are 29, 14, 9, 12, 0.1, 0.2, 15, 14, and 2, then from Figures 1(j)–1(l), its unique positive equilibrium point (0.9391896591799592, 0.9495483631730844, 0.8598773120562713) is stable and its corresponding attractor is shown in Figure 1(v). For (7), if $\alpha_i\,(i = 46, \ldots, 54)$, respectively, are 29, 14, 1.9, 12, 0.1, 1.2, 15, 14, and 2, then from Figures 1(m)–1(o), its unique positive equilibrium point (1.22564, 0.8167047, 1.007332) is stable and its corresponding attractor is shown in Figure 1(v). Finally, if $\alpha_i\,(i = 55, \ldots, 63)$, respectively, are 9, 4, 129, 12, 0.1, 25, 14, 14, and 2, then from Figures 1(p)–1(r), the unique positive equilibrium point (0.09228515208223682, 9.184938317317343, 0.0000975782517529335) of system (8) is stable and its corresponding attractor is shown in Figure 1(w). For more results on dynamical properties of difference equations, we refer the reader to [7, 8] and the references cited therein.

Data Availability

All the data utilized in this article have been included and the sources from where they were adopted are cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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